On the Properties of a Class of $P$-Valent Functions

Defined by Salagean Differential Operator

Ajab Akbarally, Shaharuddin Cik Soh and Mardhiyah Ismail

Department of Mathematics
Faculty of Computer and Mathematical Sciences,
Universiti Teknologi MARA,
40450 Shah Alam, Selangor, Malaysia
ajab@tmsk.uitm.edu.my.
shahar@tmsk.uitm.edu.my.
diah_iman87@yahoo.com.

Abstract

Salagean [3] defined the operator $D^\lambda f(z) = z + \sum_{k=2}^\infty k^\lambda a_k z^k$ where $\lambda \in N_0 = \{0\} \cup N$.

Let $S_p$ be the class of $p$-valent functions which are analytic in the unit disk $U = \{z : |z| < 1\}$ and can be written in the form $f(z) = z^p + \sum_{k=1}^\infty a_{p+k} z^{p+k}$. We define a new class of functions, $S_p(A,B,b,\lambda)$ where functions in this class satisfy the condition

$$1 + \frac{1}{b} \left( \frac{z(D^{\lambda+p} f(z))'}{D^{\lambda+p} f(z)} - p \right) < \frac{1 + A\lambda}{1 + B\lambda}$$

where $\prec$ denote subordination, $b$ is any non-zero complex number. $A$ and $B$ are the arbitrary constants with $-1 \leq B < A \leq 1$. $D^{\lambda+p}$ is an extended Salagean operator defined by Eker & Seker in [4] as

$$D^{\lambda+p} f(z) = D(D^{\lambda+p-1} f(z)) = z^p + \sum_{k=1}^{\infty} \left( \frac{p+k}{p} \right)^\lambda a_{p+k} z^{p+k}$$

where $\lambda \in N_0 = \{0\} \cup N$.

Coefficient bounds, growth and distortion theorems and theorems on radius for the class $S_p(A,B,b,\lambda)$ are found.

Keywords: Analytic, $p$-valent, subordination, Salagean operator, coefficient bounds, distortion theorems, radius theorems
1 Introduction

Let $S$ be the class of analytic univalent functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

that are defined in the open unit disk $U = \{z : |z| < 1\}$.

Let $S_p$ denote the class of analytic $p$-valent functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}.$$  \hspace{1cm} (1.2)

For $f(z) \in S$, Salagean in [3] introduced the following operator:

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = Df(z) = zf'(z)$$
$$D^\lambda f(z) = D(D^{\lambda-1} f(z)),$$ \hspace{1cm} ($\lambda \in \mathbb{N} = \{1, 2, 3, \ldots\}$).

We note that

$$D^\lambda f(z) = z + \sum_{k=2}^{\infty} k^\lambda a_k z^k,$$ \hspace{1cm} ($\lambda \in \mathbb{N} \cup \{0\}$).

Following Eker & Seker in [4] we can write the following equalities for the functions $f(z) \in S_p$:

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = D(f(z)) = \frac{z}{p} f'(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right) a_{p+k} z^{p+k}$$
$$D^{\lambda+p} f(z) = D(D^{\lambda+p-1} f(z)) = z^p + \sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^\lambda a_{p+k} z^{p+k}, \hspace{0.5cm} \lambda \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}.$$ 

Let $S_p(A, B, b, \lambda)$ denote the subclass of $S_p$ that consists of functions $f(z)$ which satisfy the condition
where $\prec$ denotes subordination, $b$ is any non-zero complex number. $A$ and $B$ are the arbitrary fixed numbers, $-1 \leq B < A \leq 1$ and $z \in U$. This class is due to the class $M_p(A,B,b,n)$ defined by Ajab & Maslina in [1] using Ruscheweyh derivatives.

By specializing the parameters for $A, B, b, n$ and $\lambda$, the following subclasses studied by various earlier authors are obtained.

i. $S_1(1, -1, b, 0) = C(b, 1)$ which was studied by Wiatrowski in [5].

ii. $S_1(A, B, b, 0) = C(A, B, b)$ which was studied by Ravichandran et al. in [2].

## 2 Coefficient Estimates

**Theorem 2.1.** Let $f$ be a function given by (1.2), then $f \in S_p(A,B,b,\lambda)$ if it satisfies the condition:

$$
\sum_{k=1}^{\infty} \left[ k + \left| b(A-B) - Bk \right| \right] \left( \frac{p + k}{p} \right)^{\lambda} \left| a_{p+k} \right| \leq 1.
$$

(2.1)

**Proof.** Let $f \in S_p(A,B,b,\lambda)$, then by the definition of subordination, we can write (1.3) as

$$
1 + \frac{1}{b} \left( \frac{z(D^{\lambda+p} f(z))'}{D^{\lambda+p} f(z)} - p \right) \prec 1 + \frac{A}{B} z
$$

(1.3)

which gives

$$
\frac{z(D^{\lambda+p} f(z))'}{D^{\lambda+p} f(z)} - p = b(A-B) - B \left( \frac{z(D^{\lambda+p} f(z))'}{D^{\lambda+p} f(z)} - p \right) w(z).
$$

(2.2)
From (2.2), we obtain
\[
\begin{align*}
\frac{pz^p + \sum_{k=1}^{\infty} \frac{p + k}{p^k} a_{p+k} z^{p+k}}{z^p + \sum_{k=1}^{\infty} \frac{p + k}{p^k} a_{p+k} z^{p+k}} - p \\
= \left\{ b(A - B) - B \left\{ \frac{pz^p + \sum_{k=1}^{\infty} \frac{p + k}{p^k} a_{p+k} z^{p+k}}{z^p + \sum_{k=1}^{\infty} \frac{p + k}{p^k} a_{p+k} z^{p+k}} - p \right\} \right\} \lambda \end{align*}
\]
which yields
\[
\begin{align*}
\frac{\sum_{k=1}^{\infty} \left( \frac{p + k}{p} \right)^k a_{p+k} z^k}{1 + \sum_{k=1}^{\infty} \left( \frac{p + k}{p} \right)^k a_{p+k} z^k} &= \left\{ b(A - B) - B \left\{ \frac{\sum_{k=1}^{\infty} \left( \frac{p + k}{p} \right)^k a_{p+k} z^k}{1 + \sum_{k=1}^{\infty} \left( \frac{p + k}{p} \right)^k a_{p+k} z^k} \right\} \right\} \lambda \\
&= b(A - B) - B \sum_{k=1}^{\infty} \left[ B_k - b(A - B) \right] \left( \frac{p + k}{p} \right)^k a_{p+k} z^k.
\end{align*}
\]
Since \( |\lambda| \leq 1 \),
\[
\left| \sum_{k=1}^{\infty} \left( \frac{p + k}{p} \right)^k a_{p+k} z^k \right| \leq b(A - B) - B \sum_{k=1}^{\infty} \left[ B_k - b(A - B) \right] \left( \frac{p + k}{p} \right)^k a_{p+k} z^k.
\]
Letting \( |z| \to 1^- \) through real values, we have
\[
\left| \sum_{k=1}^{\infty} \left[ k + \left| b(A - B) - Bk \right| \right] \left( \frac{p + k}{p} \right)^k a_{p+k} \right| \leq \left| b \right| (A - B)
\]
and therefore,
\[
\left| \sum_{k=1}^{\infty} \left[ k + \left| b(A - B) - Bk \right| \right] \left( \frac{p + k}{p} \right)^k a_{p+k} \right| \leq \left| b \right| (A - B)
\]

\[\leq 1\]
which is the required assertion.

3 Growth and Distortion Theorems

Theorem 3.1. If \( f \in S_p(A, B, b, \lambda) \), then

\[
\begin{align*}
  &r^p - r^{p+1} \left\{ \frac{|b|(A-B)}{\left[ 1 + |b(A-B) - B| \left( \frac{p+1}{p} \right) \right]} \right\} \leq |f(z)| \\
  &\leq r^p + r^{p+1} \left\{ \frac{|b|(A-B)}{\left[ 1 + |b(A-B) - B| \left( \frac{p+1}{p} \right) \right]} \right\} \quad (|z| = r < 1).
\end{align*}
\]

Equality is attained for

\[
f(z) = z^p + z^{p+1} \left\{ \frac{|b|(A-B)}{\left[ 1 + |b(A-B) - B| \left( \frac{p+1}{p} \right) \right]} \right\}.
\]

Proof. From Theorem 2.1, we have that

\[
\sum_{k=1}^{\infty} |a_{p+k}| \leq \left\{ \frac{|b|(A-B)}{\left[ 1 + |b(A-B) - B| \left( \frac{p+1}{p} \right) \right]} \right\}.
\]

(3.1)

From (1.2) and (3.1), it follows that

\[
|f(z)| = \left| z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \right|
\]
\[
\leq |z|^p + \sum_{k=1}^{\infty} |a_{p+k}| |z|^{p+k}
\]
\[
\leq r^p + r^{p+1} \sum_{k=1}^{\infty} |a_{p+k}|
\]
\[
\leq r^p + r^{p+1} \left\{ \frac{|b|(A-B)}{\left[1 + |b(A-B) - B| \left( \frac{p+1}{p} \right)^{1/2} \right]} \right\}
\]

Similarly,
\[
|f(z)| \geq |z|^p - \sum_{k=1}^{\infty} |a_{p+k}| |z|^{p+k}
\]
\[
\geq r^p - r^{p+1} \sum_{k=1}^{\infty} |a_{p+k}|
\]
\[
\geq r^p - r^{p+1} \left\{ \frac{|b|(A-B)}{\left[1 + |b(A-B) - B| \left( \frac{p+1}{p} \right)^{1/2} \right]} \right\}
\]

**Theorem 3.2.** If \( f \in S_p(A, B, b, \lambda), \) then
\[
p^r r^{p-1} - r^p \left\{ \frac{(p+1)|b|(A-B)}{\left[1 + |b(A-B) - B| \left( \frac{p^2}{(p+1)^{1/2}} \right) \right]} \right\} \leq |f'(z)|
\]
\[
\leq pr^{p-1} + r^p \left\{ \frac{(p+1)|b|(A-B)}{\left[1 + |b(A-B) - B| \left( \frac{p^2}{(p+1)^{1/2}} \right) \right]} \right\}
\]

with equality for
\[ f(z) = z^p + z^{p+1} \frac{|b|(A-B)}{\left[ 1 + |b(A-B) - B| \left( \frac{p+1}{p} \right)^{2} \right]} \, . \]

**Proof.** From (3.1), we have that

\[
\sum_{k=1}^{\infty} (p + k) |a_{p+k}| \leq \frac{(p+1)|b|(A-B)}{\left[ 1 + |b(A-B) - B| \left( \frac{p+1}{p} \right)^{2} \right]} .
\]

From (1.2) and (3.2), it follows that

\[
|f'(z)| = \left| pz^{p-1} + \sum_{k=1}^{\infty} (p + k) a_{p+k} z^{p+k-1} \right|
\]
\[
\leq p|z|^{p-1} + \sum_{k=1}^{\infty} (p + k) |a_{p+k}| |z|^{p+k-1}
\]
\[
\leq pr^{p-1} + r^p \sum_{k=1}^{\infty} (p + k) |a_{p+k}|
\]
\[
\leq pr^{p-1} + r^p \frac{(p+1)|b|(A-B)}{\left[ 1 + |b(A-B) - B| \left( \frac{p+1}{p} \right)^{2} \right]} .
\]

Similarly,

\[
|f'(z)| \geq p|z|^{p-1} - \sum_{k=1}^{\infty} (p + k) |a_{p+k}| |z|^{p+k-1}
\]
\[
\geq pr^{p-1} - r^p \sum_{k=1}^{\infty} (p + k) |a_{p+k}|
\]
\[
\geq pr^{p-1} - r^p \frac{(p+1)|b|(A-B)}{\left[ 1 + |b(A-B) - B| \left( \frac{p+1}{p} \right)^{2} \right]} .
\]
3 Radii of Close-to-Convexity, Starlikeness and Convexity

Let $S(p), K(p)$ and $C(p)$ denote the class of $p$-valently starlike, convex and close-to-convex respectively which are regular in $S_p$.

A function of the form (1.2) is said to be $p$-valently starlike of order $\delta$ in $|z|<1$ if

$$\Re\left\{\frac{zf''(z)}{f'(z)}\right\} > \delta \quad (0 \leq \delta < p, p \in N).$$

A function is said to be in the class $K(p)$ of order $\delta$ if it satisfies the condition

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \delta \quad (0 \leq \delta < p, p \in N).$$

A function is said to be in the class $C(p)$ of order $\delta$ if

$$\Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \delta \quad (0 \leq \delta < p, p \in N).$$

**Theorem 4.1.** If $f \in S_p(A,B,b,\lambda)$, then $f$ is close-to-convex of order $\delta$ in $|z|<r_1(p,A,B,b,\lambda,\delta)$ where

$$r_1(p,A,B,b,\lambda,\delta) = \inf_k \left[ \frac{(p-\delta)[k + |b(A-B) - Bk|\left(\frac{p+k}{p}\right)]^{\lambda}}{(p+k)[|b|(A-B)]} \right]^{\frac{1}{k}}$$

**Proof.** It suffices to prove that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{k=1}^{\infty} (p+k)\left|a_{p+k}\right| |z|^k \leq p - \delta. \quad (4.4)$$

From (2.1) we have

$$\sum_{k=1}^{\infty} \left[ k + |b(A-B)| - Bk \left(\frac{p+k}{p}\right) \right]^{\lambda} \left|a_{p+k}\right| \leq |b|(A-B). \quad (4.5)$$

Hence (4.4) is proven true if
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\[
\frac{(p + k)z^k}{p - \delta} \leq \frac{\left[ k + |b(A - B) - Bk|\left(\frac{p + k}{p}\right)^{\lambda}\right]}{|b|(A - B)}. \tag{4.6}
\]

Solving (4.6) for $|z|$ we obtain

\[
|z| \leq \frac{\left( p - \delta \right) \left[ k + |b(A - B) - Bk|\left(\frac{p + k}{p}\right)^{\lambda}\right]^{\frac{1}{k}}}{\left( p + k \right)|b|(A - B)}.
\]

**Theorem 4.2.** If $f \in S_p(A, B, b, \lambda)$, then $f$ is starlike of order $\delta$ in $|z| < r_2(p, A, B, b, \lambda, \delta)$ where

\[
r_2(p, A, B, b, \lambda, \delta) = \inf_k \left[ \frac{(p - \delta)\left[ k + |b(A - B) - Bk|\left(\frac{p + k}{p}\right)^{\lambda}\right]^{\frac{1}{k}}}{(k - p + \delta)|b|(A - B)} \right].
\]

**Proof.** We have to show that

\[
\frac{|zf'(z)|}{f(z)} - p \leq \frac{\sum_{k=1}^{\infty} k|a_{p+k}|z^k}{1 + \sum_{k=1}^{\infty} |a_{p+k}|z^k} \leq p - \delta. \tag{4.7}
\]

From (4.5) we see that (4.7) holds true if

\[
\frac{(k - p + \delta)|z|^k}{p - \delta} \leq \frac{\left[ k + |b(A - B) - Bk|\left(\frac{p + k}{p}\right)^{\lambda}\right]}{|b|(A - B)}. \tag{4.8}
\]

It follows from (4.8) that
\[
|z| \leq \left\{ \frac{(p-\delta)^{k + \|b(A-B) - Bk\| \left( \frac{p+k}{p} \right)^{\lambda}}}{(k-p+\delta)\|b(A-B)\|} \right\}^{\frac{1}{1/\lambda}}.
\]

**Theorem 4.3.** If \( f \in S_p(A,B,b,\lambda) \), then \( f \) is convex of order \( \delta \) in \( |z| < r_\delta(p,A,B,b,\lambda,\delta) \) where

\[
r_\delta(p,A,B,b,\lambda,\delta) = \inf_k \left[ \frac{p(p-\delta)^{k + \|b(A-B) - Bk\| \left( \frac{p+k}{p} \right)^{\lambda}}}{(p+k)(k-p+\delta)\|b(A-B)\|} \right]^{\frac{1}{1/\lambda}}.
\]

**Proof.** We have to prove that

\[
\left| \frac{z f''(z)}{f'(z)} - p \right| \leq \frac{\sum_{k=1}^{\infty} (p+k)\|a_{p+k}\| z^k}{p + \sum_{k=1}^{\infty} (p+k)\|a_{p+k}\| z^k} \leq p - \delta. \quad (4.9)
\]

From (4.5) we see that (4.9) is true if

\[
(p+k)(k-p+\delta)|z|^\lambda \leq \frac{\left( k + \|b(A-B) - Bk\| \left( \frac{p+k}{p} \right)^{\lambda} \right)^{\frac{1}{1/\lambda}}}{(p+k)(k-p+\delta)\|b(A-B)\|} \quad (4.10)
\]

Solving (4.10) we obtain

\[
|z| \leq \left\{ \frac{p(p-\delta)^{k + \|b(A-B) - Bk\| \left( \frac{p+k}{p} \right)^{\lambda}}}{(p+k)(k-p+\delta)\|b(A-B)\|} \right\}^{\frac{1}{1/\lambda}}.
\]
References


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