Fredholm Integral Equations of the First Kind
Solved by Using the Homotopy Perturbation Method

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Abstract

In this research, a numerical solution for solving the Fredholm integral equations is considered. An application of homotopy perturbation method is applied to solve the Fredholm integral equations. The results reveal that the homotopy perturbation method is very effective and simple and gives the exact solution. For illustration and more explanation of the idea, three examples are provided.

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1 Introduction

Homotopy perturbation method introduced by He [3, 4, 5, 6] has been used by many mathematicians and engineers to solve various functional equations. In this method the solution is considered as the sum of an infinite series which converges rapidly to the accurate solutions. We are frequently faced with the problem of determining the solution of integral equations, one of these integral equations is the Fredholm integral equation where defined in [7, 8, 9]. In recent years, a large amount of literature developed concerning the modified decomposition method introduced by Wazwaz by applying it to a large size of applications in applied sciences. A new perturbation method called homotopy perturbation method (HPM) was proposed by He in 1997 and systematical description in 2000 which is, in fact, coupling of the traditional perturbation method and homotopy in topology [10]. Until recently, the application of the HPM [8] in integral equations has been developed by scientists and engineers, because this method is the most effective and convenient ones for both weakly and strongly integral equations. In this paper, this method is applied for the
Fredholm integral equations.
The Fredholm integral equations are given by
\[ \phi(x) = f(x) + \int_a^b K(x, y)\phi(y)dy, \quad a \leq x \leq b. \] (1)

\( \phi(x) \) is an unknown function that will be determined, \( K(x, y) \) is the kernel of
the integral equation, \( f(x) \) is an analytic function.

2 Preliminary Notes

This section is devoted to reviewing HPM for solving the Fredholm integral equation. To illustrate the HPM, we consider (1) as
\[ L(u) = u(x) - f(x) - \int_a^b K(x, y)u(y)dy = 0, \quad a \leq x \leq b. \] (2)

With solution \( u(x) = \phi(x) \), we define the homotopy \( H(u, p) \) by
\[ H(u, 0) = F(u), \quad H(u, 1) = L(u), \]
where \( F(u) \) is a functional operator with solution, say, \( u_0 \), which can be obtained easily. We may choose a convex homotopy
\[ H(u, p) = (1 - p)F(u) + pL(u) = 0, \] (3)
and continuously trace an implicitly defined curve from a starting point \( H(u_0, 0) \)
to a solution function \( H(U, 1) \). The embedding parameter \( p \) monotonically increase from zero to unit as the trivial problem \( L(u) = 0 \). The embedding parameter \( p \in [0, 1] \) can be considered as an expanding parameter.

The HPM uses the homotopy parameter \( p \) as expanding parameter to obtain
\[ u = u_0 + pu_1 + p^2u_2 + \cdots. \] (4)

When \( p \to 1 \), (4) corresponds to (3) becomes the approximate solution of (2),
i.e.,
\[ U = \lim_{p \to 1} u = u_0 + u_1 + u_2 + \cdots. \] (5)
The series (5) is convergent for most cases, and also the rate of convergent
depends on \( L(u) \). Taking \( F(u) = u(x) - f(x) \), and substituting (4) in (3) and
equating the terms with identical power of \( p \), we obtain
\[ p^0 : u_0 - f(x) = 0 \implies u_0 = f(x), \]
\[ p^1 : u_1 - \int_a^b K(x, y)u_0(y)dy = 0, \]
Fredholm integral equations

\[ u_1 = \int_a^b K(x, y)u_0(y) \, dy = 0, \]

and in general we have

\[ u_0(x) = f(x), \]

\[ u_{n+1}(x) = \int_a^b K(x, y)u_n(y) \, dy, \quad n = 1, 2, 3, \ldots \]

which is the standard Adomian’s decomposition method.

3 Numerical Example

This section contained three example of Fredholm integral equations.

Example 3.1 Consider the following Fredholm integral equation

\[ \phi(x) = 5 \cos x - \frac{1}{4} \cos x \sin^2(1) + 0.1 \int_0^1 \sin y \cos x \phi(y) \, dy. \]  

(6)

Exact solution of this equation is \( \phi(x) = 5 \cos x \).

We define

\[ F(u) = u(x) - 5 \cos(x), \]

\[ L(u) = u(x) - 5 \cos(x) + \frac{1}{4} \cos x \sin^2(1) - 0.1 \int_0^1 \sin y \cos x \, u(y) \, dy, \]

and substituting \( F(u) \) and \( L(u) \) in (3) and equating the terms with identical power of \( p \), we obtain

\[ p^0 : \quad u_0(x) = 5 \cos(x), \]

\[ p^1 : \quad u_1(x) = -\frac{1}{4} \cos x \sin^2(1) + 0.1 \int_0^1 \sin y \cos x \, u_0(y) \, dy = 0, \]
\[ p^{k+2} : \ u_{k+2}(x) = -\frac{1}{4} \cos x \ \sin^2(1) + 0.1 \int_0^1 \sin y \cos x \ u_{k+1}(y)dy = 0, \]

such that \( k \geq 0 \). With using (5) we have

\[ U(x) = u(x) = \lim_{p \to 1} u = u_0(x) + u_1(x) + u_2(x) + \cdots = 5 \cos(x). \]

**Example 3.2** Consider the following Fredholm integral equation

\[ \phi(x) = x^3 + 2x + \frac{1}{20} \left( -\frac{26}{15} x - \frac{13}{15} x^2 \right) + 0.05 \int_0^1 y \left( x^2 + 2x \right) \phi(y)dy. \quad (7) \]

Exact solution of this equation is \( \phi(x) = x^3 + 2x \).

We define

\[ F(u) = u(x) - x^3 - 2x, \]
\[ L(u) = u(x) - x^3 - 2x - \frac{1}{20} \left( -\frac{26}{15} x - \frac{13}{15} x^2 \right) - 0.05 \int_0^1 y \left( x^2 + 2x \right) u(y)dy, \]

and substituting \( F(u) \) and \( L(u) \) in (3) and equating the terms whit identical power of \( p \), we obtain

\[ p^0 : \ u_0(x) = x^3 + 2x, \]

\[ p^1 : \ u_1(x) = -\frac{1}{20} \left( -\frac{26}{15} x - \frac{13}{15} x^2 \right) - 0.05 \int_0^1 y \left( x^2 + 2x \right) u_0(y)dy, \]

\[ p^{k+2} : \ u_{k+2}(x) = -\frac{1}{20} \left( -\frac{26}{15} x - \frac{13}{15} x^2 \right) - 0.05 \int_0^1 y \left( x^2 + 2x \right) u_{k+1}(y)dy = 0, \]

such that \( k \geq 0 \). With using (5) we have

\[ U(x) = u(x) = \lim_{p \to 1} u = u_0(x) + u_1(x) + u_2(x) + \cdots = x^3 + 2x. \]
Example 3.3 Consider the following Fredholm integral equation
\[
\phi(x) = -\frac{x^{0.75}}{40} + x^2 + 0.1 \int_0^1 x^{0.75} y \phi(y) dy.
\] (8)

Exact solution of this equation is \( \phi(x) = x^2 \).
We define
\[
F(u) = u(x) - x^2,
\]
\[
L(u) = u(x) - x^2 + \frac{x^{0.75}}{40} - x^2 - 0.1 \int_0^1 x^{0.75} y u(y) dy,
\]
and substituting \( F(u) \) and \( L(u) \) in (3) and equating the terms whit identical power of \( p \), we obtain

\[ p^0 : u_0(x) = x^2, \]

\[ p^1 : u_1(x) = \frac{x^{0.75}}{40} - 0.1 \int_0^1 x^{0.75} y u_0(y) dy, \]

\[ p^{k+2} : u_{k+2}(x) = \frac{x^{0.75}}{40} - 0.1 \int_0^1 x^{0.75} y u_{k+1}(y) dy = 0, \]

such that \( k \geq 0 \). With using (5) we have
\[
U(x) = u(x) = \lim_{p \to 1} u = u_0(x) + u_1(x) + u_2(x) + \cdots = x^2.
\]

Conclusion
In this paper, He’s homotopy perturbation method has been Exactly and suc-
cessfully applied to Finding the solution of Fredholm integral equations of the
second kind has been shown. The approximate solutions obtained by the ho-
motopy perturbation method are compared with exact solutions. It can be
concluded that the He’s homotopy perturbation method is Very Strong and
effective and Applicable technique in finding exact solutions for wide classes
of problems.
References


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