Generalized Closed Sets in Bigeneralized Topological Spaces

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Abstract

The purpose of the present paper is to introduce the concept of \( \mu_{(m,n)} \)-closed sets in bigeneralized topological spaces and study some of their properties. We introduce the notion of \( g_{(m,n)} \)-continuous functions on bigeneralized topological spaces and investigate some of their characterizations.

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1 Introduction

Generalized closed sets in a topological space were introduced by Levin [9] in order to extend many of the important properties of closed sets to a larger family. For instance, it was shown that compactness, normality and completeness in a uniform space are inherited by generalized closed subsets. The study of bitopological spaces was first initiated by Kelly [8] and thereafter a large number of papers have been done to generalize the topological concepts to bitopological setting. Fukutake [7] introduced generalized closed sets and pairwise generalized closure operator in bitopological spaces. He defined a set \( A \) of a bitopological space \( X \) to be \( \tau_i\tau_j \)-generalized closed sets if \( \tau_j\text{-Cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-open in \( X \). Also, he defined a new closure operator and strongly pairwise \( T_{1\frac{1}{2}} \)-space. Á. Császár [3] introduced the concepts of generalized neighborhood systems and generalized topological spaces. He also introduced the concepts of continuous functions and associated interior and closure operators on generalized neighborhood systems and generalized...
topological spaces. In particular, he investigated characterizations for the generalized continuous function by using a closure operator defined on generalized neighborhood systems. C. Boonpok [2] introduced the concept of bigeneralized topological spaces and studied \((m, n)\)-closed sets and \((m, n)\)-open sets in bigeneralized topological spaces.

In this paper, we introduce the notions of \(\mu_{(m,n)}\)-closed sets in bigeneralized topological spaces and study some of their properties. We introduce the notion of \(g_{(m,n)}\)-continuous functions on bigeneralized topological spaces and investigate some of their characterizations.

## 2 Preliminaries

We recall some basic definitions and notations. Let \(X\) be a set and denote \(\text{exp}X\) the power set of \(X\). A subset \(\mu\) of \(\text{exp}X\) is said to be a generalized topology (briefly GT) on \(X\) if \(\emptyset \in \mu\) and an arbitrary union of elements of \(\mu\) belongs to \(\mu\) [3]. Let \(\mu\) be a GT on \(X\), the elements of \(\mu\) are called \(\mu\)-open sets and the complements of \(\mu\)-open sets are called \(\mu\)-closed sets. Let \(M_\mu\) denote the union of all elements of \(\mu\). We say \(\mu\) is strong [4] if \(M_\mu = X\). If \(A \subseteq X\), then \(i_\mu(A)\) denotes the union of all \(\mu\)-open sets contained in \(A\) and \(c_\mu(A)\) is the intersections of all \(\mu\)-closed sets containing \(A\) [5]. According to [6], for \(A \subseteq X\) and \(x \in X\), we have \(x \in c_\mu(A)\) iff \(x \in M \in \mu\) implies \(M \cap A \neq \emptyset\).

Let \((X, \mu)\) be a generalized topological space. A subset \(M\) of \(X\) is said to be \(\mu\)-semi-open iff \(M \subseteq c_\mu(i_\mu(M))\), \(\mu\)-preopen iff \(M \subseteq i_\mu(c_\mu(M))\), \(\mu\)-\(\alpha\)-open iff \(M \subseteq i_\mu(c_\mu(i_\mu(M)))\), finally \(\mu\)-\(\beta\)-open iff \(M \subseteq c_\mu(i_\mu(c_\mu(M)))\) [5]. Let \((X, \mu)\) and \((Y, \mu')\) be generalized topological spaces. A map \(f : (X, \mu) \rightarrow (Y, \mu')\) is said to be \((\mu, \mu')\)-continuous iff \(M' \in \mu'\) implies \(f^{-1}(M') \in \mu\) [3].

**Theorem 2.1.** [3] Let \((X, \mu)\) be a generalized topological space. Then

1. \(c_\mu(A) = X - i_\mu(X - A)\);
2. \(i_\mu(A) = X - c_\mu(X - A)\).

**Proposition 2.2.** [10] Let \((X, \mu)\) be a generalized topological space. For subsets \(A\) and \(B\) of \(X\), the following properties holds:

1. \(c_\mu(X - A) = X - i_\mu(A)\) and \(i_\mu(X - A) = X - c_\mu(A)\);
2. If \((X - A) \in \mu\), then \(c_\mu(A) = A\) and if \(A \in \mu\), then \(i_\mu(A) = A\);
3. If \(A \subseteq B\), then \(c_\mu(A) \subseteq c_\mu(B)\) and \(i_\mu(A) \subseteq i_\mu(B)\);
4. \(A \subseteq c_\mu(A)\) and \(i_\mu(A) \subseteq A\);
5. \(c_\mu(c_\mu(A)) = c_\mu(A)\) and \(i_\mu(i_\mu(A)) = i_\mu(A)\).
**Definition 2.3.** [2] Let $X$ be a nonempty set and let $\mu_1, \mu_2$ be generalized topologies on $X$. A triple $(X, \mu_1, \mu_2)$ is said to be a bigeneralized topological space (briefly $BGTS$).

Let $(X, \mu_1, \mu_2)$ be a bigeneralized topological space and $A$ a subset of $X$. The closure of $A$ and the interior of $A$ with respect to $\mu_m$ are denoted by $c_{\mu_m}(A)$ and $i_{\mu_m}(A)$, respectively, for $m = 1, 2$.

**Definition 2.4.** [2] A subset $A$ of a bigeneralized topological space $(X, \mu_1, \mu_2)$ is called $(m, n)$-closed if $c_{\mu_n}(c_{\mu_m}(A)) = A$, where $m, n = 1, 2$ and $m \neq n$. The complement of $(m, n)$-closed set is called $(m, n)$-open.

**Proposition 2.5.** [2] Let $(X, \mu_1, \mu_2)$ be a bigeneralized topological space and $A$ a subset of $X$. Then $A$ is $(m, n)$-closed if and only if $A$ is both $\mu$-closed in $(X, \mu_m)$ and $(X, \mu_n)$.

**Proposition 2.6.** [2] Let $(X, \mu_1, \mu_2)$ be a bigeneralized topological space. Then $A$ is $(m, n)$-open if and only if $i_{\mu_n}(i_{\mu_m}(A)) = A$.

# 3 Generalized closed sets

In this section, we introduce $\mu_{(m,n)}$-closed sets in bigeneralized topological space and study some of their properties.

**Definition 3.1.** A subset $A$ of a bigeneralized topological space $(X, \mu_1, \mu_2)$ is said to be $(m, n)$ generalized closed (briefly $\mu_{(m,n)}$-closed) set if $c_{\mu_n}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\mu_m$-open set in $X$, where $m, n = 1, 2$ and $m \neq n$. The complement of $\mu_{(m,n)}$-closed set is said to be $(m, n)$ generalized open (briefly $\mu_{(m,n)}$-open) set.

The family of all $\mu_{(m,n)}$-closed (resp. $\mu_{(m,n)}$-open) sets of $(X, \mu_1, \mu_2)$ is denote by $\mu_{(m,n)}-C(X)$ (resp. $\mu_{(m,n)}-O(X)$), where $m, n = 1, 2$ and $m \neq n$.

A subset $A$ of a bigeneralized topological space $(X, \mu_1, \mu_2)$ is called pairwise $\mu$-closed if $A$ is $\mu_{(1,2)}$-closed and $\mu_{(2,1)}$-closed. The complement of pairwise $\mu$-closed set is called pairwise $\mu$-open.

**Remark 1.** Every $(m, n)$-closed set is $\mu_{(m,n)}$-closed. The converse is not true as can be seen from the following example.

**Example 3.2.** Let $X = \{a, b, c\}$. Consider two generalized topologies $\mu_1 = \emptyset, \{a\}, \{a, b\}, X$ and $\mu_2 = \emptyset, \{c\}, \{b, c\}$ on $X$. Then $\{a\}$ is $\mu_{(1,2)}$-closed but is not $(1, 2)$-closed.

**Proposition 3.3.** Let $(X, \mu_1, \mu_2)$ be a bigeneralized topological space and $A$ a subset of $X$. If $A$ is $\mu_n$-closed, then $A$ is $\mu_{(m,n)}$-closed, where $m, n = 1, 2$ and $m \neq n$. 
Remark 2. The union of two $\mu_{(m,n)}$-closed sets is not a $\mu_{(m,n)}$-closed set in general as can be seen from the following example.

Example 3.4. Let $X = \{a, b, c, d\}$. Consider two generalized topologies $\mu_1 = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\mu_2 = \{\emptyset, \{a, b, d\}, \{b, c, d\}, X\}$ on $X$. Then $\{a\}$ and $\{c\}$ are $\mu_{(1,2)}$-closed but $\{a\} \cap \{c\} = \{a, c\}$ is not $\mu_{(1,2)}$-closed.

Remark 3. The intersection of two $\mu_{(m,n)}$-closed sets is not a $\mu_{(m,n)}$-closed set in general as can be seen from the following example.

Example 3.5. Let $X = \{a, b, c\}$. Consider two generalized topologies $\mu_1 = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\mu_2 = \{\emptyset, \{c\}, \{b, c\}\}$ on $X$. Then $\{a\}$ and $\{b\}$ are $\mu_{(1,2)}$-closed but $\{a\} \cup \{b\} = \emptyset$ is not $\mu_{(1,2)}$-closed.

Remark 4. The intersection of two $\mu_{(m,n)}$-open sets is not a $\mu_{(m,n)}$-open set in general as can be seen from the following example.

Example 3.6. Let $X = \{a, b, c, d\}$. Consider two generalized topologies $\mu_1 = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\mu_2 = \{\emptyset, \{a, b, d\}, \{b, c, d\}, X\}$ on $X$. Then $\{a, b, d\}$ and $\{b, c, d\}$ are $\mu_{(1,2)}$-open but $\{a, b, d\} \cap \{b, c, d\} = \{b, d\}$ is not $\mu_{(1,2)}$-open.

Remark 5. The union of two $\mu_{(m,n)}$-open sets is not a $\mu_{(m,n)}$-open set in general as can be seen from the following example.

Example 3.7. Let $X = \{a, b, c\}$. Consider two generalized topologies $\mu_1 = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\mu_2 = \{\emptyset, \{c\}, \{b, c\}\}$ on $X$. Then $\{a\}$ and $\{b, c\}$ are $\mu_{(1,2)}$-open but $\{a\} \cup \{b, c\} = \{b, c\}$ is not $\mu_{(1,2)}$-open.

Proposition 3.8. Let $(X, \mu_1, \mu_2)$ be a bigeneralized topological space. If $A$ is $\mu_{(m,n)}$-closed and $F$ is $(m,n)$-closed, then $A \cap F$ is $\mu_{(m,n)}$-closed, where $m, n = 1, 2$ and $m \neq n$.

Proof. Let $A \cap F \subseteq U$ and $U$ is $\mu_m$-open set. Then $A \subseteq U \cup (X - F)$ and so $c_{\mu_n}(A) \subseteq U \cup (X - F)$. Therefore, $c_{\mu_n}(A) \cap F \subseteq U$. Since $F$ is $(m,n)$-closed, we obtain $c_{\mu_n}(A \cap F) \subseteq c_{\mu_n}(A) \cap c_{\mu_n}(F) \subseteq U$. Hence, $A \cap F$ is $\mu_{(m,n)}$-closed. \qed

Remark 6. $\mu_{(1,2)}$-$C(X)$ is generally not equal to $\mu_{(2,1)}$-$C(X)$ as can be seen from the following example.

Example 3.9. Let $X = \{a, b, c\}$. Consider two generalized topologies $\mu_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\mu_2 = \{\emptyset, \{c\}, \{b, c\}\}$ on $X$. Then $\mu_{(1,2)}$-$C(X) = \{\{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Thus $\mu_{(1,2)}$-$C(X) \neq \mu_{(2,1)}$-$C(X)$.

Proposition 3.10. Let $\mu_1$ and $\mu_2$ be generalized topologies on $X$. If $\mu_1 \subseteq \mu_2$, then $\mu_{(2,1)}$-$C(X) \subseteq \mu_{(1,2)}$-$C(X)$. 
Proposition 3.11. For each element $x$ of a bigeneralized topological space $(X, \mu_1, \mu_2)$, $\{x\}$ is $\mu_m$-closed or $X - \{x\}$ is $\mu_{(m,n)}$-closed, where $m, n = 1, 2$ and $m \neq n$.

Proof. Let $x \in X$ and the singleton $\{x\}$ be not $\mu_m$ closed. Then $X - \{x\}$ is not $\mu_m$-open, if $x \in \mu_m$, then $X$ is only $\mu_m$-open set which contains $X - \{x\}$, hence $X - \{x\}$ is $\mu_{(m,n)}$-closed and if $x \notin \mu_m$, then $X - \{x\}$ is $\mu_{(m,n)}$-closed. □

Proposition 3.12. Let $A$ be a subset of a bigeneralized topological space $(X, \mu_1, \mu_2)$. If $A$ is $\mu_{(m,n)}$-closed, then $c_{\mu_n}(A) - A$ contains no nonempty $\mu_m$-closed set, where $m, n = 1, 2$ and $m \neq n$.

Proof. Let $A$ be a $\mu_{(m,n)}$-closed set and $F \neq \emptyset$ is $\mu_m$-closed set such that $F \subseteq c_{\mu_n}(A) - A$. Since $A \in \mu_{(m,n)}-C(X)$, we have $c_{\mu_n}(A) \subseteq X - F$. Thus $F \subseteq c_{\mu_n}(A) \cap (X - c_{\mu_n}(A)) = \emptyset$, this is a contradiction. Then $c_{\mu_n}(A) - A$ contains no nonempty $\mu_m$-closed set. □

The converse of the above proposition is not true as seen from the following example.

Example 3.13. Let $X = \{a, b, c\}$. Consider two generalized topologies $\mu_1 = \{\emptyset, \{a\}, \{a, c\}\}$ and $\mu_2 = \{\emptyset, \{b\}, X\}$ on $X$. If $A = \{a\}$, then $c_{\mu_2}(A) - A = \{c\}$ does not contain any nonempty $\mu_1$-closed set. But $A$ is not $\mu_{(1,2)}$-closed.

Proposition 3.14. Let $\mu_1$ and $\mu_2$ be generalized topologies on $X$. If $A$ is a $\mu_{(m,n)}$-closed set, then $c_{\mu_n}(\{x\}) \cap A \neq \emptyset$ holds for each $x \in c_{\mu_n}(A)$, where $m, n = 1, 2$ and $m \neq n$.

Proof. Let $x \in c_{\mu_n}(A)$. Suppose that $c_{\mu_n}(\{x\}) \cap A = \emptyset$. Then $A \subseteq X - c_{\mu_m}(\{x\})$. Since $A$ is $\mu_{(m,n)}$-closed and $X - c_{\mu_m}(\{x\})$ is $\mu_m$-open. Thus $c_{\mu_n}(A) \subseteq X - c_{\mu_m}(\{x\})$. Hence, $c_{\mu_n}(A) \cap c_{\mu_m}(\{x\}) = \emptyset$. This is a contradiction. □

Proposition 3.15. If $A$ is a $\mu_{(m,n)}$-closed set of $(X, \mu_1, \mu_2)$ such that $A \subseteq B \subseteq c_{\mu_n}(A)$, then $B$ is $\mu_{(m,n)}$-closed set, where $m, n = 1, 2$ and $m \neq n$.

Proof. Let $A$ be a $\mu_{(m,n)}$-closed set and $A \subseteq B \subseteq c_{\mu_n}(A)$. Let $B \subseteq U$ and $U$ is $\mu_m$-open. Then $A \subseteq U$. Since $A$ is $\mu_{(m,n)}$-closed, we have $c_{\mu_n}(A) \subseteq U$. Since $B \subseteq c_{\mu_n}(A)$, then $c_{\mu_n}(B) \subseteq c_{\mu_n}(A) \subseteq U$. Hence, $B$ is $\mu_{(m,n)}$-closed. □

Theorem 3.16. For a bigeneralized topological space $(X, \mu_1, \mu_2)$, $\mu_m-O(X) \subseteq \mu_n-C(X)$ if and only if every subset of $X$ is $\mu_{(m,n)}$-closed set, where $m, n = 1, 2$ and $m \neq n$.

Proof. Suppose that $\mu_m-O(X) \subseteq \mu_n-C(X)$. Let $A$ be a subset of $X$ such that $A \subseteq U$, where $U \in \mu_m-O(X)$. Then $c_{\mu_n}(A) \subseteq c_{\mu_n}(U) = U$ and hence $A$ is $\mu_{(m,n)}$-closed.

Conversely, suppose that every subset of $X$ is $\mu_{(m,n)}$-closed. Let $U \in \mu_m-O(X)$. Since $U$ is $\mu_{(m,n)}$-closed, we have $c_{\mu_n}(U) \subseteq U$. Therefore, $U \in \mu_n-C(X)$ and hence $\mu_m-O(X) \subseteq \mu_n-C(X)$. □
Theorem 3.17. A subset $A$ of a bigeneralized topological space $(X, \mu_1, \mu_2)$ is $\mu_{(m,n)}$-open if and only if every subset $F$ of $X$, $F \subseteq i_{\mu} (A)$ whenever $F$ is $\mu_m$-closed and $F \subseteq A$, where $m, n = 1, 2$ and $m \neq n$.

Proof. Suppose that $A$ is $\mu_{(m,n)}$-open. We shall show that $F \subseteq i_{\mu} (A)$ whenever $F$ is $\mu_m$-closed and $F \subseteq A$. Let $F \subseteq A$ and $F$ is $\mu_m$-closed. Then $X - A \subseteq X - F$ and $X - F$ is $\mu_m$-open, we have $X - A$ is $\mu_{(m,n)}$-closed, then $c_{\mu} (X - A) \subseteq X - F$. Thus $X - i_{\mu} (A) \subseteq X - F$ and hence $F \subseteq i_{\mu} (A)$.

Conversely, suppose that $F \subseteq i_{\mu} (A)$ whenever $F$ is $\mu_m$-closed and $F \subseteq A$. Let $X - A \subseteq U$ and $U$ is $\mu_m$-open. Then $X - U \subseteq A$ and $X - U$ is $\mu_m$-closed. By assumption, we have $X - U \subseteq i_{\mu} (A)$, then $X - i_{\mu} (A) \subseteq U$. Therefore, $c_{\mu} (X - A) \subseteq U$. Thus $X - A$ is $\mu_{(m,n)}$-closed. Hence, $A$ is $\mu_{(m,n)}$-open. \qed

Theorem 3.18. Let $A$ and $B$ be subsets of a bigeneralized topological space $(X, \mu_1, \mu_2)$ such that $i_{\mu} (A) \subseteq B \subseteq A$. If $A$ is $\mu_{(m,n)}$-open, $B$ is $\mu_{(m,n)}$-open, where $m, n = 1, 2$ and $m \neq n$.

Proof. Suppose that $i_{\mu} (A) \subseteq B \subseteq A$. Let $F$ be $\mu_m$-closed such that $F \subseteq B$. Since $A$ is $\mu_{(m,n)}$-open, $F \subseteq i_{\mu} (A)$. Since $i_{\mu} (A) \subseteq B$, we have $i_{\mu} (i_{\mu} (A)) \subseteq i_{\mu} (B)$. Consequently, $i_{\mu} (A) \subseteq i_{\mu} (B)$. Hence, $F \subseteq i_{\mu} (B)$. Therefore, $B$ is $\mu_{(m,n)}$-open. \qed

Proposition 3.19. If a subset $A$ of a bigeneralized topological space $(X, \mu_1, \mu_2)$ is $\mu_{(m,n)}$-closed, then $c_{\mu} (A) - A$ is $\mu_{(m,n)}$-open, where $m, n = 1, 2$ and $m \neq n$.

Proof. Suppose that $A$ is $\mu_{(m,n)}$-closed. We shall show that $c_{\mu} (A) - A$ is $\mu_{(m,n)}$-open. Let $X - (c_{\mu} (A) - A) \subseteq U$ and $U$ is $\mu_m$-open. Then $X - U \subseteq (c_{\mu} (A) - A)$ and $X - U$ is $\mu_m$-closed. Thus, we have $c_{\mu} (A) - A$ does not contain nonempty $\mu_m$-closed set by Proposition 3.12. Consequently, $X - U = \emptyset$, then $U = X$. Therefore, $c_{\mu} (X - (c_{\mu} (A) - A)) \subseteq U$ so we obtain $X - (c_{\mu} (A) - A)$ is $\mu_{(m,n)}$-closed. Hence, $c_{\mu} (A) - A$ is $\mu_{(m,n)}$-open. \qed

4 Generalized continuous functions

In this section, we introduce $g_{(m,n)}$-continuous functions on bigeneralized topological spaces and investigate some of their properties.

Definition 4.1. Let $(X, \mu_1, \mu_2)$ and $(Y, \nu_1, \nu_2)$ be bigeneralized topological spaces. A function $f : (X, \mu_1, \mu_2) \rightarrow (Y, \nu_1, \nu_2)$ is said to be $(m, n)$-generalized continuous (briefly $g_{(m,n)}$-continuous) if $f^{-1}(F)$ is $\mu_{(m,n)}$-closed in $X$ for every $\mu_n$-closed $F$ of $Y$, where $m, n = 1, 2$ and $m \neq n$.

A function $f : (X, \mu_1, \mu_2) \rightarrow (Y, \nu_1, \nu_2)$ is said to be pairwise generalized continuous (briefly pairwise $g$-continuous) if $f$ is $g_{(1,2)}$-continuous and $g_{(2,1)}$-continuous.
**Theorem 4.2.** For an injective function \( f : (X, m^1_X, m^2_X) \to (Y, m^1_Y, m^2_Y) \), the following properties are equivalent:

(1) \( f \) is \( g_{(m,n)} \)-continuous,

(2) For each \( x \in X \) and for every \( \mu_n \)-open set \( V \) containing \( f(x) \), there exists a \( \mu_{(m,n)} \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \);

(3) \( f(c_{\mu_X}(A)) \subseteq c_{\mu_Y}(f(A)) \) for every subset \( A \) of \( X \);

(4) \( c_{\mu_X}(f^{-1}(B)) \subseteq f^{-1}(c_{\mu_Y}(B)) \) for every subset \( B \) of \( Y \).

**Proof.**

(1) \( \Rightarrow \) (2): Let \( x \in X \) and \( V \) be a \( \mu_n \)-open subset of \( Y \) containing \( f(x) \). Then by (1), \( f^{-1}(V) \) is \( \mu_{(m,n)} \)-open of \( X \) containing \( x \). If \( U = f^{-1}(V) \), then \( f(U) \subseteq V \).

(2) \( \Rightarrow \) (3): Let \( A \) be a subset of \( X \) and \( f(x) \notin c_{\mu_Y}(f(A)) \). Then, there exists a \( \mu_n \)-open subset \( V \) of \( Y \) containing \( f(x) \) such that \( V \cap f(A) = \emptyset \). Then by (2), there exists a \( \mu_{(m,n)} \)-open set such that \( f(x) \notin f(U) \subseteq V \). Hence, \( f(U) \cap f(A) = \emptyset \) implies \( U \cap A = \emptyset \). Consequently, \( x \notin c_{\mu_X}(A) \) and \( f(x) \notin f(c_{\mu_Y}(A)) \).

(3) \( \Rightarrow \) (4): Let \( B \) be a subset of \( Y \). By (3), we obtain \( f(c_{\mu_X}(f^{-1}(B))) \subseteq c_{\mu_Y}(f(f^{-1}(B))) \). Thus \( c_{\mu_X}(f^{-1}(B)) \subseteq f^{-1}(c_{\mu_Y}(B)) \).

(4) \( \Rightarrow \) (1): Let \( F \) be a \( \mu_n \)-closed subset of \( Y \). Let \( U \) be a \( \mu_m \)-open subset of \( X \) such that \( f^{-1}(F) \subseteq U \). Since \( c_{\mu_Y}(F) = F \) and by (4), \( c_{\mu_X}(f^{-1}(F)) \subseteq U \). Hence, \( f \) is \( g_{(m,n)} \)-continuous. \( \square \)

**Definition 4.3.** Let \((X, \mu^1_X, \mu^2_X)\) and \((Y, \mu^1_Y, \mu^2_Y)\) be generalized topological spaces. A function \( f : (X, \mu^1_X, \mu^2_X) \to (Y, \mu^1_Y, \mu^2_Y) \) is said to be \( g_m \)-continuous if \( f^{-1}(F) \) is \( \mu_{mn} \)-closed in \( X \) for every \( \mu_{mn} \)-closed \( F \) of \( Y \), for \( m = 1, 2 \).

**Definition 4.4.** Let \((X, \mu^1_X, \mu^2_X)\) and \((Y, \mu^1_Y, \mu^2_Y)\) be generalized topological spaces. A function \( f : (X, \mu^1_X, \mu^2_X) \to (Y, \mu^1_Y, \mu^2_Y) \) is said to be \( g_m \)-open if \( f(F) \) is \( \mu_{mn} \)-open (resp. \( \mu_{mn} \)-closed) for every \( \mu_{mn} \)-open (resp. \( \mu_{mn} \)-closed) \( F \) of \( X \), for \( m = 1, 2 \).

**Proposition 4.5.** If \( f : (X, \mu^1_X, \mu^2_X) \to (Y, \mu^1_Y, \mu^2_Y) \) is \( g_m \)-continuous and \( g_m \)-closed, then \( f(A) \) is \( \mu_{mn} \)-closed subset of \( Y \) for every \( \mu_{mn} \)-closed subset \( A \) of \( X \), where \( m, n = 1, 2 \) and \( m \neq n \).

**Proof.** Let \( U \) be a \( \mu_{mn} \)-open subset of \( Y \) such that \( f(A) \subseteq U \). Then \( A \subseteq f^{-1}(U) \) and \( f^{-1}(U) \) is \( \mu_{mn} \)-open subset of \( X \). Since \( A \) is \( \mu_{mn} \)-closed, \( c_{\mu_Y}(A) \subseteq f^{-1}(U) \) and hence \( f(c_{\mu_X}(A)) \subseteq U \). Therefore, we have \( c_{\mu_Y}(f(A)) \subseteq c_{\mu_Y}(f(c_{\mu_X}(A))) = f(c_{\mu_X}(A)) \subseteq U \). Therefore, \( f(A) \) is \( \mu_{mn} \)-closed subset of \( Y \). \( \square \)

**Lemma 4.6.** If \( f : (X, \mu^1_X, \mu^2_X) \to (Y, \mu^1_Y, \mu^2_Y) \) is \( g_m \)-closed, then for each subset \( S \) of \( Y \) and each \( \mu_{mn} \)-open subset \( U \) of \( X \) containing \( f^{-1}(S) \), there exists a \( \mu_{mn} \)-open subset \( V \) of \( Y \) such that \( f^{-1}(V) \subseteq U \).
Proposition 4.7. If \( f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2) \) is injective, \( g_m \)-closed and \( g_{(m,n)} \)-continuous, then \( f^{-1}(B) \) is \( \mu_{(m,n)} \)-closed subset of \( X \) for every \( \mu_{(m,n)} \)-closed subset \( B \) of \( Y \), where \( m, n = 1, 2 \) and \( m \neq n \).

Proof. Let \( B \) be a \( \mu_{(m,n)} \)-closed subset of \( Y \). Let \( U \) be a \( \mu_m \)-open subset of \( X \) such that \( f^{-1}(B) \subseteq U \). Since \( f \) is \( g_m \)-closed and by Lemma 4.6, there exists a \( \mu_m \)-open subset of \( Y \) such that \( B \subseteq V \) and \( f^{-1}(V) \subseteq U \). Since \( B \) is \( \mu_{(m,n)} \)-closed set and \( B \subseteq V \), then \( c_{\mu_Y^m}(B) \subseteq V \). Consequently, \( f^{-1}(c_{\mu_Y^m}(B)) \subseteq f^{-1}(V) \subseteq U \). By Theorem 4.2, \( c_{\mu_X^m}(f^{-1}(B)) \subseteq f^{-1}(c_{\mu_Y^m}(B)) \subseteq U \) and hence \( f^{-1}(B) \) is \( \mu_{(m,n)} \)-closed subset of \( X \). \( \Box \)

Definition 4.8. A bigeneralized topological space \((X, \mu_X^1, \mu_X^2)\) is said to be \( \mu_{(m,n)} \)-\( T_\frac{1}{2} \) space if, for every \( \mu_{(m,n)} \)-closed set is \( \mu_n \)-closed, where \( m, n = 1, 2 \) and \( m \neq n \).

Definition 4.9. A bigeneralized topological space \((X, \mu_X^1, \mu_X^2)\) is said to be pairwise \( \mu \)-\( T_\frac{1}{2} \) space if it is both \( \mu_{(1,2)} \)-\( T_\frac{1}{2} \) space and \( \mu_{(2,1)} \)-\( T_\frac{1}{2} \) space.

Theorem 4.10. A bigeneralized topological space is a \( \mu_{(m,n)} \)-\( T_\frac{1}{2} \) space if and only if \( \{x\} \) is \( \mu_n \)-open or \( \mu_m \)-closed for each \( x \in X \), where \( m, n = 1, 2 \) and \( m \neq n \).

Proof. Suppose that \( \{x\} \) is not \( \mu_m \)-closed. Then \( X - \{x\} \) is \( \mu_{(m,n)} \)-closed by Proposition 3.11. Since \( X \) is \( \mu_{(1,2)} \)-\( T_\frac{1}{2} \) space, \( X - \{x\} \) is \( \mu_n \)-closed. Hence, \( \{x\} \) is \( \mu_n \)-open.

Conversely, let \( F \) be a \( \mu_{(m,n)} \)-closed set. By assumption, \( \{x\} \) is \( \mu_n \)-open or \( \mu_m \)-closed for any \( x \in c_{\mu_n}(F) \). Case (i) Suppose that \( \{x\} \) is \( \mu_n \)-open. Since \( \{x\} \cap F \neq \emptyset \), we have \( x \in F \). Case (ii) Suppose that \( \{x\} \) is \( \mu_m \)-closed. If \( x \notin F \), then \( \{x\} \subseteq c_{\mu_m}(F) - F \), which is a contradiction to Proposition 3.12. Therefore, \( x \in F \). Thus is both cases, we conclude that \( F \) is \( \mu_n \)-closed. Hence, \((X, \mu_X^1, \mu_X^2)\) is a \( \mu_{(m,n)} \)-\( T_\frac{1}{2} \) space. \( \Box \)

Definition 4.11. Let \((X, \mu_X^1, \mu_X^2)\) and \((Y, \mu_Y^1, \mu_Y^2)\) be generalized topological spaces. A function \( f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2) \) is said to be \( g_{(m,n)} \)-irresolute if \( f^{-1}(F) \) is \( \mu_{(m,n)} \)-closed in \( X \) for every \( \mu_{(m,n)} \)-closed \( F \) of \( Y \), where \( m, n = 1, 2 \) and \( m \neq n \).

Proposition 4.12. Let \( f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2) \) and \( g : (Y, \mu_Y^1, \mu_Y^2) \to (Z, \mu_Z^1, \mu_Z^2) \) be functions, the following properties holds:

(i) If \( f \) is \( g_{(m,n)} \)-irresolute and \( g \) is \( g_{(m,n)} \)-continuous, then \( g \circ f \) is \( g_{(m,n)} \)-continuous;

(ii) If \( f \) and \( g \) are \( g_{(m,n)} \)-irresolute, then \( g \circ f \) is \( g_{(m,n)} \)-irresolute;
(iii) Let $(Y, \mu^1_Y, \mu^2_Y)$ be a $\mu_{(m,n)}$-$T _{\frac{1}{2}}$ space. If $f$ and $g$ are $g_{(m,n)}$-continuous, then $g \circ f$ is $g_{(m,n)}$-continuous.

**Proof.** (i) Let $F$ be a $\mu_n$-closed subset of $Z$. Since $g$ is $g_{(m,n)}$-continuous, then $g^{-1}(F)$ is $\mu_{(m,n)}$-closed subset of $Y$. Since $f$ is $g_{(m,n)}$-irresolute, then $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F))$ is $\mu_{(m,n)}$-closed subset of $X$. Hence, $g \circ f$ is $g_{(m,n)}$-continuous.

(ii) Let $F$ be a $\mu_{(m,n)}$-closed closed subset of $Z$. Since $g$ is $g_{(m,n)}$-irresolute, then $g^{-1}(F)$ is $\mu_{(m,n)}$-closed subset of $Y$. Since $f$ is $g_{(m,n)}$-irresolute, then $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F))$ is $\mu_{(m,n)}$-closed subset of $X$. Hence, $g \circ f$ is $g_{(m,n)}$-irresolute.

(iii) Let $F$ be a $\mu_n$-closed closed subset of $Z$. Since $g$ is $g_{(m,n)}$-continuous, then $g^{-1}(F)$ is $\mu_{(m,n)}$-closed subset of $Y$. Since $(Y, \mu^1_Y, \mu^2_Y)$ is a $\mu_{(m,n)}$-$T _{\frac{1}{2}}$ space, then $g^{-1}(F)$ is $\mu_n$-closed subset of $Y$. Since $f$ is $g_{(m,n)}$-continuous, then $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F))$ is $\mu_{(m,n)}$-closed subset of $X$. Hence, $g \circ f$ is $g_{(m,n)}$-continuous. □

**Proposition 4.13.** Let $(X, \mu^1_X, \mu^2_X)$ be a $\mu_{(m,n)}$-$T _{\frac{1}{2}}$ space. If $f : (X, \mu^1_X, \mu^2_X) \rightarrow (Y, \mu^1_Y, \mu^2_Y)$ is surjective, $g_n$-closed and $g_{(m,n)}$-irresolute, then $(Y, \mu^1_Y, \mu^2_Y)$ is a $\mu_{(m,n)}$-$T _{\frac{1}{2}}$ space, where $m, n = 1, 2$ and $m \neq n$.

**Proof.** Let $F$ be a $\mu_{(m,n)}$-closed subset of $Y$. Since $f$ is $g_{(m,n)}$-irresolute, we have $f^{-1}(F)$ is a $\mu_{(m,n)}$-closed subset of $X$. Since $(X, \mu^1_X, \mu^2_X)$ is a $\mu_{(m,n)}$-$T _{\frac{1}{2}}$ space, $f^{-1}(F)$ is a $\mu_n$-closed subset of $X$. It follows by assumption that $F$ is a $\mu_n$-closed subset of $Y$, Hence, $(Y, \mu^1_Y, \mu^2_Y)$ is a $\mu_{(m,n)}$-$T _{\frac{1}{2}}$ space. □

**Proposition 4.14.** Let $(X, \mu^1_X, \mu^2_X)$ be a $\mu_{(m,n)}$-$T _{\frac{1}{2}}$ space. If $f : (X, \mu^1_X, \mu^2_X) \rightarrow (Y, \mu^1_Y, \mu^2_Y)$ is bijective, $g_n$-open and $g_{(m,n)}$-irresolute, then $(Y, \mu^1_Y, \mu^2_Y)$ is a $\mu_{(m,n)}$-$T _{\frac{1}{2}}$ space, where $m, n = 1, 2$ and $m \neq n$.

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**References**


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