New Exact Analytic Solutions for Stable and Unstable Nonlinear Schrödinger Equations with a Cubic Nonlinearity

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Abstract

The Exp-Function method is used for constructing new exact analytic solutions of stable and unstable nonlinear Schrödinger (SNLS and UNLS) equations with a cubic nonlinearity. The obtained solutions has not appeared in literature as far as we know. It is shown that many previous known results can be recovered as special cases of our results.

Mathematics Subject Classification: 35C07, 35G20, 35Q55

Keywords: Nonlinear Schrödinger equation, Cubic nonlinearity, Exp-function method, Exact solutions

1 Introduction

The nonlinear Schrödinger equation (NLS) arises in various contexts in the description of nonlinear waves [1] such as propagation of a laser beam in a medium whose index of a refraction is sensitive to the wave amplitude, water waves at the free surface of an ideal fluid, and the plasma waves. It provides a canonical description of the envelope dynamics of a scalar dispersive wave train with a small amplitude, slowly modulated in space and time, propagating in a conservative system [2]. Benney and Newell [2] used a multiple-scales approach to obtain slow time dependence of the wave properties, and arrived at the NLS equation. The same NLS equation was derived by Zakharov [3], using small amplitude expansion of the Fourier representation. Specific applications to deep water wave evolution were obtained via Whitham’s theory [4-7].

A part of nonlinear water waves, the derivation of the NLS equation has received considerable attention in various problems in plasma physics, fluid
and solid mechanics. Taniuti and Washimi [8] and Watanbe [9] have used the method of multiple-scales same to investigate the modulational instability of a small, but finite amplitude, dispersive hydromagnetic wave propagating in a cold plasma without dissipation. They showed that, in a frame of reference moving downstream with the group velocity, the slow evolution in the complex amplitude of the wave satisfies the NLS equation. Several authors including [10-13] have employed the method of multiple-scales to show that the evolution of wave trains satisfies the NLS equation. During the mid-1970 the NLS equation was widely accepted as the equation for description of the evolution of weakly nonlinear wave trains on deep water.

One among the most investigated topics in quantum mechanics regards the nonlinear generalizations of the Schrödinger equation. In the last decades many nonlinear extensions of the Schrödinger equation have been proposed in literature either to explore the fundamental aspects of quantum mechanics, with the usual linear theory representing only a limiting case, or to describe particular physical phenomenologies. Prequantum classical statistical field theory induces both linear and nonlinear generalizations of Schrödinger’s equation [14]. The statistical mechanics of a complex field whose dynamics is governed by NLS equation was investigated by Lebowitz et al [15]. Such fields describe, in suitable idealizations, Langmuir waves in a plasma and a propagating laser field in a nonlinear medium.

The conventional SNLS equation reads

\[ iu_t + u_{xx} + 2|u|^2u = 0. \]

(1)

The function \( u \) is a complex valued function of the spatial coordinate \( x \) and the time \( t \).

Interchange of space \( x \) and time \( t \) in Eq. (1) leads to the UNLS equation,

\[ iu_x + u_{tt} + 2|u|^2u = 0. \]

(2)

Equation (2) describes the nonlinear modulation of a high frequency mode in electron beam plasmas [16-18]. It may be considered as a prototype amplitude equation for the soliton phenomena in unstable systems. It occurs for the lossless symmetric two-stream plasma instability [19] and the two-layer baroclinic instability [20]. It has been shown that it also describes the nonlinear modulation of waves in the Rayleigh-Taylor problem [21]. Several authors [22-26] constructed travelling wave solutions to Eq. (1) via Bäcklund transformations, Tanh, Sine-cosine, Extended tanh methods.

In this paper we construct exact analytic solutions to the following general forms of Eqs. (1) and (2) using the Exp-function method.

\[ iu_t + \alpha u_{xx} + \beta |u|^2u = 0, \]

(3)
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\[ iu_x + \alpha u_{tt} + \beta |u|^2 u = 0, \]  

(4)

where \(\alpha\) and \(\beta\) are real constants.

Direct use of this method is suggested, not only for its convenience, but because applying Exp-function leads to obtaining more general solutions.

The paper is organized as follows. In section 2, we briefly review the Exp-Function method for constructing exact solutions of nonlinear partial differential equations. In section 3, Exact solution classes to SNLS and UNLS equations are obtained. The paper is summarized in section 4.

2 Review of the Exp-function method

The Exp-function method was first proposed by Wu and He [27] and systematically studied in [28]. Furthermore, He [29] presented an excellent study on the concepts of the recently developed asymptotic methods including Exp-Function. In addition, this method was successfully applied to KdV equation with variable coefficients [30], high-dimensional nonlinear evolution equation [31], Burgers and combine KdV-mKdV (Extended KdV) equations [32], etc.

We now present briefly the main steps of the Exp-function method that will be applied. A traveling wave transformation \(u = u(\xi), \xi = k(x - \nu t)\) converts a partial differential equation (PDE)

\[ \phi = (u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \cdots) = 0, \]  

(5)

into an ordinary differential equation (ODE)

\[ \phi = (u, -k\nu u’, ku’, k^2u”, k^2\nu u”, k^2\nu^2u”, \cdots) = 0, \]  

(6)

where the prime denotes differentiation with respect to \(\xi\).

The Exp-function method is based on the assumption that traveling wave solutions can be expressed in the following form [27, 28, 33-35]:

\[ u(\xi) = \frac{\sum_{n=-p}^{q} a_n \exp(n\xi)}{\sum_{n=-r}^{s} b_m \exp(m\xi)}, \]  

(7)

where \(p, q, r\) and \(s\) are positive integers which are the unknowns to be later determined and \(a_n\) and \(b_m\) are unknown constants. By differentiating of Eq. (7) with respect to \(\xi\), introducing the result into Eq. (6) and equating the arranged coefficients of the same powered \(\exp(\xi)\) to zero, a system of algebraic equations is constructed. Solving the algebraic equations yields exact travelling wave solutions to the nonlinear PDE (5).
3 Exact solution classes to SNLS and UNLS equations

In this section, we seek new exact solutions for Eqs. (3) and (4). We assume that Eq. (3) admits solutions in the following form:

\[ u = v(\xi)e^{i(ax+bt+c)}, \]

where \( v = v(\xi) \) is a real function, \( \xi = k(x - 2aat) + \xi_0 \), and \( a, b, \) and \( c \) are real constants.

Substitution of Eq. (8) into Eq. (3) yields the following ODE

\[ \alpha k^2 v'' - (b + \alpha a^2)v + \beta v^3 = 0. \]  \hspace{1cm} (9)

Inserting Eq. (7) into Eq. (9) and balancing the linear term of highest (lowest) order with the highest (lowest) order nonlinear term gives \( p = r \) and \( q = s \). In what follows, we consider some choices for \( p, r, q, \) and \( s \).

**Case 1:** \( p = r = 1, q = s = 1 \)

In this case Eq. (7) reduces to

\[ v = \frac{a_{-1}\exp(-\xi) + a_0 + a_1\exp(\xi)}{b_{-1}\exp(-\xi) + b_0 + b_1\exp(\xi)}. \]

Substituting Eq. (10) into Eq. (9), we obtain

\[ \frac{1}{A} \sum_{j=-3}^{3} C_j e^{j\xi} = 0, \]

where

\[ A = [b_{-1}\exp(-\xi) + b_0 + b_1\exp(\xi)]^3, \]

and \( C_n \) are the coefficients of \( \exp(n\xi) \). The coefficients of \( \exp(n\xi) \) must be zero, therefore we have

\[ \begin{align*}
C_{-3} &= a_{-1}(\beta a_{-1}^2 - \delta b_{-1}^2) = 0, \\
C_{-2} &= 3\beta a_{-1}^2 a_0 - \delta a_0 b_{-1}^2 + \alpha k^2 a_0 b_{-1}^2 - 2\delta a_{-1} b_{-1} b_0 - \alpha k^2 a_{-1} b_{-1} b_0 = 0, \\
C_{-1} &= 3\beta a_{-1} a_0^2 + 3\beta a_{-1} a_1 - \delta a_1 b_{-1}^2 + 4\alpha k^2 a_1 b_{-1}^2 - 2\delta a_0 b_{-1} b_0 - \alpha k^2 a_0 b_{-1} b_0 \\
&\quad - \delta a_{-1} b_{-1}^2 + \alpha k^2 a_{-1} b_{-1}^2 - 2\delta a_{-1} b_{-1} b_1 - 4\alpha k^2 a_{-1} b_{-1} b_1 = 0, \\
C_0 &= \beta a_0^3 + 6\beta a_{-1} a_0 a_1 - 2\delta a_1 b_{-1} b_0 + 3\alpha k^2 a_1 b_{-1} b_0 - \delta a_0 b_{-1}^2 - 2\delta a_0 b_{-1} b_1 \\
&\quad - 6\alpha k^2 a_0 b_{-1} b_1 - 2\delta a_{-1} b_{-1} b_1 + 3\alpha k^2 a_{-1} b_{-1} b_1 = 0, \\
C_1 &= 3\beta a_0^2 a_1 + 3\beta a_{-1} a_1^2 - \delta a_1 b_{-1}^2 + \alpha k^2 a_1 b_{-1}^2 - 2\delta a_1 b_{-1} b_1 - 4\alpha k^2 a_1 b_{-1} b_1 \\
&\quad - 2\delta a_0 b_0 b_1 - \alpha k^2 a_0 b_0 b_1 - \delta a_{-1} b_{-1}^2 + 4\alpha k^2 a_{-1} b_{-1}^2 = 0, \\
C_2 &= 3\beta a_0 a_1^2 - 2\delta a_1 b_0 b_1 - \alpha k^2 a_1 b_0 b_1 - \delta a_0 b_{-1}^2 + \alpha k^2 a_0 b_{-1}^2 = 0, \\
C_3 &= a_1(\beta a_1^2 - \delta b_1^2) = 0.
\end{align*} \]

(13)
where \(\delta = b + \alpha a^2\).
Solving this system of algebraic equations, using Mathematica, we obtain the following solution classes:

Case 1.1:

\[
a_{-1} = \sqrt{\frac{\delta}{\beta}} b_{-1}, \quad a_1 = -\sqrt{\frac{\delta}{\beta}} b_1, \quad a_0 = \pm \sqrt{\frac{\delta (b_0^2 - 4b_{-1}b_1)}{\beta}}, \quad \delta = b + \alpha a^2 = -\frac{\alpha k^2}{2},
\]

(14)

\[
a_{-1} = -\sqrt{\frac{\delta}{\beta}} b_{-1}, \quad a_1 = \sqrt{\frac{\delta}{\beta}} b_1, \quad a_0 = \pm \sqrt{\frac{\delta (b_0^2 - 4b_{-1}b_1)}{\beta}}, \quad \delta = b + \alpha a^2 = -\frac{\alpha k^2}{2},
\]

(15)

where \(\delta \beta > 0\), and \(b_0, b_{-1}\), and \(b_1\) are arbitrary constant parameters such that \(b_0^2 > 4b_{-1}b_1\).

Substituting Eqs. (14) and (15) into Eq. (10), we obtain the following generalized solitary solutions to Eq. (9):

\[
v = \pm \sqrt{-\frac{\alpha k^2 b_{-1} \exp(-\xi) \pm \sqrt{b_0^2 - 4b_{-1}b_1 - b_1 \exp(\xi)}}{2\beta}} \frac{b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi)}{b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi)}, \quad \alpha \beta < 0, \quad b_0^2 > 4b_{-1}b_1.
\]

(16)

Thus we obtain the following exact solution class to Eq. (3):

\[
u_1 = \pm \sqrt{-\frac{\alpha k^2 b_{-1} \exp(-\xi) \pm \sqrt{b_0^2 - 4b_{-1}b_1 - b_1 \exp(\xi)}}{2\beta}} \frac{b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi)}{b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi)} e^{[\alpha x - \alpha (a^2 + \frac{\beta}{2}) t + c]}, \quad \alpha \beta < 0, \quad b_0^2 > 4b_{-1}b_1.
\]

(17)

where \(\xi = k(x - 2\alpha at) + \xi_0\).

Note that if \(u\) is a solution of Eq. (3), then \(iu\) is also a solution of it. Hence, Eq. (3) has another solution in the form

\[
u_2 = \pm i \sqrt{-\frac{\alpha k^2 b_{-1} \exp(-\xi) \pm \sqrt{b_0^2 - 4b_{-1}b_1 - b_1 \exp(\xi)}}{2\beta}} \frac{b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi)}{b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi)} e^{[\alpha x - \alpha (a^2 + \frac{\beta}{2}) t + c]}, \quad \alpha \beta < 0, \quad b_0^2 > 4b_{-1}b_1.
\]

(18)

By the same procedure, we obtain the following exact solution classes to Eq. (4):

\[
u_1 = \pm \sqrt{-\frac{\alpha k^2 b_{-1} \exp(-\xi) \pm \sqrt{b_0^2 - 4b_{-1}b_1 - b_1 \exp(\xi)}}{2\beta}} \frac{b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi)}{b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi)} e^{[\beta t - \beta (a^2 + \frac{\beta}{2}) x + c]}, \quad \alpha \beta < 0, \quad b_0^2 > 4b_{-1}b_1.
\]

(19)
Case 1.2: 

\[ u_2 = \pm i \sqrt{\frac{\alpha k^2 b_{-1} \exp(-\xi) \pm b_0^2 - 4b_{-1} b_1 - b_1 \exp(\xi)}{2\beta}} \frac{\exp(-\xi) + b_0 + b_1 \exp(\xi)}{b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi)} e^{i \frac{\beta}{2} x + \xi_0}, \]

where \( \xi = k(2\alpha bx - t) + \xi_0. \)

Solutions (17) and (18) are more general than those obtained in [36]. The solutions were constructed in [36] can be obtained as special cases of solution (17) by setting \( a = 0, \alpha = 1, c = 0, \) and \( \xi_0 = 0, \) then Eq. (17) becomes

\[ u = \pm \sqrt{\frac{-k^2 b_{-1} \exp(-kx) \pm b_0^2 - 4b_{-1} b_1 - b_1 \exp(kx)}{2\beta}} \frac{\exp(-kx) + b_0 + b_1 \exp(kx)}{b_{-1} \exp(-kx) + b_0 + b_1 \exp(kx)} e^{-i \frac{k^2 x}{2}}, \]

where \( \beta < 0, \quad b_0^2 > 4b_{-1} b_1. \) (21)

Gharib [37] constructed soliton solutions to Eq. (2) via tanh-method. Those solutions are incorrect because of the invalid treatment by the author [37] with the system of algebraic equations which he obtained. We can obtain exact solution classes to Eq. (2) from Eqs. (19) and (20) by takeing \( \alpha = 1 \) and \( \beta = 2. \)

Case 1.2:

\[ a_{-1} = \pm \sqrt{\frac{\delta}{\beta}} b_{-1}, \quad a_1 = \mp \sqrt{\frac{\delta}{\beta}} b_1, \quad a_0 = b_0 = 0, \quad \delta = -2\alpha k^2, \] (22)

where \( \delta \beta > 0, \) and \( b_{-1}, \) and \( b_1 \) are arbitrary constant parameters.

Using Eq. (22), we obtain the following exact solution classes to Eq. (3):

\[ u_1 = \pm \sqrt{\frac{-2\alpha k^2 b_1 \exp(\xi) - b_{-1} \exp(-\xi)}{\beta}} e^{i \frac{a(x-\alpha(a^2+2k^2)t+c)}{2}} e^{i \frac{\beta}{2} x + \xi_0}, \quad \alpha \beta < 0, \] (23)

\[ u_2 = \pm i \sqrt{\frac{-2\alpha k^2 b_1 \exp(\xi) - b_{-1} \exp(-\xi)}{\beta}} e^{i \frac{a(x-\alpha(a^2+2k^2)t+c)}{2}} e^{i \frac{\beta}{2} x + \xi_0}, \quad \alpha \beta < 0, \] (24)

with \( \xi = k(x-2\alpha at) + \xi_0, \) and the following exact solution classes to Eq. (4):

\[ u_1 = \pm \sqrt{\frac{-2\alpha k^2 b_1 \exp(\xi) - b_{-1} \exp(-\xi)}{\beta}} e^{i \frac{\beta}{2} x + \xi_0} e^{i \frac{\beta}{2} x + \xi_0}, \quad \alpha \beta < 0, \] (25)

\[ u_2 = \pm i \sqrt{\frac{-2\alpha k^2 b_1 \exp(\xi) - b_{-1} \exp(-\xi)}{\beta}} e^{i \frac{\beta}{2} x + \xi_0} e^{i \frac{\beta}{2} x + \xi_0}, \quad \alpha \beta < 0, \] (26)
with \( \xi = k(2\alpha bx - t) + \xi_0 \).

Solutions (23) and (24) are more general than those obtained in Refs. [24, 38] as we show below.

If we set \( a = 0, b_1 = b_1, \alpha = 1, c = 0, \) and \( \xi_0 = 0 \), solution (23) becomes [24]:

\[
u = \pm \sqrt{\frac{-2k^2}{\beta}} \tanh(kx) e^{-2ik^2t}, \quad \beta < 0.
\]

If we set \( a = 0, b_1 = -b_1, \alpha = 1, c = 0, \) and \( \xi_0 = 0 \), solution (23) becomes [24]:

\[
u = \pm \sqrt{\frac{-2k^2}{\beta}} \coth(kx) e^{-2ik^2t}, \quad \beta < 0.
\]

If we replace \( k \) by \( ik \) in the above procedure, solutions (27) and (28) become [24]:

\[
u = \pm i \sqrt{\frac{2k^2}{\beta}} \tan(kx) e^{2ik^2t}, \quad \beta > 0,
\]

and

\[
u = \pm i \sqrt{\frac{2k^2}{\beta}} \cot(kx) e^{2ik^2t}, \quad \beta > 0.
\]

If we set \( \alpha = 1, b_1 = b_1, c = 0, \) and \( \xi_0 = 0 \), solution (23) becomes [38]:

\[
u = \pm \sqrt{\frac{-2k^2}{\beta}} \tanh[kx - 2at] e^{[ax - (a^2 + 2k^2)t]}, \quad \beta < 0.
\]

**Case 2: \( p = r = 2, q = s = 2 \)**

In this case Eq. (7) reduces to

\[
v = \frac{a_{-2} \exp(-2\xi) + a_{-1} \exp(-\xi) + a_0 + a_1 \exp(\xi) + a_2 \exp(2\xi)}{b_{-2} \exp(-2\xi) + b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi) + b_2 \exp(2\xi)}.
\]

Substituting Eq. (32) into Eq. (9), we obtain an equation of the form

\[
\frac{1}{A} \sum_{j=-6}^{6} C_j e^{j\xi} = 0, \quad A = \left( \sum_{j=-2}^{2} b_j e^{j\xi} \right)^3.
\]
Equating the coefficients \( C_n \) of \( \exp(n\xi) \) to zero, we obtain a system of nonlinear algebraic equations. To seek solutions for that algebraic system, we consider the following cases.

**Case 2.1:**

\[
\begin{aligned}
b_{-2} &= \frac{a_{-1}(\delta b_0 \pm \sqrt{\delta \eta_1})}{2\delta a_1}, \\
b_2 &= \frac{a_1(\delta b_0 \pm \sqrt{\delta \eta_1})}{2\delta a_1}, \\
b_{-1} &= \frac{a_0(\delta b_0 \pm \sqrt{\delta \eta_1})}{2\delta a_1}, \\
b_1 &= \frac{a_0(\delta b_0 \pm \sqrt{\delta \eta_1})}{2\delta a_1}, \\
a_{-2} &= 0, \\a_2 &= 0, \\\delta &= b + \alpha a^2 = \alpha k^2, \\\eta_1 &= b_0^2 \delta - \frac{\beta a_{-1} a_1}{2},
\end{aligned}
\]

where \( a_0, a_1 \neq 0, a_{-1} \neq 0 \), and \( b_0 \) are arbitrary constant parameters.

Substituting Eq. (34) into Eq. (32), we obtain the following solutions to Eq. (3):

\[
\begin{aligned}
&u_1 = \pm \frac{[a_{-1} \exp(-\xi) + a_0 + a_1 \exp(\xi)]e^{[ax + \alpha (k^2 - a^2) + \epsilon]}}{A_1 [a_{-1} \exp(-2\xi) + a_0 \exp(-\xi)] + b_0 + B_1 [a_0 \exp(\xi) + a_1 \exp(2\xi)]}, \\
&u_2 = \pm \frac{[a_{-1} \exp(-\xi) + a_0 + a_1 \exp(\xi)]e^{[ax + \alpha (k^2 - a^2) + \epsilon]}}{A_2 [a_{-1} \exp(-2\xi) + a_0 \exp(-\xi)] + b_0 + B_2 [a_0 \exp(\xi) + a_1 \exp(2\xi)]}, \\
&u_3 = \pm i \frac{[a_{-1} \exp(-\xi) + a_0 + a_1 \exp(\xi)]e^{[ax + \alpha (k^2 - a^2) + \epsilon]}}{A_1 [a_{-1} \exp(-2\xi) + a_0 \exp(-\xi)] + b_0 + B_1 [a_0 \exp(\xi) + a_1 \exp(2\xi)]}, \\
&u_4 = \pm i \frac{[a_{-1} \exp(-\xi) + a_0 + a_1 \exp(\xi)]e^{[ax + \alpha (k^2 - a^2) + \epsilon]}}{A_2 [a_{-1} \exp(-2\xi) + a_0 \exp(-\xi)] + b_0 + B_2 [a_0 \exp(\xi) + a_1 \exp(2\xi)]},
\end{aligned}
\]

where

\[
\begin{aligned}
A_1 &= \frac{b_0 + \sqrt{b_0^2 - \frac{\beta a_{-1} a_1}{2\alpha k^2}}}{2a_1}, \\
B_1 &= \frac{b_0 - \sqrt{b_0^2 - \frac{\beta a_{-1} a_1}{2\alpha k^2}}}{2a_1}, \\
A_2 &= \frac{b_0 - \sqrt{b_0^2 - \frac{\beta a_{-1} a_1}{2\alpha k^2}}}{2a_1}, \\
B_2 &= \frac{b_0 + \sqrt{b_0^2 - \frac{\beta a_{-1} a_1}{2\alpha k^2}}}{2a_1}, \\
b_0^2 &= \frac{\beta a_{-1} a_1}{2\alpha k^2}, \\
\end{aligned}
\]

and \( \xi = k(x - 2\alpha t) + \xi_0 \).
For Eq. (4), we obtain the following solutions:

\[
\begin{align*}
    u_1 &= \pm \frac{[a_1 \exp(-\xi) + a_0 + a_1 \exp(\xi)] e^{i[bt + \alpha(k^2 - b^2)x + c]}}{A_1 [a_1 \exp(-2\xi) + a_0 \exp(-\xi)] + b_0 + B_1 [a_0 \exp(\xi) + a_1 \exp(2\xi)]}, \\
    u_2 &= \pm \frac{[a_1 \exp(-\xi) + a_0 + a_1 \exp(\xi)] e^{i[bt + \alpha(k^2 - b^2)x + c]}}{A_2 [a_1 \exp(-2\xi) + a_0 \exp(-\xi)] + b_0 + B_2 [a_0 \exp(\xi) + a_1 \exp(2\xi)]}, \\
    u_3 &= \pm \frac{i [a_1 \exp(-\xi) + a_0 + a_1 \exp(\xi)] e^{i[bt + \alpha(k^2 - b^2)x + c]}}{A_1 [a_1 \exp(-2\xi) + a_0 \exp(-\xi)] + b_0 + B_1 [a_0 \exp(\xi) + a_1 \exp(2\xi)]}, \\
    u_4 &= \pm \frac{i [a_1 \exp(-\xi) + a_0 + a_1 \exp(\xi)] e^{i[bt + \alpha(k^2 - b^2)x + c]}}{A_2 [a_1 \exp(-2\xi) + a_0 \exp(-\xi)] + b_0 + B_2 [a_0 \exp(\xi) + a_1 \exp(2\xi)]},
\end{align*}
\]

with \( \xi = k(2\alpha bx - t) + \xi_0 \).

Solutions (35), (36), (40) and (41) are more general than those obtained in Refs. [22, 24, 36, 38] as we show below.

If we set \( b = 0, a_0 = 0, b_0 = 0, a_{-1} = -a_1, \alpha = 1, \beta = 2, c = 0, \) and \( \xi_0 = 0 \), solutions (40) and (41) become [22]:

\[
u = \pm k \sech(kt) e^{ik^2x}.
\]

If we set \( a = 0, a_0 = 0, b_0 = 0, a_{-1} = -a_1, \alpha = 1, c = 0, \) and \( \xi_0 = 0 \), solutions (35) and (36) become [24]:

\[
u = \pm \sqrt{\frac{2k^2}{\beta}} \sech(kx) e^{ik^2t}, \quad \beta > 0.
\]

If we set \( a = 0, a_0 = 0, b_0 = 0, a_{-1} = a_1, \alpha = 1, c = 0, \) and \( \xi_0 = 0 \), solutions (35) and (36) become [24]:

\[
u = \pm i \sqrt{\frac{2k^2}{\beta}} \coth(kx) e^{ik^2t}, \quad \beta > 0.
\]

If we replace \( k \) by \( ik \) in the above procedure, solutions (45) and (46) become [24]:

\[
u = \pm \sqrt{\frac{-2k^2}{\beta}} \sec(kx) e^{-ik^2t}, \quad \beta < 0,
\]

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If we set \( a = 0, a_0 = 0, \alpha = 1, c = 0, \) and \( \xi_0 = 0, \) Eqs. (35) and (36) become [36]:

\[
   u = \pm \frac{[a_{-1} \exp(-kx) + a_1 \exp(kx)]e^{ik^2t}}{A_1 a_{-1} \exp(-2kx) + b_0 + B_1 a_1 \exp(2kx)},
\]

(49)

\[
   u = \pm \frac{[a_{-1} \exp(-kx) + a_1 \exp(kx)]e^{ik^2t}}{A_2 a_{-1} \exp(-2kx) + b_0 + B_2 a_1 \exp(2kx)}.
\]

(50)

If we set \( a_0 = 0, b_0 = 0, a_{-1} = -a_1, \alpha = 1, c = 0, \) and \( \xi_0 = 0, \) solutions (35) and (36) become [39]:

\[
   u = \pm \sqrt{\frac{2k^2}{\beta} \sech[k(x-2at)]} e^{i(ax+(k^2-a^2)t)}, \quad \beta > 0,
\]

(51)

**Case 2.2:**

\[
   a_{-2} = \sqrt{\frac{\delta}{\beta}} b_{-2}, \quad a_2 = -\sqrt{\frac{\delta}{\beta}} b_2, \quad a_0 = \pm \frac{\sqrt{\delta(b_0^2 - 4b_{-2}b_2)}}{\beta},
\]

(52)

\[
   a_{-2} = -\sqrt{\frac{\delta}{\beta}} b_{-2}, \quad a_2 = \sqrt{\frac{\delta}{\beta}} b_2, \quad a_0 = \pm \frac{\sqrt{\delta(b_0^2 - 4b_{-2}b_2)}}{\beta},
\]

(53)

where \( \delta \beta > 0, \) and \( b_0, b_{-2}, \) and \( b_2 \) are arbitrary constant parameters such that 
\( b_0^2 > 4b_{-2}b_2. \)

Substituting Eqs. (52) and (53) into Eq. (32), we obtain the following solutions to Eq. (3):

\[
   u_1 = \pm \frac{\sqrt{2\alpha k^2 [b_{-2} \exp(-2\xi) + \sqrt{b_0^2 - 4b_{-2} b_2} b_2 \exp(2\xi)] e^{i(ax-\alpha(a^2+2k^2)t+c)}}}{b_{-2} \exp(-2\xi) + b_0 + b_2 \exp(2\xi)},
\]

(54)

\( \alpha \beta < 0, \quad b_0^2 > 4b_{-2}b_2, \)
with \( \xi = k(x - 2\alpha at) + \xi_0 \), and the following solutions to Eq. (4):

\begin{align*}
\text{Case 2.3:} \quad u_2 &= \pm \sqrt{-\frac{2\alpha k^2 [b_{-2} \exp(-2\xi) - \sqrt{b_0^2 - 4b_{-2} b_2 - b_2 \exp(2\xi)] e^{i[\alpha x - \alpha (a^2 + 2k^2) t + c]}}}{\beta}} \\
&\quad \quad \frac{b_{-2} \exp(-2\xi) + b_0 + b_2 \exp(2\xi)}{b_{-2} \exp(-2\xi) + b_0 + b_2 \exp(2\xi)} \\
&\quad \quad \alpha \beta < 0, \quad b_0^2 > 4b_{-2} b_2, \quad (55)
\end{align*}

\begin{align*}
\text{Case 2.3:} \quad u_3 &= \pm i \sqrt{-\frac{2\alpha k^2 [b_{-2} \exp(-2\xi) + \sqrt{b_0^2 - 4b_{-2} b_2 - b_2 \exp(2\xi)] e^{i[\alpha x - \alpha (a^2 + 2k^2) t + c]}}}{\beta}} \\
&\quad \quad \frac{b_{-2} \exp(-2\xi) + b_0 + b_2 \exp(2\xi)}{b_{-2} \exp(-2\xi) + b_0 + b_2 \exp(2\xi)} \\
&\quad \quad \alpha \beta < 0, \quad b_0^2 > 4b_{-2} b_2, \quad (56)
\end{align*}

\begin{align*}
\text{Case 2.3:} \quad u_4 &= \pm i \sqrt{-\frac{2\alpha k^2 [b_{-2} \exp(-2\xi) - \sqrt{b_0^2 - 4b_{-2} b_2 - b_2 \exp(2\xi)] e^{i[\alpha x - \alpha (a^2 + 2k^2) t + c]}}}{\beta}} \\
&\quad \quad \frac{b_{-2} \exp(-2\xi) + b_0 + b_2 \exp(2\xi)}{b_{-2} \exp(-2\xi) + b_0 + b_2 \exp(2\xi)} \\
&\quad \quad \alpha \beta < 0, \quad b_0^2 > 4b_{-2} b_2, \quad (57)
\end{align*}

with \( \xi = k(x - 2\alpha at) + \xi_0 \).
Substituting Eqs. (62) and (63) into Eq. (32), we obtain the following solutions:

\[ a_1 = -\frac{(\sqrt{3}a_1 - \sqrt{\beta b_1})(\sqrt{3}a_1 + \sqrt{\beta b_1})^2}{8\sqrt{3}\delta b_2}, \quad a_2 = -\sqrt{\frac{\beta}{\delta}} b_2, \]

\[ b_0 = \frac{\delta b_1 - \beta a_1^2}{2\delta b_2}, \quad b_1 = \frac{-(\sqrt{3}a_1 - \sqrt{\beta b_1})(\sqrt{3}a_1 + \sqrt{\beta b_1})^2}{8\delta^{3/2}b_2}, \]

\[ a_0 = a_2 = b_2 = 0, \quad \delta = -\frac{\alpha k^2}{2}, \quad (62) \]

\[ a_1 = -\frac{(\sqrt{3}a_1 - \sqrt{\beta b_1})^2(\sqrt{3}a_1 + \sqrt{\beta b_1})}{8\sqrt{3}\delta b_2}, \quad a_2 = \sqrt{\frac{\beta}{\delta}} b_2, \]

\[ b_0 = \frac{\delta b_1 - \beta a_1^2}{2\delta b_2}, \quad b_1 = \frac{(\sqrt{3}a_1 - \sqrt{\beta b_1})^2(\sqrt{3}a_1 + \sqrt{\beta b_1})}{8\delta^{3/2}b_2}, \]

\[ a_0 = a_2 = b_2 = 0, \quad \delta = -\frac{\alpha k^2}{2}, \quad (63) \]

where \( \alpha < 0, \beta > 0, \) and \( b_2 \neq 0, b_1, \) and \( a_1 \) are arbitrary constant parameters.

Substituting Eqs. (62) and (63) into Eq. (32), we obtain the following solutions to Eq. (3):

\[ u_1 = \pm \frac{\left[ -\sqrt{\frac{\beta}{\delta}} b_2 \exp(-2\xi) + a_{-1} \exp(-\xi) - C_1 \exp(\xi) \right] e^{iax-\alpha(a^2+\xi^2)t+c}}{b_{-2} \exp(-2\xi) + b_{-1} \exp(-\xi) + \frac{\delta b_1 - \beta a_1^2}{2\delta b_2} - C_1 \exp(\xi)}, \]

\[ \alpha < 0, \beta > 0, \quad (64) \]

\[ u_2 = \pm \frac{\left[ \sqrt{\frac{\beta}{\delta}} b_2 \exp(-2\xi) + a_{-1} \exp(-\xi) - \frac{D_1}{\sqrt{\beta}} \exp(\xi) \right] e^{iax-\alpha(a^2+\xi^2)t+c}}{b_{-2} \exp(-2\xi) + b_{-1} \exp(-\xi) + \frac{\delta b_1 - \beta a_1^2}{2\delta b_2} + \frac{D_1}{\sqrt{\beta}} \exp(\xi)}, \]

\[ \alpha < 0, \beta > 0, \quad (65) \]

\[ \begin{aligned} u_3 &= \pm \left[ -\sqrt{\frac{\beta}{\delta}} b_2 \exp(-2\xi) + a_{-1} \exp(-\xi) - \frac{C_1}{\sqrt{\beta}} \exp(\xi) \right] e^{iax-\alpha(a^2+\xi^2)t+c} \\
&= \frac{b_{-2} \exp(-2\xi) + b_{-1} \exp(-\xi) + \frac{\delta b_1 - \beta a_1^2}{2\delta b_2} - C_1 \exp(\xi)}, \end{aligned} \]

\[ \alpha < 0, \beta > 0, \quad (66) \]
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\[ u_4 = \pm i \frac{\sqrt{2} \alpha b_2 \exp(-2\xi) + a_{-1} \exp(-\xi) - \frac{D_1}{\sqrt{\beta}} \exp(\xi)}{b_2 \exp(-2\xi) + b_{-1} \exp(-\xi) + \frac{\delta b_1 - \beta a_{-1}}{2b_{-2}} + \frac{D_1}{\sqrt{\beta}} \exp(\xi)} e^{iax - \alpha(a^2 + b^2) t + c}, \]

\[ \alpha < 0, \quad \beta > 0, \quad (67) \]

with \( \xi = k(x - 2\alpha t) + \xi_0 \) and

\[ C_1 = \frac{(\sqrt{\beta}a_{-1} - \sqrt{\delta}b_{-1})(\sqrt{\beta}a_{-1} + \sqrt{\delta}b_{-1})^2}{8\delta b^2_{-2}}, \]

\[ D_1 = \frac{(\sqrt{\beta}a_{-1} - \sqrt{\delta}b_{-1})(\sqrt{\beta}a_{-1} + \sqrt{\delta}b_{-1})}{8\delta b^2_{-2}}, \quad (68) \]

and the following solutions to Eq. (4):

\[ u_1 = \pm \frac{\sqrt{2} \alpha b_2 \exp(-2\xi) + a_{-1} \exp(-\xi) - \frac{C_1}{\sqrt{\beta}} \exp(\xi)}{b_2 \exp(-2\xi) + b_{-1} \exp(-\xi) + \frac{\delta b_1 - \beta a_{-1}}{2b_{-2}} - \frac{C_1}{\sqrt{\beta}} \exp(\xi)} e^{ibt - \alpha(b^2 + \frac{k^2}{2}) x + c}, \]

\[ \alpha < 0, \quad \beta > 0, \quad (69) \]

\[ u_2 = \pm \frac{\sqrt{2} \alpha b_2 \exp(-2\xi) + a_{-1} \exp(-\xi) - \frac{D_1}{\sqrt{\beta}} \exp(\xi)}{b_2 \exp(-2\xi) + b_{-1} \exp(-\xi) + \frac{\delta b_1 - \beta a_{-1}}{2b_{-2}} + \frac{D_1}{\sqrt{\beta}} \exp(\xi)} e^{ibt - \alpha(b^2 + \frac{k^2}{2}) x + c}, \]

\[ \alpha < 0, \quad \beta > 0, \quad (70) \]

\[ u_3 = \pm \frac{i \sqrt{2} \alpha b_2 \exp(-2\xi) + a_{-1} \exp(-\xi) - \frac{C_1}{\sqrt{\beta}} \exp(\xi)}{b_2 \exp(-2\xi) + b_{-1} \exp(-\xi) + \frac{\delta b_1 - \beta a_{-1}}{2b_{-2}} - \frac{C_1}{\sqrt{\beta}} \exp(\xi)} e^{ibt - \alpha(b^2 + \frac{k^2}{2}) x + c}, \]

\[ \alpha < 0, \quad \beta > 0, \quad (71) \]

\[ u_4 = \pm \frac{i \sqrt{2} \alpha b_2 \exp(-2\xi) + a_{-1} \exp(-\xi) - \frac{D_1}{\sqrt{\beta}} \exp(\xi)}{b_2 \exp(-2\xi) + b_{-1} \exp(-\xi) + \frac{\delta b_1 - \beta a_{-1}}{2b_{-2}} + \frac{D_1}{\sqrt{\beta}} \exp(\xi)} e^{ibt - \alpha(b^2 + \frac{k^2}{2}) x + c}, \]

\[ \alpha < 0, \quad \beta > 0, \quad (72) \]

with \( \xi = k(2\alpha bx - t) + \xi_0 \).
Case 2.4:

\[
a_{-2} = -\frac{\sqrt{\beta a_0} - \sqrt{\beta} b_0)(\sqrt{\beta a_0} + \sqrt{\beta} b_0)^2}{8\sqrt{\beta} b_1^2}, \quad a_1 = -\sqrt{\frac{\delta}{\beta} b_1}, \quad b_{-1} = \frac{\delta b_0^2 - \beta a_0^2}{2\delta b_1},
\]

\[
b_{-2} = -\frac{\sqrt{\beta a_0} - \sqrt{\beta} b_0)(\sqrt{\beta a_0} + \sqrt{\beta} b_0)^2}{8\sqrt{\beta} b_1^2}, \quad a_{-1} = a_2 = b_2 = 0, \quad \delta = -\frac{\alpha k^2}{2}, \tag{73}
\]

\[
a_{-2} = -\frac{\sqrt{\beta a_0} - \sqrt{\beta} b_0)(\sqrt{\beta a_0} + \sqrt{\beta} b_0)^2}{8\sqrt{\beta} b_1^2}, \quad a_1 = \sqrt{\frac{\delta}{\beta} b_1}, \quad b_{-1} = \frac{\delta b_0^2 - \beta a_0^2}{2\delta b_1},
\]

\[
b_{-2} = \frac{\sqrt{\beta a_0} - \sqrt{\beta} b_0)(\sqrt{\beta a_0} + \sqrt{\beta} b_0)^2}{8\sqrt{\beta} b_1^2}, \quad a_{-1} = a_2 = b_2 = 0, \quad \delta = -\frac{\alpha k^2}{2}, \tag{74}
\]

where \(\alpha < 0, \beta > 0,\) and \(b_1 \neq 0, b_0,\) and \(a_0\) are arbitrary constant parameters. Substituting Eqs. (73) and (74) into Eq. (32), we obtain the following solutions to Eq. (3):

\[
u_1 = \pm \frac{-\frac{c_2}{\sqrt{\beta}} \exp(-2\xi) + a_0 - \sqrt{\frac{\delta}{\beta} b_1} \exp(\xi)}{-\frac{c_2}{\sqrt{\beta}} \exp(-2\xi) + \frac{\delta b_0^2 - \beta a_0^2}{2\delta b_1} \exp(-\xi) + b_0 + b_1 \exp(\xi)}, \quad \alpha < 0, \quad \beta > 0, \tag{75}
\]

\[
u_2 = \pm \frac{-\frac{c_2}{\sqrt{\beta}} \exp(-2\xi) + a_0 + \sqrt{\frac{\delta}{\beta} b_1} \exp(\xi)}{-\frac{c_2}{\sqrt{\beta}} \exp(-2\xi) + \frac{\delta b_0^2 - \beta a_0^2}{2\delta b_1} \exp(-\xi) + b_0 + b_1 \exp(\xi)}, \quad \alpha < 0, \quad \beta > 0, \tag{76}
\]

\[
u_3 = \pm \frac{-\frac{c_2}{\sqrt{\beta}} \exp(-2\xi) + a_0 - \sqrt{\frac{\delta}{\beta} b_1} \exp(\xi)}{-\frac{c_2}{\sqrt{\beta}} \exp(-2\xi) + \frac{\delta b_0^2 - \beta a_0^2}{2\delta b_1} \exp(-\xi) + b_0 + b_1 \exp(\xi)}, \quad \alpha < 0, \quad \beta > 0, \tag{77}
\]

\[
u_4 = \pm \frac{-\frac{c_2}{\sqrt{\beta}} \exp(-2\xi) + a_0 + \sqrt{\frac{\delta}{\beta} b_1} \exp(\xi)}{-\frac{c_2}{\sqrt{\beta}} \exp(-2\xi) + \frac{\delta b_0^2 - \beta a_0^2}{2\delta b_1} \exp(-\xi) + b_0 + b_1 \exp(\xi)}, \quad \alpha < 0, \quad \beta > 0, \tag{78}
\]
with $\xi = k(x - 2\alpha at) + \xi_0$ and

$$C_2 = \frac{1}{8\delta b_1^2} (\sqrt{\beta a_0} - \sqrt{\delta b_0})(\sqrt{\beta a_0} + \sqrt{\delta b_0})^2,$$

$$D_2 = \frac{1}{8\delta b_1^2} (\sqrt{\beta a_0} - \sqrt{\delta b_0})(\sqrt{\beta a_0} + \sqrt{\delta b_0}),$$

and the following solutions to Eq. (4):

$$u_1 = \pm \left[ -\frac{C_2}{\sqrt{\beta}} \exp(-2\xi) + a_0 - \sqrt{\frac{\delta}{\beta}} b_1 \exp(\xi) \right] e^{i\left[b\alpha \left( \beta^2 + \frac{k^2}{4} \right) x + c\right]},$$

$$u_2 = \pm \left[ -\frac{D_2}{\sqrt{\beta}} \exp(-2\xi) + a_0 + \sqrt{\frac{\delta}{\beta}} b_1 \exp(\xi) \right] e^{i\left[b\alpha \left( \beta^2 + \frac{k^2}{4} \right) x + c\right]},$$

$$u_3 = \pm \left[ -\frac{C_2}{\sqrt{\beta}} \exp(-2\xi) + a_0 - \sqrt{\frac{\delta}{\beta}} b_1 \exp(\xi) \right] e^{i\left[b\alpha \left( \beta^2 + \frac{k^2}{4} \right) x + c\right]},$$

$$u_4 = \pm \left[ -\frac{D_2}{\sqrt{\beta}} \exp(-2\xi) + a_0 + \sqrt{\frac{\delta}{\beta}} b_1 \exp(\xi) \right] e^{i\left[b\alpha \left( \beta^2 + \frac{k^2}{4} \right) x + c\right]},$$

with $\xi = k(2\alpha bx - t) + \xi_0$.

**Case 2.5:**

$$a_{-2} = -\frac{\delta b_1^2}{4\beta a_0}, \quad b_{-2} = \pm \sqrt{\frac{\delta}{\beta}} \frac{b_1}{4a_0}, \quad b_0 = \pm \sqrt{\frac{\beta}{\delta}}, \quad \delta = -\frac{\alpha k^2}{2},$$

$$a_{-1} = a_{1} = a_{2} = b_1 = b_2 = 0,$$

(84)
where $\delta \beta > 0$, and $a_0 \neq 0$ and $b_{-1}$ are arbitrary constant parameters.
Substituting Eq. (84) into Eq. (32), we obtain the following solutions to Eq. (3):

$$u_1 = \pm \left( a_0 - \frac{\delta b^2}{4\delta a_0} \exp(-2\xi) \right) e^{i\left(ax - \alpha (a^2 + \frac{\xi^2}{2}) t + \gamma \right)} \pm \sqrt{\frac{\delta b^2}{\beta 4\delta a_0}} \exp(-2\xi) + b_{-1} \exp(-\xi) \pm \sqrt{\frac{\beta}{\delta}}, \quad \alpha \beta < 0, \quad (85)$$

$$u_2 = \pm \left( a_0 - \frac{\delta b^2}{4\delta a_0} \exp(-2\xi) \right) e^{i\left(ax - \alpha (a^2 + \frac{\xi^2}{2}) t + \gamma \right)} \pm \sqrt{\frac{\delta b^2}{\beta 4\delta a_0}} \exp(-2\xi) + b_{-1} \exp(-\xi) \pm \sqrt{\frac{\beta}{\delta}}, \quad \alpha \beta < 0, \quad (86)$$

with $\xi = k(x - 2\alpha at) + \xi_0$, and the following solutions to Eq. (4):

$$u_1 = \pm \left( a_0 - \frac{\delta b^2}{4\delta a_0} \exp(-2\xi) \right) e^{i\left[bt - \alpha (a^2 + \frac{\xi^2}{2}) x + \gamma \right]} \pm \sqrt{\frac{\delta b^2}{\beta 4\delta a_0}} \exp(-2\xi) + b_{-1} \exp(-\xi) \pm \sqrt{\frac{\beta}{\delta}}, \quad \alpha \beta < 0, \quad (87)$$

$$u_2 = \pm \left( a_0 - \frac{\delta b^2}{4\delta a_0} \exp(-2\xi) \right) e^{i\left[bt - \alpha (a^2 + \frac{\xi^2}{2}) x + \gamma \right]} \pm \sqrt{\frac{\delta b^2}{\beta 4\delta a_0}} \exp(-2\xi) + b_{-1} \exp(-\xi) \pm \sqrt{\frac{\beta}{\delta}}, \quad \alpha \beta < 0, \quad (88)$$

with $\xi = k(2abx - t) + \xi_0$.

**Case 3:** $p = r = 3, q = s = 3$

As mentioned above, the trial function in Eq. (7) is

$$v = \sum_{j=-3}^{3} a_j e^{j\xi} \frac{\sum_{j=-3}^{3} b_j e^{j\xi}}{\sum_{j=-3}^{3} b_j e^{j\xi}}. \quad (89)$$

Substituting Eq. (89) into Eq. (9), we obtain system of algebraic equations. For solving that system we consider the following cases:

**Case 3.1:**

$$b_{-3} = \frac{2b_{-1} b_1^2 + \sqrt{2\delta^2 b_1^4 b_{-1}^4 - \beta \delta a_0^2 b_{-1}^2 b_1^2}}{4\delta b_1^4}, \quad b_3 = \frac{2b_{-1} b_1^2 + \sqrt{2\delta^2 b_1^4 b_{-1}^4 - \beta \delta a_0^2 b_{-1}^2 b_1^2}}{4\delta b_{-1}^4},$$

$$a_{-2} = \frac{a_0 b_{-1}}{b_1}, \quad a_2 = \frac{a_0 b_1}{b_{-1}}, \quad a_{-3} = a_{-1} = a_1 = a_3 = b_{-2} = b_2 = b_0 = 0, \quad \delta = -\frac{\alpha k^2}{2}. \quad (90)$$
where \(b_{-1} \neq 0\), \(b_1 \neq 0\) and \(a_0\) are arbitrary constant parameters.

From Eq. (90), we obtain the following solutions to Eq. (3)

\[
u_1 = \pm \frac{\left[\frac{ab_{-1}}{b_1} \exp(-2\xi) + a_0 + \frac{ab_1}{b_{-1}} \exp(2\xi)\right] e^{i[x + \alpha(k^2 - a^2)t + c]}}{2b^2, b_1^2 + 2\sqrt{2\xi}} \exp(-3\xi) + b_{-1} \exp(-\xi) + b_1 \exp(\xi) + \frac{2b^2, b_1^2 - 2\sqrt{2\xi}}{4b_1^2} \exp(3\xi)},
\]

(91)

\[
u_2 = \pm \frac{\left[\frac{ab_{-1}}{b_1} \exp(-2\xi) + a_0 + \frac{ab_1}{b_{-1}} \exp(2\xi)\right] e^{i[x + \alpha(k^2 - a^2)t + c]}}{2b^2, b_1^2 - 2\sqrt{2\xi}} \exp(-3\xi) + b_{-1} \exp(-\xi) + b_1 \exp(\xi) + \frac{2b^2, b_1^2 + 2\sqrt{2\xi}}{4b_1^2} \exp(3\xi)},
\]

(92)

\[
u_3 = \pm \frac{i \left[\frac{ab_{-1}}{b_1} \exp(-2\xi) + a_0 + \frac{ab_1}{b_{-1}} \exp(2\xi)\right] e^{i[x + \alpha(k^2 - a^2)t + c]}}{2b^2, b_1^2 + 2\sqrt{2\xi}} \exp(-3\xi) + b_{-1} \exp(-\xi) + b_1 \exp(\xi) + \frac{2b^2, b_1^2 - 2\sqrt{2\xi}}{4b_1^2} \exp(3\xi)},
\]

(93)

\[
u_4 = \pm \frac{i \left[\frac{ab_{-1}}{b_1} \exp(-2\xi) + a_0 + \frac{ab_1}{b_{-1}} \exp(2\xi)\right] e^{i[x + \alpha(k^2 - a^2)t + c]}}{2b^2, b_1^2 - 2\sqrt{2\xi}} \exp(-3\xi) + b_{-1} \exp(-\xi) + b_1 \exp(\xi) + \frac{2b^2, b_1^2 + 2\sqrt{2\xi}}{4b_1^2} \exp(3\xi)},
\]

(94)

where

\[
C = 2\delta^2 b_{-1}^2 b_1^4 - \beta \delta a_0^2 b_{-1} b_1^3, \quad \frac{\beta a_0^2}{\alpha k^2 b_{-1} b_1} > -1,
\]

and \(\xi = k(x - 2aat) + \xi_0\).

For Eq. (4), we obtain the following solutions:

\[
u_1 = \pm \frac{\left[\frac{ab_{-1}}{b_1} \exp(-2\xi) + a_0 + \frac{ab_1}{b_{-1}} \exp(2\xi)\right] e^{i[b + \alpha(k^2 - b^2)x + c]}}{2b^2, b_1^2 + 2\sqrt{2\xi}} \exp(-3\xi) + b_{-1} \exp(-\xi) + b_1 \exp(\xi) + \frac{2b^2, b_1^2 - 2\sqrt{2\xi}}{4b_1^2} \exp(3\xi)},
\]

(96)

\[
u_2 = \pm \frac{\left[\frac{ab_{-1}}{b_1} \exp(-2\xi) + a_0 + \frac{ab_1}{b_{-1}} \exp(2\xi)\right] e^{i[b + \alpha(k^2 - b^2)x + c]}}{2b^2, b_1^2 - 2\sqrt{2\xi}} \exp(-3\xi) + b_{-1} \exp(-\xi) + b_1 \exp(\xi) + \frac{2b^2, b_1^2 + 2\sqrt{2\xi}}{4b_1^2} \exp(3\xi)},
\]

(97)
\[ u_3 = \pm \frac{i \left[ \frac{a_0 b_{-1}}{b_1} \exp(-2\xi) + a_0 + \frac{a_0 b_1}{b_{-1}} \exp(2\xi) \right] e^{i|b_2 - \alpha (k^2 - b^2) x| + c}}{\frac{2b_1^2 \delta \sqrt{2\xi}}{4b_2} \exp(-3\xi) + b_{-1} \exp(-\xi) + b_1 \exp(\xi) + \frac{2b_1^2 \delta \sqrt{2\xi}}{4b_2} \exp(3\xi)}, \]

(98)

\[ u_4 = \pm \frac{i \left[ \frac{a_0 b_{-1}}{b_1} \exp(-2\xi) + a_0 + \frac{a_0 b_1}{b_{-1}} \exp(2\xi) \right] e^{i|b_2 - \alpha (k^2 - b^2) x| + c}}{\frac{2b_1^2 \delta \sqrt{2\xi}}{4b_2} \exp(-3\xi) + b_{-1} \exp(-\xi) + b_1 \exp(\xi) + \frac{2b_1^2 \delta \sqrt{2\xi}}{4b_2} \exp(3\xi)}, \]

(99)

with \( \xi = k(2\alpha b x - t) + \xi_0 \).

Case 3.2:

\[ a_{-2} = 2 \sqrt{2b_{-3}^5} b_3^{1/6} \sqrt{\frac{\delta}{\beta}}, \quad a_2 = 2 \sqrt{2b_{-3}^1} b_3^{5/6} \sqrt{\frac{\delta}{\beta}}, \quad a_0 = 2 \sqrt{\frac{2b_{-3} b_3^3}{\beta}}, \]

(100)

where \( \delta \beta > 0 \), and \( b_{-3} \) and \( b_1 \) are arbitrary constant parameters such that \( b_{-3} b_3 > 0 \).

From Eq. (100), we obtain the following solutions to Eq. (3)

\[ u_1 = \pm \frac{2 \sqrt{2b_{-3}^5} b_3^{1/6} \sqrt{\frac{\delta}{\beta}} \exp(-2\xi) + 2 \sqrt{\frac{2b_{-3} b_3^3}{\beta}} + 2 \sqrt{2b_{-3}^1} b_3^{5/6} \sqrt{\frac{\delta}{\beta}} \exp(2\xi)}{b_{-3} \exp(-3\xi) + b_3 \exp(3\xi)} \times e^{i|a_2 - \alpha (k^2 - a^2) x| + c}, \]

(101)

\[ u_2 = \pm \frac{i \left[ 2 \sqrt{2b_{-3}^5} b_3^{1/6} \sqrt{\frac{\delta}{\beta}} \exp(-2\xi) + 2 \sqrt{\frac{2b_{-3} b_3^3}{\beta}} + 2 \sqrt{2b_{-3}^1} b_3^{5/6} \sqrt{\frac{\delta}{\beta}} \exp(2\xi) \right]}{b_{-3} \exp(-3\xi) + b_3 \exp(3\xi)} \times e^{i|a_2 - \alpha (k^2 - a^2) x| + c}, \]

(102)

with \( \xi = k(x - 2\alpha at) + \xi_0 \), and the following solutions to Eq. (4):

\[ u_1 = \pm \frac{2 \sqrt{2b_{-3}^5} b_3^{1/6} \sqrt{\frac{\delta}{\beta}} \exp(-2\xi) + 2 \sqrt{\frac{2b_{-3} b_3^3}{\beta}} + 2 \sqrt{2b_{-3}^1} b_3^{5/6} \sqrt{\frac{\delta}{\beta}} \exp(2\xi)}{b_{-3} \exp(-3\xi) + b_3 \exp(3\xi)} \times e^{i|b_2 - \alpha (k^2 - b^2) x| + c}, \]

(103)
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\[ u_2 = \pm \frac{i \left[ 2 \sqrt{2b_{-3} b_3^{1/6}} \sqrt{\frac{2}{3}} \exp(-2\xi) + 2 \sqrt{\frac{2b_{-3} b_3}{3}} + 2 \sqrt{2b_{-3} b_3^{5/6}} \frac{2}{3} \exp(2\xi) \right]}{b_{-3} \exp(-3\xi) + b_3 \exp(3\xi)} \times e^{i[b t + \alpha(k^2 - b^2)x + \zeta]}, \quad (104) \]

with \( \xi = k(2 \alpha b x - t) + \xi_0 \).

4 Summary

We have obtained several exact solutions to SNLS and UNLS equations using the Exp-function method. The expression of the Exp-function is more general than the trigonometric and hyperbolic functions, so we can find solutions that are more general in the Exp-function method. Thus, the Exp-function method is more convenient and effective than that methods using the trigonometric or hyperbolic functions.

With the help of symbolic computations, we have obtained a broad class of analytical solutions to NLS and UNLS equations. From our results, many previous known results of NLS and UNLS equations obtained by some authors [22, 24, 36, 38] can be recovered by means of some suitable selections of the arbitrary constants as we have explained in section 3. Also, in section 3 (case 1.1), we have commented on the solutions constructed in [37].

References


Received: November, 2010