Some Generalised Growth Properties of Composite Entire Functions Involving their Maximum Terms in Terms of Slowly Changing Functions

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Abstract

In the paper we introduce some definitions and obtain some new results on the comparative growth properties of composite entire and meromorphic functions.

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1 Introduction, Definitions and Notations.

Let $f$ be an entire function defined in the open complex plane $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic
functions which are available in [5] and [3]. In the sequel we use the following notations:

\[ \log^{[k]} x = \log(\log^{[k-1]} x) \] for \( k = 1, 2, 3 \ldots \) and \( \log^{[0]} x = x \)

and

\[ \exp^{[k]} x = \exp(\exp^{[k-1]} x) \] for \( k = 1, 2, 3 \ldots \) and \( \exp^{[0]} x = x \).

The order and lower order of an entire function \( f \) are defined in the following way:

**Definition 1** The order \( \rho_f \) and lower order \( \lambda_f \) of an entire function \( f \) are defined as follows:

\[
\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log r}.
\]

Somasundaram and Thamizharasi [4] introduced the notion of \( L \)-order and \( L \)-lower order for entire functions where \( L = \omega(r) \) is a positive continuous function increasing slowly i.e., \( L(ar) \sim L(r) \) as \( r \to \infty \) for every positive constant \( a \). The more generalised concept for \( L \)-order and \( L \)-lower order for entire functions are \( L^* \)-order and \( L^* \)-lower order respectively. Their definitions are as follows:

**Definition 2** [4] The \( L^* \)-order \( \rho^{L^*}_f \) and \( L^* \)-lower order \( \lambda^{L^*}_f \) of an entire function \( f \) are given by

\[
\rho^{L^*}_f = \limsup_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log \left[ r \exp^{[p]} \omega(r) \right]} \quad \text{and} \quad \lambda^{L^*}_f = \liminf_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log \left[ r \exp^{[p]} \omega(r) \right]}.
\]

In the line of Definition 2 we may introduce the following definitions:

**Definition 3** The \( t \)-th generalised \( pL^* \)-order with rate \( p \) denoted by \( \rho^{(t)}_{L^*} \) and \( t \)-th generalised \( pL^* \)-lower order with rate \( p \) denoted by \( \lambda^{L^*}_{(p)} \) of an entire function \( f \) are defined in the following way:

\[
\rho^{(t)}_{L^*} = \limsup_{r \to \infty} \frac{\log^{[t+1]} M(r, f)}{\log \left[ r \exp^{[p]} \omega(r) \right]} \quad \text{and} \quad \lambda^{(t)}_{L^*} = \liminf_{r \to \infty} \frac{\log^{[t+1]} M(r, f)}{\log \left[ r \exp^{[p]} \omega(r) \right]}.
\]

where both \( p \) and \( t \) are positive integers.

If \( f \) is meromorphic one can easily verify that

\[
\rho^{(t)}_{L^*} = \limsup_{r \to \infty} \frac{\log^{[t]} T(r, f)}{\log \left[ r \exp^{[p]} \omega(r) \right]} \quad \text{and} \quad \lambda^{(t)}_{L^*} = \liminf_{r \to \infty} \frac{\log^{[t]} T(r, f)}{\log \left[ r \exp^{[p]} \omega(r) \right]}.
\]

where both \( p \) and \( t \) are positive integers.

The maximum term \( \mu(r, f) \) of \( f = \sum_{n=0}^{\infty} a_n z^n \) on \( |z| = r \) is defined by

\[
\mu(r, f) = \max_{n \geq 0} (|a_n| r^n).
\]
Since for $0 \leq r < R$,
\[ \mu(r, f) \leq M(r, f) \leq \frac{R}{R - r} \mu(R, f) \]
it is easy to see that
\[ \rho_f = \limsup_{r \to \infty} \frac{\log^2 \mu(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} \mu(r, f)}{\log r}. \]

With the help of the notion of maximum terms of entire functions Definition 3 can be alternatively stated as follows:

**Definition 4** The $t$-th generalised $pL^*$-order with rate $p$ and $t$-th generalised $pL^*$-lower order with rate $p$ of an entire function $f$ respectively denoted by $(t)^{(p)} \rho_f^{L^*}$ and $(t)^{(p)} \lambda_f^{L^*}$ are defined as:
\[ (t)^{(p)} \rho_f^{L^*} = \limsup_{r \to \infty} \frac{\log^{[t+1]} \mu(r, f)}{\log [r \exp^{[p]} L(r)]} \quad \text{and} \quad (t)^{(p)} \lambda_f^{L^*} = \liminf_{r \to \infty} \frac{\log^{[t+1]} \mu(r, f)}{\log [r \exp^{[p]} L(r)]}. \]

Datta and Kar [2], Datta and Mondal [1] studied on some growth relationships of composite entire functions. In this paper we further investigate their results and obtain few generalised theorems on the comparative growth properties of composition of two entire and meromorphic functions.

## 2 Theorems.

In this section we present the main results of the paper.

**Theorem 1** Let $f$ and $g$ be two entire functions such that $0 < (t)^{(p)} \lambda_f^{L^*} < \infty$ and $0 < (t)^{(p)} \rho_f^{L^*} < \infty$. Then

\[ (i) \quad \liminf_{r \to \infty} \frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} \leq \frac{(t)^{(p)} \rho_f^{L^*}}{(t)^{(p)} \rho_g^{L^*}} \leq \limsup_{r \to \infty} \frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} \]

\[ (ii) \quad \frac{(t)^{(p)} \lambda_f^{L^*}}{(t)^{(p)} \rho_f^{L^*}} \leq \liminf_{r \to \infty} \frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} \leq \frac{(t)^{(p)} \lambda_f^{L^*}}{(t)^{(p)} \rho_f^{L^*}} \leq \limsup_{r \to \infty} \frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} \]

and
From (4) and (5) we get for a sequence of values of $r$
that
\[
\lim_{r \to \infty} \frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} \leq \min \left\{ \frac{(t)\lambda^L_{fog}}{(t)\rho^*_{fog}}, \frac{(t)\lambda^L_{g}}{(t)\rho^*_{g}} \right\}.
\]
Combining (1) and (2) we get for a sequence of values of $r$ tending to infinity
\[
\lim_{r \to \infty} \frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} \leq \frac{(t)\rho^*_{fog} + \varepsilon}{(t)\rho^*_{g} - \varepsilon}.
\]
As $\varepsilon(> 0)$ is arbitrary, we have
\[
\liminf_{r \to \infty} \frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} \leq \frac{(t)\lambda^L_{fog}}{(t)\rho^*_{fog}}.
\]
Also, for a sequence of values of $r$ tending to infinity we have
\[
\limsup_{r \to \infty} \frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} \geq \frac{(t)\rho^*_{fog} - \varepsilon}{(t)\rho^*_{g} + \varepsilon}.
\]
From (4) and (5) we get for a sequence of values of $r$ tending to infinity that
\[
\limsup_{r \to \infty} \frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} \geq \frac{(t)\rho^*_{fog}}{(t)\rho^*_{g}}.
\]
Thus combining (3) and (6) we get the first part of Theorem 1.

\((ii)\) For sufficiently large values of \( r \), we have

\[
\log^{[t+1]} M(r, f \circ g) \geq \left( \frac{(t)}{(p)} \lambda_{f \circ g}^* - \varepsilon \right) \log[r \exp[p] L(r)].
\] (7)

Hence from (5) and (7) we get for sufficiently large values of \( r \) that

\[
\frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} \geq \frac{(t)}{(p)} \lambda_{f \circ g}^* - \varepsilon.
\]

As \( \varepsilon > 0 \) is arbitrary, we have

\[
\liminf_{r \to \infty} \frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} \geq \frac{(t)}{(p)} \lambda_{f \circ g}^*.
\] (8)

Now for all sufficiently large values of \( r \), we have

\[
\log^{[t+1]} M(r, g) \geq \left( \frac{(t)}{(p)} \lambda_{g}^* - \varepsilon \right) \log[r \exp[p] L(r)].
\] (9)

Hence from (1) and (9) we get for all sufficiently large values of \( r \) that

\[
\frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} \leq \frac{(t)}{(p)} \lambda_{g}^* + \varepsilon.
\]

As \( \varepsilon > 0 \) is arbitrary, we have

\[
\limsup_{r \to \infty} \frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} \leq \frac{(t)}{(p)} \lambda_{g}^*.
\] (10)

For a sequence of values of \( r \) tending to infinity we have

\[
\log^{[t+1]} M(r, g) \leq \left( \frac{(t)}{(p)} \lambda_{g}^* + \varepsilon \right) \log[r \exp[p] L(r)].
\] (11)

From (7) and (11) we get for a sequence of values of \( r \) tending to infinity that

\[
\frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} \geq \frac{(t)}{(p)} \lambda_{f \circ g}^* - \varepsilon.
\]

As \( \varepsilon > 0 \) is arbitrary, we have

\[
\limsup_{r \to \infty} \frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} \geq \frac{(t)}{(p)} \lambda_{f \circ g}^*.
\] (12)
Now for a sequence of values of $r$ tending to infinity we have
\[
\log^{[t+1]} M(r, f \circ g) \leq \left( \frac{(t)}{(p)} \lambda_{f \circ g}^* + \varepsilon \right) \log[r \exp^{[p]} L(r)].
\] (13)

Hence from (9) and (13) we get for a sequence of values of $r$ tending to infinity that
\[
\frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} \leq \frac{(t)}{(p)} \lambda_{f \circ g}^* + \varepsilon.
\]

As $\varepsilon(>0)$ is arbitrary, we have
\[
\liminf_{r \to \infty} \frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} \leq \frac{(t)}{(p)} \lambda_{f \circ g}^*.
\] (14)

Combining (8), (10), (12) and (14) we get the second part of Theorem 1.

(iii) Combining (i) and (ii) we get the third part of Theorem 1.$\blacksquare$

**Remark 1** If we take the left factor as meromorphic and right factor as entire, Theorem 1 is still valid with maximum modulus replaced by Nevanlinna’s characteristic function as we see in the following theorem.

**Theorem 2** Let $f$ be meromorphic and $g$ be entire such that $0 < \frac{(t)}{(p)} \lambda_{f \circ g}^* \leq \frac{(t)}{(p)} \rho_{f \circ g} < \infty$ and $0 < \frac{(t)}{(p)} \lambda_{g}^* \leq \frac{(t)}{(p)} \rho_{g} < \infty$. Then

(i) \[
\liminf_{r \to \infty} \frac{\log^{[t]} T(r, f \circ g)}{\log^{[t]} T(r, g)} \leq \frac{(t)}{(p)} \lambda_{f \circ g}^* \leq \frac{(t)}{(p)} \rho_{f \circ g} \leq \limsup_{r \to \infty} \frac{\log^{[t]} T(r, f \circ g)}{\log^{[t]} T(r, g)}
\]

(ii) \[
\frac{(t)}{(p)} \lambda_{f \circ g}^* \leq \liminf_{r \to \infty} \frac{\log^{[t]} T(r, f \circ g)}{\log^{[t]} T(r, g)} \leq \frac{(t)}{(p)} \lambda_{f \circ g}^* \leq \limsup_{r \to \infty} \frac{\log^{[t]} T(r, f \circ g)}{\log^{[t]} T(r, g)} \leq \frac{(t)}{(p)} \rho_{f \circ g} \leq \frac{(t)}{(p)} \lambda_{g}^*
\]

and

(iii) \[
\frac{(t)}{(p)} \lambda_{f \circ g}^* \leq \liminf_{r \to \infty} \frac{\log^{[t]} T(r, f \circ g)}{\log^{[t]} T(r, g)} \leq \min \left\{ \frac{(t)}{(p)} \lambda_{f \circ g}^*, \frac{(t)}{(p)} \rho_{f \circ g} \right\} \leq \max \left\{ \frac{(t)}{(p)} \lambda_{g}^*, \frac{(t)}{(p)} \rho_{g} \right\} \leq \limsup_{r \to \infty} \frac{\log^{[t]} T(r, f \circ g)}{\log^{[t]} T(r, g)} \leq \frac{(t)}{(p)} \rho_{f \circ g} \leq \frac{(t)}{(p)} \lambda_{g}^*
\]
where both $p$ and $t$ are positive integers.

Proof of Theorem 2 can be carried out in the line of Theorem 1 and so is omitted.

**Theorem 3** Let $f$ and $g$ be two entire functions such that $\rho_{f \circ g}^{(t)} \rho_{L^*}^{(p)} = \infty$ and $\rho_{g}^{(t)} \rho_{L^*}^{(p)} < \infty$. Then

$$\limsup_{r \to \infty} \frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} = \infty$$

where both $p$ and $t$ are positive integers.

**Proof** If possible let us suppose that

$$\limsup_{r \to \infty} \frac{\log^{[t+1]} M(r, f \circ g)}{\log^{[t+1]} M(r, g)} < \infty.$$ 

Then there exists $A(>0)$ such that for all sufficiently large values of $r$, we have

$$\log^{[t+1]} M(r, f \circ g) \leq A \log^{[t+1]} M(r, g).$$

(15)

Now from the definition of $\rho_{g}^{(t)} \rho_{L^*}^{(p)}$ we get for all sufficiently large values of $r$ and for arbitrary $\varepsilon(>0)$ that

$$\log^{[t+1]} M(r, g) \leq (\rho_{g}^{(t)} \rho_{L^*}^{(p)} + \varepsilon) \log[r \exp[p] L(r)].$$

(16)

Thus from (15) and (16) we get for all sufficiently large values of $r$ that

$$\frac{\log^{[t+1]} M(r, f \circ g)}{\log[r \exp[p] L(r)]} \leq A(\rho_{g}^{(t)} \rho_{L^*}^{(p)} + \varepsilon)$$

i.e., $\rho_{f \circ g}^{(t)} \rho_{L^*}^{(p)} < \infty$, which is a contradiction. This completes the proof. $\blacksquare$

**Remark 2** If we take $\rho_{f}^{(t)} \rho_{L^*}^{(p)} < \infty$ instead of $\rho_{g}^{(t)} \rho_{L^*}^{(p)} < \infty$ in Theorem 3 and the other conditions remain the same then the theorem remains valid with $g$ replaced by $f$ in the denominator.

**Remark 3** If we take $f$ as meromorphic and $g$ as entire, Theorem 3 is still valid with maximum modulus replaced by Nevanlinna’s characteristic function as we see in the following theorem.

**Theorem 4** Let $f$ be meromorphic and $g$ be entire such that $\rho_{f \circ g}^{(t)} \rho_{L^*}^{(p)} = \infty$ and $\rho_{g}^{(t)} \rho_{L^*}^{(p)} < \infty$. Then

$$\limsup_{r \to \infty} \frac{\log^{[t]} T(r, f \circ g)}{\log^{[t]} T(r, g)} = \infty$$
where both $p$ and $t$ are positive integers.

Proof of Theorem 4 can be carried out in the line of Theorem 3.

In the line of Theorem 1 we may prove the following theorem.

**Theorem 5** Let $f$ and $g$ be two entire functions such that $0 < \frac{(t)}{(p)} \lambda_{fog}^L < \infty$ and $0 < \frac{(t)}{(p)} \lambda_y^L \leq \frac{(t)}{(p)} \rho_g^L < \infty$. Then

\[
\liminf_{r \to \infty} \frac{\log^{[t+1]} \mu(r, f \circ g)}{\log^{[t+1]} \mu(r, g)} \leq \frac{(t)}{(p)} \rho_{fog}^L \leq \limsup_{r \to \infty} \frac{\log^{[t+1]} \mu(r, f \circ g)}{\log^{[t+1]} \mu(r, g)}
\]

\[
\liminf_{r \to \infty} \frac{\log^{[t+1]} \mu(r, f \circ g)}{\log^{[t+1]} \mu(r, g)} \leq \frac{(t)}{(p)} \lambda_{fog}^L \leq \limsup_{r \to \infty} \frac{\log^{[t+1]} \mu(r, f \circ g)}{\log^{[t+1]} \mu(r, g)}
\]

and

\[
\liminf_{r \to \infty} \frac{\log^{[t+1]} \mu(r, f \circ g)}{\log^{[t+1]} \mu(r, g)} \leq \min \left\{ \frac{(t)}{(p)} \lambda_{fog}^L, \frac{(t)}{(p)} \lambda_y^L \right\} \leq \limsup_{r \to \infty} \frac{\log^{[t+1]} \mu(r, f \circ g)}{\log^{[t+1]} \mu(r, g)} \leq \frac{(t)}{(p)} \rho_{fog}^L \leq \min \left\{ \frac{(t)}{(p)} \lambda_{fog}^L, \frac{(t)}{(p)} \lambda_y^L \right\}
\]

where both $p$ and $t$ are positive integers.

The following Theorem can be carried out in line of Theorem 3.

**Theorem 6** Let $f$ and $g$ be two entire functions such that $\frac{(t)}{(p)} \rho_{fog}^L = \infty$ and $\frac{(t)}{(p)} \rho_g^L < \infty$. Then

\[
\limsup_{r \to \infty} \frac{\log^{[t+1]} \mu(r, f \circ g)}{\log^{[t+1]} \mu(r, g)} = \infty
\]

where both $p$ and $t$ are positive integers.

**References**


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