Relative $L$-Ritt Order and Related Comparative Growth Properties of Entire Dirichlet Series

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Abstract

In the paper we establish some results on the comparative growth properties related to generalised $L^*$-Ritt order of entire Dirichlet series.

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1 Introduction, Definitions and Notations.

During the past decades, several authors {cf. [1], [2], [4] and [5]} made close investigations on the properties of entire Dirichlet series related to Ritt order. Let \( f(s) \) be an entire function of the complex variable \( s = \sigma + it \) defined by everywhere absolutely convergent Dirichlet series
\[
\sum_{n=1}^{\infty} a_n e^{\lambda_n s} \tag{A}
\]
where \( 0 < \lambda_n < \lambda_{n+1} (n \geq 1) \), \( \lambda_n \to \infty \) as \( n \to \infty \) and \( a_n \)'s are complex constants. If \( \sigma_c \) and \( \sigma_a \) denote respectively the abscissa of convergence and absolute convergence of (A) then in this clearly \( \sigma_c = \sigma_a = \infty \).

Let \( F(\sigma) = \text{lub}_{-\infty < t < \infty} |f(\sigma + it)| \).

Then the Ritt order \([3]\) of \( f(s) \) denoted by \( \rho(f) \) is given by
\[
\rho(f) = \limsup_{\sigma \to \infty} \frac{\log \log F(\sigma)}{\sigma} = \limsup_{\sigma \to \infty} \frac{\log[2] F(\sigma)}{\sigma}.
\]

In other words
\[
\rho(f) = \inf \{ \mu > 0 : \log F(\sigma) < \exp(\sigma \mu) \text{ for all } \sigma > R(\mu) \}.
\]

Similarly the lower Ritt order of \( f(s) \) denoted by \( \lambda(f) \) may be defined. In the paper we prove some results on the comparative growth properties related to the \( L \)-Ritt order of entire Dirichlet series where \( L \equiv L(\sigma) \) is a positive continuous function increasing slowly i.e. \( L(a\sigma) \sim L(\sigma) \) as \( \sigma \to \infty \) for every constant ‘\( a \)’. In the paper we do not explain the standard definitions and notations in the theory of entire functions as those are available in [6]. The following definitions are well known.

**Definition 1** The \( L \)-Ritt order \( \rho^{L}_f \equiv \rho^{L}(f) \) and the \( L \)-Ritt lower order (or equivalently lower \( L \)-Ritt order) \( \lambda^{L}_f \equiv \lambda^{L}(f) \) of \( f(s) \) are defined as follows respectively
\[
\rho^{L}(f) = \limsup_{\sigma \to \infty} \frac{\log[2] F(\sigma)}{\sigma L(\sigma)} \quad \text{and} \quad \lambda^{L}(f) = \liminf_{\sigma \to \infty} \frac{\log[2] F(\sigma)}{\sigma L(\sigma)}.
\]

where \( \log[^k] x = \log (\log[^{k-1}] x) \) for \( k = 1, 2, 3, \ldots \text{and } \log[^0] x = x \). Similarly one can define the relative \( L \)-Ritt order and relative lower \( L \)-Ritt order of \( f(s) \).

**Definition 2** The relative \( L \)-Ritt order \( \rho^{L}_g(f) \) and the relative lower \( L \)-Ritt order \( \lambda^{L}_g(f) \) of \( f(s) \) with respect to entire \( g(s) \) are respectively defined as
\[
\rho^{L}_g(f) = \limsup_{\sigma \to \infty} \frac{G^{-1} \log F(\sigma)}{\sigma L(\sigma)} \quad \text{and} \quad \lambda^{L}_g(f) = \liminf_{\sigma \to \infty} \frac{G^{-1} \log F(\sigma)}{\sigma L(\sigma)}.
\]

Analogously one can define the following.
The hyper $L$-Ritt order $\overline{\rho}^L_f \equiv \overline{\rho}^L(f)$ and the hyper $L$-Ritt lower order (or equivalently hyper lower $L$-Ritt order) $\overline{\lambda}^L_f \equiv \lambda^L(f)$ of $f(s)$ are defined respectively as follows

$$\overline{\rho}^L(f) = \limsup_{\sigma \to \infty} \frac{\log^3 F(\sigma)}{\sigma L(\sigma)} \quad \text{and} \quad \overline{\lambda}^L(f) = \liminf_{\sigma \to \infty} \frac{\log^3 F(\sigma)}{\sigma L(\sigma)}.$$ 

The relative hyper $L$-Ritt order $\overline{\rho}^L_g(f)$ and the relative hyper lower $L$-Ritt order $\overline{\lambda}^L_g(f)$ of $f(s)$ with respect to entire $g(s)$ are respectively defined as

$$\overline{\rho}^L_g(f) = \limsup_{\sigma \to \infty} \frac{G^{-1} \log^2 F(\sigma)}{\sigma L(\sigma)} \quad \text{and} \quad \overline{\lambda}^L_g(f) = \liminf_{\sigma \to \infty} \frac{G^{-1} \log^2 F(\sigma)}{\sigma L(\sigma)}.$$ 

The more generalised concept of $L$-Ritt order (lower $L$-Ritt order) and relative $L$-Ritt order (relative lower $L$-Ritt order) are respectively $L^*\text{-Ritt order (lower } L^*\text{-Ritt order)}$ and relative $L^*\text{-Ritt order (relative lower } L^*\text{-Ritt order).} \text{ We may now state the following definitions.}$

The $L^*\text{-Ritt order } \rho^L_f \equiv \rho^L(f) \text{ and the } L^*\text{-Ritt lower order (or equivalently lower } L^*\text{-Ritt order) } \lambda^L_f \equiv \lambda^L(f) \text{ of } f(s) \text{ are defined respectively as follows}$

$$\rho^L_f = \limsup_{\sigma \to \infty} \frac{\log^2 F(\sigma)}{\sigma \exp L(\sigma)} \quad \text{and} \quad \lambda^L_f = \liminf_{\sigma \to \infty} \frac{\log^2 F(\sigma)}{\sigma \exp L(\sigma)}.$$ 

The relative $L^*\text{-Ritt order } \rho^L_g(f) \text{ and the relative lower } L^*\text{-Ritt order } \lambda^L_g(f) \text{ of } f(s) \text{ with respect to entire } g(s) \text{ are respectively defined as}$

$$\rho^L_g(f) = \limsup_{\sigma \to \infty} \frac{G^{-1} \log F(\sigma)}{\sigma \exp L(\sigma)} \quad \text{and} \quad \lambda^L_g(f) = \liminf_{\sigma \to \infty} \frac{G^{-1} \log F(\sigma)}{\sigma \exp L(\sigma)}.$$ 

The hyper $L^*\text{-Ritt order } \overline{\rho}^L_f \equiv \overline{\rho}^L(f) \text{ and the hyper } L^*\text{-Ritt lower order (hyper lower } L^*\text{-Ritt order) } \overline{\lambda}^L_f \equiv \lambda^L(f) \text{ of } f(s) \text{ are defined respectively as follows}$

$$\overline{\rho}^L(f) = \limsup_{\sigma \to \infty} \frac{\log^3 F(\sigma)}{\sigma \exp L(\sigma)} \quad \text{and} \quad \overline{\lambda}^L(f) = \liminf_{\sigma \to \infty} \frac{\log^3 F(\sigma)}{\sigma \exp L(\sigma)}.$$
The generalised relative lower $L$-Ritt order $\rho^L_\lambda(f)$ and the generalised relative lower $L^*$-Ritt order $\overline{\lambda}^L_\lambda(f)$ of $f(s)$ with respect to entire $g(s)$ are respectively defined as

$$
\rho^L_\lambda(f) = \limsup_{\sigma \to \infty} \frac{G^{-1} \log^k F(\sigma)}{\sigma \exp L(\sigma)} \quad \text{and} \quad \overline{\lambda}^L_\lambda(f) = \liminf_{\sigma \to \infty} \frac{G^{-1} \log^k F(\sigma)}{\sigma \exp L(\sigma)}.
$$

Generalising our notion we may state the following definitions.

**Definition 9** The generalised $L$-Ritt order $^{(k)}\rho^L_\lambda(f)$ and the generalised $L$-Ritt lower order (generalised lower $L$-Ritt order) $^{(k)}\lambda^L_\lambda(f)$ are defined respectively as follows.

$$
^{(k)}\rho^L_\lambda(f) = \limsup_{\sigma \to \infty} \frac{\log^k F(\sigma)}{\sigma L(\sigma)} \quad \text{and} \quad ^{(k)}\lambda^L_\lambda(f) = \liminf_{\sigma \to \infty} \frac{\log^k F(\sigma)}{\sigma L(\sigma)}
$$

where $k = 2, 3, ...$

**Definition 10** The generalised $L^*$-Ritt order $^{(k)}\rho^L_\lambda(f)$ and the generalised $L^*$-Ritt lower order (or equivalently generalised lower $L^*$-Ritt order) $^{(k)}\lambda^L_\lambda(f)$ are respectively defined as

$$
^{(k)}\rho^L_\lambda(f) = \limsup_{\sigma \to \infty} \frac{\log^k F(\sigma)}{\sigma \exp L(\sigma)} \quad \text{and} \quad ^{(k)}\lambda^L_\lambda(f) = \liminf_{\sigma \to \infty} \frac{\log^k F(\sigma)}{\sigma \exp L(\sigma)}
$$

where $k = 2, 3, ...$

**Definition 11** The generalised relative $L$-Ritt order $^{(k)}\rho^L_\lambda(f)$ and the generalised relative lower $L$-Ritt order $^{(k)}\lambda^L_\lambda(f)$ of $f(s)$ with respect to entire $g(s)$ are respectively defined as

$$
^{(k)}\rho^L_\lambda(f) = \limsup_{\sigma \to \infty} \frac{G^{-1} \log^k F(\sigma)}{\sigma L(\sigma)} \quad \text{and} \quad ^{(k)}\lambda^L_\lambda(f) = \liminf_{\sigma \to \infty} \frac{G^{-1} \log^k F(\sigma)}{\sigma L(\sigma)}
$$

where $k = 1, 2, 3, ...$

**Definition 12** The generalised relative $L^*$-Ritt order $^{(k)}\rho^L_\lambda(f)$ and the generalised relative lower $L^*$-Ritt order $^{(k)}\lambda^L_\lambda(f)$ of $f(s)$ with respect to $g(s)$ are respectively defined as follows

$$
^{(k)}\rho^L_\lambda(f) = \limsup_{\sigma \to \infty} \frac{G^{-1} \log^k F(\sigma)}{\sigma \exp L(\sigma)} \quad \text{and} \quad ^{(k)}\lambda^L_\lambda(f) = \liminf_{\sigma \to \infty} \frac{G^{-1} \log^k F(\sigma)}{\sigma \exp L(\sigma)}
$$

where $k = 1, 2, 3, ...$
2 Theorems.

In this section we present the main results of the paper.

**Theorem 1** Let $f$ and $g$ be two entire functions such that $0 < (k)\lambda^{L^*}(fog) \leq (k)\rho^{L^*}(fog) < \infty$ and $0 < (k)\lambda^{L^*}(g) \leq (k)\rho^{L^*}(g) < \infty$. Then

\[
\frac{(k)\lambda^{L^*}(fog)}{(k)\rho^{L^*}(g)} \leq \liminf_{\sigma \to \infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)}
\]

\[
\leq \min \left\{ \frac{(k)\lambda^{L^*}(fog)}{(k)\lambda^{L^*}(g)}, \frac{(k)\rho^{L^*}(fog)}{(k)\rho^{L^*}(g)} \right\}
\]

\[
\leq \max \left\{ \frac{(k)\lambda^{L^*}(fog)}{(k)\lambda^{L^*}(g)}, \frac{(k)\rho^{L^*}(fog)}{(k)\rho^{L^*}(g)} \right\}
\]

\[
\leq \limsup_{\sigma \to \infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{(k)\rho^{L^*}(fog)}{(k)\lambda^{L^*}(g)}
\]

where $k = 1, 2, 3, ...$

**Proof.** From the definition of generalised $L^*$-Ritt order and generalised $L^*$-lower Ritt order of entire $g$ we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $\sigma$,

\[
\log^{[k]} G(\sigma) \leq (k)\rho^{L^*}(g) + \varepsilon)\sigma \exp L(\sigma)
\]

(1)

and $\log^{[k]} G(\sigma) \geq (k)\lambda^{L^*}(g) - \varepsilon)\sigma \exp L(\sigma)$.

(2)

Also for a sequence of values of $\sigma$ tending to infinity,

\[
\log^{[k]} G(\sigma) \leq (k)\lambda^{L^*}(g) + \varepsilon)\sigma \exp L(\sigma)
\]

(3)

and $\log^{[k]} G(\sigma) \geq (k)\rho^{L^*}(g) - \varepsilon)\sigma \exp L(\sigma)$.

(4)

Now again from the definition of generalised $L^*$-Ritt order and generalised $L^*$-lower Ritt order of the composite function $fog$ we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $\sigma$,

\[
\log^{[k]} FoG(\sigma) \leq (k)\rho^{L^*}(fog) + \varepsilon)\sigma \exp L(\sigma)
\]

(5)

and $\log^{[k]} FoG(\sigma) \geq (k)\lambda^{L^*}(fog) - \varepsilon)\sigma \exp L(\sigma)$.

(6)
Again for a sequence of values of $\sigma$ tending to infinity

\[
\log^{[k]} F o G(\sigma) \leq (^{(k)}\lambda^{L^*}(fog) + \varepsilon)\sigma \exp L(\sigma)
\]  

and \[\log^{[k]} F o G(\sigma) \geq (^{(k)}\rho^{L^*}(fog) - \varepsilon)\sigma \exp L(\sigma)\].

Now from (1) and (6) it follows for all sufficiently large values of $\sigma$, \[\frac{\log^{[k]} F o G(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{^{(k)}\lambda^{L^*}(fog) - \varepsilon}{^{(k)}\rho^{L^*}(g) + \varepsilon} .\]  

As $\varepsilon(>0)$ is arbitrary, we obtain that \[\liminf_{\sigma \to \infty} \frac{\log^{[k]} F o G(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{^{(k)}\lambda^{L^*}(fog)}{^{(k)}\rho^{L^*}(g)} .\]  

Again combining (2) and (7) we get for a sequence of values of $\sigma$ tending to infinity, \[\frac{\log^{[k]} F o G(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{^{(k)}\lambda^{L^*}(fog) + \varepsilon}{^{(k)}\lambda^{L^*}(g) - \varepsilon} .\]  

Since $\varepsilon(>0)$ is arbitrary, it follows that \[\liminf_{\sigma \to \infty} \frac{\log^{[k]} F o G(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{^{(k)}\lambda^{L^*}(fog)}{^{(k)}\lambda^{L^*}(g)} .\]  

Similarly from (4) and (5) it follows for a sequence of values of $\sigma$ tending to infinity that \[\frac{\log^{[k]} F o G(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{^{(k)}\rho^{L^*}(fog) + \varepsilon}{^{(k)}\rho^{L^*}(g) - \varepsilon} .\]  

As $\varepsilon(>0)$ is arbitrary, we obtain that \[\liminf_{\sigma \to \infty} \frac{\log^{[k]} F o G(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{^{(k)}\rho^{L^*}(fog)}{^{(k)}\rho^{L^*}(g)} .\]  

Now combining (9), (10) and (11) we get that

\[
\frac{^{(k)}\lambda^{L^*}(fog)}{^{(k)}\rho^{L^*}(g)} \leq \liminf_{\sigma \to \infty} \frac{\log^{[k]} F o G(\sigma)}{\log^{[k]} G(\sigma)}
\]
As \( \varepsilon \) tends to infinity, we get that
\[
\leq \min \left\{ \frac{(k)\lambda^L(fog)}{(k)\lambda^L(g)}, \frac{(k)\rho^L(fog)}{(k)\rho^L(g)} \right\}. \tag{12}
\]

Now from (3) and (6) we obtain for a sequence of values of \( \sigma \) tending to infinity,
\[
\frac{\log^{[k]} FOG(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{(k)\lambda^L(fog) - \varepsilon}{(k)\lambda^L(g) + \varepsilon}.
\]
Choosing \( \varepsilon \to 0 \) we get that
\[
\limsup_{\sigma \to \infty} \frac{\log^{[k]} FOG(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{(k)\lambda^L(fog)}{(k)\lambda^L(g)} \tag{13}
\]
Again from (2) and (5) it follows for all sufficiently large values of \( \sigma \),
\[
\frac{\log^{[k]} FOG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{(k)\rho^L(fog) + \varepsilon}{(k)\lambda^L(g) - \varepsilon}.
\]
As \( \varepsilon(>0) \) is arbitrary, we obtain that
\[
\limsup_{\sigma \to \infty} \frac{\log^{[k]} FOG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{(k)\rho^L(fog)}{(k)\lambda^L(g)} \tag{14}
\]
Similarly combining (1) and (8) we get for a sequence of values of \( \sigma \) tending to infinity,
\[
\frac{\log^{[k]} FOG(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{(k)\rho^L(fog) - \varepsilon}{(k)\rho^L(g) + \varepsilon}.
\]
Since \( \varepsilon(>0) \) is arbitrary, it follows that
\[
\limsup_{\sigma \to \infty} \frac{\log^{[k]} FOG(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{(k)\rho^L(fog)}{(k)\rho^L(g)} \tag{15}
\]
Therefore combining (13), (14) and (15) we get that
\[
\max \left\{ \frac{(k)\lambda^L(fog)}{(k)\lambda^L(g)}, \frac{(k)\rho^L(fog)}{(k)\rho^L(g)} \right\} \leq \limsup_{\sigma \to \infty} \frac{\log^{[k]} FOG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{(k)\rho^L(fog)}{(k)\lambda^L(g)}. \tag{16}
\]
Thus the theorem follows from (12) and (16).

Remark 1 If we take \( 0 < (k)\lambda^L(f) \leq (k)\rho^L(f) < \infty \) instead of \( 0 < (k)\lambda^L(g) \leq (k)\rho^L(g) < \infty \) and the other conditions remain the same then also Theorem 1 holds with \( g \) replaced by \( f \) in the denominator as we see in the next theorem.
Theorem 2 Let \( f \) and \( g \) be two entire functions such that \( 0 < (k)\lambda^L(fog) \leq (k)\rho^L(fog) < \infty \) and \( 0 < (k)\lambda^L(f) \leq (k)\rho^L(f) < \infty \). Then

\[
\frac{(k)\lambda^L(fog)}{(k)\rho^L(f)} \leq \liminf_{\sigma \to \infty} \frac{\log^{[k]}F\sigma}{\log^{[k]}F_{\sigma}} \leq \min \left\{ \frac{(k)\lambda^L(fog)}{(k)\lambda^L(f)}, \frac{(k)\rho^L(fog)}{(k)\rho^L(f)} \right\} \leq \max \left\{ \frac{(k)\lambda^L(fog)}{(k)\lambda^L(g)}, \frac{(k)\rho^L(fog)}{(k)\rho^L(g)} \right\} \leq \limsup_{\sigma \to \infty} \frac{\log^{[k]}F\sigma}{\log^{[k]}F_{\sigma}} \leq \frac{(k)\rho^L(fog)}{(k)\lambda^L(f)}
\]

where \( k = 1, 2, 3, \ldots \)
In fact, Theorem 1 and Theorem 2 are the more generalised concept of Theorem 3 and Theorem 4 respectively.

Theorem 3 Let \( f \) and \( g \) be two entire functions such that \( 0 < (k)\lambda^L(fog) \leq (k)\rho^L(fog) < \infty \) and \( 0 < (k)\lambda^L(g) \leq (k)\rho^L(g) < \infty \). Then

\[
\frac{(k)\lambda^L(fog)}{(k)\rho^L(g)} \leq \liminf_{\sigma \to \infty} \frac{\log^{[k]}F\sigma}{\log^{[k]}G_{\sigma}} \leq \min \left\{ \frac{(k)\lambda^L(fog)}{(k)\lambda^L(g)}, \frac{(k)\rho^L(fog)}{(k)\rho^L(g)} \right\} \leq \max \left\{ \frac{(k)\lambda^L(fog)}{(k)\lambda^L(g)}, \frac{(k)\rho^L(fog)}{(k)\rho^L(g)} \right\} \leq \limsup_{\sigma \to \infty} \frac{\log^{[k]}F\sigma}{\log^{[k]}G_{\sigma}} \leq \frac{(k)\rho^L(fog)}{(k)\lambda^L(g)}
\]

where \( k = 1, 2, 3, \ldots \)
The proof is omitted.

Theorem 4 Let \( f \) and \( g \) be two entire functions such that \( 0 < (k)\lambda^L(fog) \leq \)
for arbitrary positive ε. Let

\[
\frac{(k)\lambda^L(f_{og})}{(k)\rho^L(f)} \leq \lim_{\sigma \to \infty} \frac{\log^k[\log^k(\sigma)]}{\log^k F(\sigma)} \leq \min \left\{ \frac{(k)\lambda^L(f)}{(k)\rho^L(f)} \right\} \\
\leq \max \left\{ \frac{(k)\rho^L(f)}{(k)\rho^L(f)} \right\} \\
\leq \lim_{\sigma \to \infty} \frac{\log^k[\log^k(\sigma)]}{\log^k F(\sigma)} \leq \frac{(k)\rho^L(f_{og})}{(k)\lambda^L(f)}
\]

where k = 1, 2, 3, ...

The proof is omitted.

Lahiri and Banerjee [1] studied on relative Ritt order of entire Dirichlet series and proved some basic theorems. In the subsequent theorems we prove something more.

**Theorem 5** Let f, g and h be three entire functions with 0 < (k)\lambda_h(f) \leq (k)\rho_h(f) < \infty and 0 < (k)\lambda_h(g) \leq (k)\rho_h(g) < \infty. Then

\[
(i) \quad \lim_{\sigma \to \infty} \frac{H^{-1}\log^k G(\sigma)}{H^{-1}\log^k F(\sigma)} \leq \frac{(k)\lambda^*_h(g)}{(k)\rho^*_h(f)} \leq \lim_{\sigma \to \infty} \frac{H^{-1}\log^k G(\sigma)}{H^{-1}\log^k F(\sigma)}
\]

and (ii) \lim_{\sigma \to \infty} \frac{H^{-1}\log^k G(\sigma)}{H^{-1}\log^k F(\sigma)} \leq \min \left\{ \frac{(k)\lambda^*_h(g)}{(k)\rho^*_h(f)} \right\} \leq \max \left\{ \frac{(k)\lambda^*_h(g)}{(k)\rho^*_h(f)} \right\} \leq \lim_{\sigma \to \infty} \frac{H^{-1}\log^k G(\sigma)}{H^{-1}\log^k F(\sigma)}.

**Proof.** From the definition of generalised relative \( L^\ast \)-Ritt order and generalised relative \( L^\ast \)-lower Ritt order of entire \( g \) with respect to entire \( h \) we have for arbitrary positive ε and for all sufficiently large values of \( \sigma \),

\[
H^{-1}\log^k G(\sigma) \leq \frac{(k)\rho^*_h(g) + \varepsilon}{\sigma} \exp L(\sigma) \tag{17}
\]

and

\[
H^{-1}\log^k G(\sigma) \geq \frac{(k)\lambda^*_h(g) - \varepsilon}{\sigma} \exp L(\sigma). \tag{18}
\]

Also for a sequence of values of \( \sigma \) tending to infinity,

\[
H^{-1}\log^k G(\sigma) \leq \frac{(k)\lambda^*_h(g) + \varepsilon}{\sigma} \exp L(\sigma) \tag{19}
\]
and $H^{-1} \log^{[k]} G(\sigma) \geq (^{(k)} \rho_h^L^*(g) - \varepsilon)\sigma \exp L(\sigma)$. \hfill (20)

Now again from the definition of generalised relative $L^*$-Ritt order and generalised relative $L^*$-lower Ritt order of entire $f$ with respect to entire $h$ we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $\sigma$,

$$H^{-1} \log^{[k]} F(\sigma) \leq (^{(k)} \rho_h^L^*(f) + \varepsilon)\sigma \exp L(\sigma)$$ \hfill (21)

and $H^{-1} \log^{[k]} F(\sigma) \geq (^{(k)} \lambda_h^L^*(f) - \varepsilon)\sigma \exp L(\sigma)$. \hfill (22)

Also for a sequence of values of $\sigma$ tending to infinity,

$$H^{-1} \log^{[k]} F(\sigma) \leq (^{(k)} \lambda_h^L^*(f) + \varepsilon)\sigma \exp L(\sigma)$$ \hfill (23)

and $H^{-1} \log^{[k]} F(\sigma) \geq (^{(k)} \rho_h^L^*(f) - \varepsilon)\sigma \exp L(\sigma)$. \hfill (24)

Now from (17) and (22) it follows for all sufficiently large values of $\sigma$,

$$\frac{H^{-1} \log^{[k]} G(\sigma)}{H^{-1} \log^{[k]} F(\sigma)} \leq \frac{(^{(k)} \rho_h^L^*(g) + \varepsilon)}{(^{(k)} \lambda_h^L^*(f) - \varepsilon)}.$$ \hfill (25)

As $\varepsilon(>0)$ is arbitrary, we obtain from above that

$$\liminf_{\sigma \to \infty} \frac{H^{-1} \log^{[k]} G(\sigma)}{H^{-1} \log^{[k]} F(\sigma)} \leq \frac{(^{(k)} \rho_h^L^*(g))}{(^{(k)} \lambda_h^L^*(f))}.$$ \hfill (26)

Again combining (18) and (23) we get for a sequence of values of $\sigma$ tending to infinity,

$$\frac{H^{-1} \log^{[k]} G(\sigma)}{H^{-1} \log^{[k]} F(\sigma)} \leq \frac{(^{(k)} \lambda_h^L^*(g) - \varepsilon)}{(^{(k)} \lambda_h^L^*(f) + \varepsilon)}.$$ \hfill (26)

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{\sigma \to \infty} \frac{H^{-1} \log^{[k]} G(\sigma)}{H^{-1} \log^{[k]} F(\sigma)} \geq \frac{(^{(k)} \lambda_h^L^*(g))}{(^{(k)} \lambda_h^L^*(f))}.$$ \hfill (26)

Similarly from (20) and (21) it follows for a sequence of values of $r$ tending to infinity,

$$\frac{H^{-1} \log^{[k]} G(\sigma)}{H^{-1} \log^{[k]} F(\sigma)} \geq \frac{(^{(k)} \rho_h^L^*(g) - \varepsilon)}{(^{(k)} \rho_h^L^*(f) + \varepsilon)}.$$
As \( \varepsilon(>0) \) is arbitrary, we obtain that
\[
\limsup_{\sigma \to \infty} \frac{H^{-1} \log[k] G(\sigma)}{H^{-1} \log[k] F(\sigma)} \geq \frac{(k) \rho_h^*(g)}{(k) \rho_h^*(f)}.
\]  
(27)

Now from (19) and (22) we obtain for a sequence of values of \( r \) tending to infinity,
\[
\frac{H^{-1} \log[k] G(\sigma)}{H^{-1} \log[k] F(\sigma)} \leq \frac{(k) \lambda_h^L(g) + \varepsilon}{(k) \lambda_h^L(f) - \varepsilon}.
\]
Choosing \( \varepsilon \to 0 \) we get that
\[
\liminf_{\sigma \to \infty} \frac{H^{-1} \log[k] G(\sigma)}{H^{-1} \log[k] F(\sigma)} \leq \frac{(k) \lambda_h^L(g)}{(k) \lambda_h^L(f)}.
\]  
(28)

Again from (18) and (23) it follows for all sufficiently large values of \( r \),
\[
\frac{H^{-1} \log[k] G(\sigma)}{H^{-1} \log[k] F(\sigma)} \geq \frac{(k) \lambda_h^L(g) - \varepsilon}{(k) \lambda_h^L(f) + \varepsilon}.
\]
As \( \varepsilon(>0) \) is arbitrary, we obtain that
\[
\limsup_{\sigma \to \infty} \frac{H^{-1} \log[k] G(\sigma)}{H^{-1} \log[k] F(\sigma)} \geq \frac{(k) \lambda_h^L(g)}{(k) \lambda_h^L(f)}. 
\]  
(29)

Similarly combining (17) and (24) we get for a sequence of values of \( r \) tending to infinity,
\[
\frac{H^{-1} \log[k] G(\sigma)}{H^{-1} \log[k] F(\sigma)} \leq \frac{(k) \rho_h^*(g) + \varepsilon}{(k) \rho_h^*(f) - \varepsilon}.
\]
Since \( \varepsilon(>0) \) is arbitrary, it follows that
\[
\liminf_{\sigma \to \infty} \frac{H^{-1} \log[k] G(\sigma)}{H^{-1} \log[k] F(\sigma)} \leq \frac{(k) \rho_h^*(g)}{(k) \rho_h^*(f)}. 
\]  
(30)

Combining (28) and (29) we obtain that
\[
\liminf_{\sigma \to \infty} \frac{H^{-1} \log[k] G(\sigma)}{H^{-1} \log[k] F(\sigma)} \leq \frac{(k) \lambda_h^L(g)}{(k) \lambda_h^L(f)} \leq \limsup_{\sigma \to \infty} \frac{H^{-1} \log[k] G(\sigma)}{H^{-1} \log[k] F(\sigma)}.
\]
This proves the first part of the theorem. Again combining (26) and (27) it follows that
\[
\limsup_{\sigma \to \infty} \frac{H^{-1} \log^{[k]} G(\sigma)}{H^{-1} \log^{[k]} F(\sigma)} \geq \max \left\{ \frac{(k) \lambda^L_{h} (g)}{(k) \lambda^L_{h} (f)}, \frac{(k) \rho^L_{h} (g)}{(k) \rho^L_{h} (f)} \right\},
\] (31)

Now combining (28) and (30) we get that
\[
\liminf_{\sigma \to \infty} \frac{H^{-1} \log^{[k]} G(\sigma)}{H^{-1} \log^{[k]} F(\sigma)} \leq \min \left\{ \frac{(k) \lambda^L_{h} (g)}{(k) \lambda^L_{h} (f)}, \frac{(k) \rho^L_{h} (g)}{(k) \rho^L_{h} (f)} \right\}.
\] (32)

Thus from (31) and (32) the second part of the theorem follows.

References


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