A Related Fixed Point Theorem in

Tree Fuzzy Metric Spaces

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Abstract

First, the implicit relations were given. A fixed point theorem for mappings satisfying implicit relations was proved. This result gives fuzzy versions of known fixed point theorems for mappings in tree metric spaces

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1. Introduction and Preliminaries


Our aim is to unify, generalize and extend all the above theorems in three fuzzy metric spaces.

The concept of fuzzy sets was introduced initially by Zadeh [18]. George and Veeramani [6] modified the concept of fuzzy metric space which was introduced by Kramosil and Michalek [10] and defined a Hausdorff topology in this space. Grabiec [7] extended the well known fixed point theorems of Banach [2] and Edelstein [4] in fuzzy metric spaces.
In this paper, using a new class of implicit relations, we prove a theorem as corollaries of which are taken the fuzzy versions of theorems of [13], [8], [9], etc.

Firstly, we will give some known definitions and lemmas.

**Definition 1.1 [18]:** A fuzzy set $A$ in $X$ is a function with domain $X$ and values in $[0,1]$.

**Definition 1.2 [16]:** A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous $t$-norm, if $([0,1], *)$ is an abelian topological monoid with the unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Two typical examples of continuous $t$-norm are $a * b = ab$ and $a * b = \min(a, b)$.

**Definition 1.3 [6]:** The 3-tuple $(X, M, *)$ is called a fuzzy metric space if $X$ is an arbitrary (non-empty) set, $*$ is a continuous $t$-norm and $M$ is a fuzzy set on $X \times (0, \infty)$ satisfying the following conditions:

1. **(FM-1)** $M(x, y, t) > 0$,
2. **(FM-2)** $M(x, y, t) = 1$ if and only if $x = y$,
3. **(FM-3)** $M(x, y, t) = M(y, x, t)$,
4. **(FM-4)** $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
5. **(FM-5)** $M(x, y, t): (0, \infty) \rightarrow [0,1]$ is continuous.

**Example 1.4 [6]:** Let $(X, d)$ be a metric space. Define $a * b = ab$ and

$$M(x, y, t) = \frac{kt^n}{kt^n + md(x, y)}, k, m, n \in \mathbb{R}^+.$$ 

Then $(X, M, *)$ is a fuzzy metric space. In the above example by taking $k = m = n = 1$ we get

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$ 

We call this fuzzy metric induced by a metric $d$ the standard fuzzy metric.

**Definition 1.5 [7].** Let $(X, M, *)$ be a fuzzy metric space. Then:

1. **(1)** A sequence $\{x_n\}$ in $X$ is said to be convergent to a point $x \in X$ (denoted by $\lim_{n \to \infty} x_n = x$) if $\lim_{t \to 0} M(x_n, x, t) = 1$ for all $t > 0$.
2. **(2)** A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if $\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$.
3. **(3)** A fuzzy metric space in which every Cauchy sequence is convergent is called complete.
Lemma 1.6 [7]. For all \( x, y \in X, M(x, y, \cdot) \) is no decreasing.

Remark 1.7: Throughout this paper, \((X, M, \ast)\) will denote the fuzzy metric space in the sense of Definition 2.3 with the following condition:

\[
\text{(FM-6)} \lim_{t \to \infty} M(x, y, t) = 1 \quad \text{for all } x, y \in X \text{ and } t > 0.
\]

Lemma 1.8 [15]. Let \((X, M, \ast)\) be a fuzzy metric space. Then \( M \) is a continuous function on \( X^2 \times (0, \infty) \).

Lemma 1.9 ([3], [11]). Let \( \{y_n\} \) be a sequence in a fuzzy metric space \((X, M, \ast)\). If there exist a number \( k \in (0, 1) \) such that

\[
M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t) \quad \text{for all } t > 0 \text{ and } n = 1, 2, \ldots \text{ then } \{y_n\} \text{ is a Cauchy sequence in } X.
\]

Lemma 1.10 [11]. Let \((X, M, \ast)\) be a fuzzy metric space. If there exists \( k \in (0, 1) \) such that

\[
M(x, y, kt) \geq M(x, y, t), \quad x \neq y.
\]

2. Implicit relations

First, we introduce and consider a new class of implicit relations which will give a general character to the main theorem 3.1.

Definition 2.1 Let \( \Phi_k \) be the set of all functions with \( k \) variables \( \varphi : [0, 1]^k \to R \) satisfying the following conditions:

\((a)\). \( \varphi \) is lower semi-continuous in each coordinate variable \( t_1, t_2, \ldots, t_k \)

\((b)\). If, for some constant \( 0 < c < 1 \) we have

\[
\varphi(u(ct), u(t), v_1(t), v_2(t), \ldots, v_{k-3}(t)) \geq 0 \quad \text{for any } t > 0 \text{ and any functions}
\]

\[
\varphi(u(ct), u(t), v_1(t), v_2(t), \ldots, v_{k-3}(t)) \geq 0 \quad \text{for any } t > 0 \text{ and any functions}
\]

\[
u, v_1, v_2, \ldots, v_{k-3} : (0, \infty) \to [0, 1] \text{ where the function } u \text{ is no decreasing and}
\]

\[
u(ct) = u(t) \Rightarrow u(t) = 1, \quad \text{then } u(ct) \geq \min\{v_1(t), v_2(t), \ldots, v_{k-3}(t)\}.
\]

Every such function \( \varphi \) will be called \( \Phi_k \)-function with constant \( c \).

Some examples of \( \Phi_k \)-function are as follows:

Example 2.2 Let \( F : R \to R \) a continuous function with \( F(1) = 0 \).

The function \( \varphi(t_1, t_2, \ldots, t_k) = t_1^p - \min\{t_2^p, t_3^p, \ldots, t_k^p\} - F(max\{t_2^p, t_3^p, \ldots, t_k^p\}) \),

where \( p > 0 \), is \( \Phi_k \)-function.

Proof: (a) is clear.

Suppose that \( 0 < c < 1 \); \( u, v_1, v_2, \ldots, v_{k-3} : (0, \infty) \to [0, 1], u \) no decreasing,

\( u(ct) = u(t) \Rightarrow u(t) = 1 \) and then
\[
\phi(u(ct),1,u(t),v_1(t),v_2(t),\ldots,v_{k-3}(t)) = \\
= u^p(ct) - \min\{1,u^p(t),v_1^p(t),v_2^p(t),\ldots,v_{k-3}^p(t)\} - \\
F(\max\{1,u^p(t),v_1^p(t),v_2^p(t),\ldots,v_{k-3}^p(t)\}) = \\
= u^p(ct) - \min\{1,u^p(t),v_1^p(t),v_2^p(t),\ldots,v_{k-3}^p(t)\} - F(1) = \\
= u^p(ct) - \min\{1,u^p(t),v_1^p(t),v_2^p(t),\ldots,v_{k-3}^p(t)\} \geq 0
\]

If \( u^p(t) < \min\{v_1^p(t),v_2^p(t),\ldots,v_{k-3}^p(t)\} \), then

\[
u^p(ct) \geq \min\{1,u^p(t),v_1^p(t),v_2^p(t),\ldots,v_{k-3}^p(t)\} = u^p(t) \quad \text{a contradiction. Therefore,}
\]

\[
u^p(ct) \geq \min\{v_1^p(t),v_2^p(t),\ldots,v_{k-3}^p(t)\},
\]

and so \( u^p(ct) \geq \min\{v_1(t),v_2(t),\ldots,v_{k-3}(t)\} \).

Similarly, if \( \phi(u(ct),u(t),1,v_1(t),v_2(t),\ldots,v_{k-3}(t)) \geq 0 \), then

\[
u^p(ct) \geq \min\{v_1(t),v_2(t),\ldots,v_{k-3}(t)\} \). The proof of (b) is completed.

**Definition 2.3** The set of all continuous functions with \( k \) variables \( f : [0,1]^k \rightarrow R \) satisfying the properties:

(a'). \( f \) is non decreasing in respect with each variable.

(b'). \( f(t,t,\ldots,t) \geq t, t \in [0,1] \)

will be noted \( F_k \) and every such function will be called a \( F_k \)-function.

Denote \( I_k = \{1,2,\ldots,k\} \). Some examples of \( F_k \)-function are as follows:

1. \( f(t_1,t_2,\ldots,t_k) = \min\{t_1,t_2,\ldots,t_k\} \)
2. \( f(t_1,t_2,\ldots,t_k) = [\min\{t_j : i,j \in I_k\}]^{\frac{1}{k}} \)
3. \( f(t_1,t_2,\ldots,t_k) = [\min\{t_1,t_2,t_3,t_4,t_5,t_6\}]^{\frac{1}{k}} \)
4. \( f(t_1,t_2,\ldots,t_k) = [\min\{t_1^p,t_2^p,\ldots,t_k^p\}]^{\frac{1}{p}}, p > 0 \)
5. \( f(t_1,t_2,\ldots,t_k) = t^* t^* \ldots t^* \) where \( * \) is a \( t \)-norm such that \( t^* t \geq t \).

For example \( a*b = \min\{a,b\} \).

6. \( f(t_1,t_2,\ldots,t_k) = (a_{t_1^p} + a_{t_2^p} + \ldots + a_{t_k^p})^{\frac{1}{p}}, \) where \( p > 0 \) and \( 0 \leq a_i, \sum_{i=1}^{k} a_i \geq 1 \)

The proof is done for the example 6:

(a’) It is obvious that the function \( f \) is no decreasing in respect with each variable

(b’) We have: \( f(t,t,\ldots,t) = (a_1 t^p + a_2 t^p + \ldots + a_k t^p)^{\frac{1}{p}} = [(a_1 + a_2 + \ldots + a_k) t^p]^{\frac{1}{p}} = \\
= (a_1 + a_2 + \ldots + a_k)^{\frac{1}{p}} t \geq t, t \in [0,1] \). The proof of (b’) is completed.

**Definition 2.4** The set of all continuous functions with \( k \) variables \( g : [0,1]^k \rightarrow R \) satisfying the property:

\( (t_1 - 1)(t_2 - 1)\ldots(t_k - 1) = 0 \Rightarrow g(t_1,t_2,\ldots,t_k) = 1, \)

(the function \( g \) takes the value 1 at the points for which at least one of the
coordinates is 1) will be noted $G_k$ and every such function will be called a $G_k$-function.

Some examples of $G_k$-function are as follows

6. $g(t_1, t_2, ..., t_k) = \max\{t_1, t_2, ..., t_k\}$

7. $g(t_1, t_2, ..., t_k) = \max\{t_1^p, t_2^p, ..., t_k^p\}, p > 0$ etc.

**Definition 2.5** Let $F$ be the set of all continuous mappings $F : R \rightarrow R$ such that $F(1) = 0$

For example $F = 0; F(t) = (t - 1)^3$ etc.

The following relationship between $G_k$-functions and $\Phi_k$-functions holds:

**Lemma 2.6** If $f \in F_{k-1}$, then the function $\varphi(t_1, t_2, ..., t_k) = t_1 - f(t_2, t_3, ..., t_k)$ is $\Phi_k$-function.

**Proof.** The condition (a) of Definition 2.1 is clear.

Suppose that $0 < c < 1, \ u, v_1, v_2, ..., v_{k-1} : (0, \infty) \rightarrow [0,1], \ u$ no decreasing, $u(ct) = u(t) \Rightarrow u(t) = 1$ and then

$$\varphi(u(ct), 1, u(t), v_1(t), v_2(t), ..., v_{k-1}(t)) =$$

$$= u(ct) - f(1, u(t), v_1(t), v_2(t), ..., v_{k-1}(t)) \geq 0 \quad (*)$$

We have $u(t) \geq \min\{v_1(t), v_2(t), ..., v_{k-1}(t)\}$ since in contrary, (if $u(t) < \min\{v_1(t), v_2(t), ..., v_{k-1}(t)\}$), by using the properties of $f$ we get:

$$f(1, u(t), v_1(t), ..., v_{k-1}(t)) \geq f(u(t), u(t), ..., u(t)) \geq u(t)$$

and by (*) it follows $u(ct) \geq u(t)$, a contradiction. Therefore, after replacing the coordinates of the point $(1, u(t), v_1(t), ..., v_{k-1}(t))$ by $\min\{v_1(t), v_2(t), ..., v_{k-1}(t)\}$ and using the properties of $f$ we get $u(ct) \geq \min\{v_1(t), v_2(t), ..., v_{k-1}(t)\}$. Similarly, if $\varphi(u(ct), u(t), 1, v_1(t), v_2(t), ..., v_{k-1}(t)) \geq 0$, then $u(ct) \geq \min\{v_1(t), v_2(t), ..., v_{k-1}(t)\}$.

The proof of (b) is completed.

**Lemma 2.7** If $f \in F_{k-1}, \ g \in G_{k-1}$ and $F \in F$, then the function $\varphi(t_1, t_2, ..., t_k) = t_1 - f(t_2, t_3, ..., t_k) - F(g(t_2, t_3, ..., t_k))$ is $\Phi_k$-function.

**Proof.** (a*) is clear.

Suppose that $0 < c < 1; \ u, v_1, v_2, ..., v_{k-1} : (0, \infty) \rightarrow [0,1]; \ u$ no decreasing, $u(ct) = u(t) \Rightarrow u(t) = 1$ and then

$$\varphi(u(ct), 1, u(t), v_1(t), v_2(t), ..., v_{k-1}(t)) =$$

$$= u(ct) - f(1, u(t), v_1(t), v_2(t), ..., v_{k-1}(t)) - F(g(1, u(t), v_1(t), v_2(t), ..., v_{k-1}(t))) =$$

$$= u(ct) - f(1, u(t), v_1(t), v_2(t), ..., v_{k-1}(t)) - F(1) =$$

$$= u(ct) - f(1, u(t), v_1(t), v_2(t), ..., v_{k-1}(t)) \geq 0$$

Further, we continue in the same way as in Lemma 2.6.

The above lemmas give us the possibility to construct other functions of type $\Phi_k$: 

Example 2.8 $\phi(t_1, t_2, \ldots, t_k) = t_1 - \left[\min\{t_{23}, t_3, t_{44}, t_{k-1}t_k\}\right]^{\frac{1}{2}} - F(\max\{t_2, t_3, \ldots, t_k\})$.

Example 2.9 $\phi(t_1, t_2, \ldots, t_k) = t_1 - (a_1t_2^p + a_2t_3^p + \ldots + a_k t_k^p)^{\frac{1}{p}} - \max\{t_2^p, t_3^p, \ldots, t_k^p\}$.

where $p > 0$ and $0 \leq a_i \sum_{i=2}^{k} a_i \geq 1$.

Example 2.10 $\phi(t_1, t_2, \ldots, t_k) = t_1 - \left[\min\{t_{23}, t_3, t_{44}, t_{k-1}t_k\}\right]^{\frac{1}{2}}$, etc.

3. Main results

Theorem 3.1 Let $(X, M_1, *),$ $(Y, M_2, *),$ $(Z, M_3, *)$ be three complete fuzzy metric spaces, $T : X \to Y,$ $S : Y \to Z$ and $R : Z \to X$ three maps satisfying:

\begin{align*}
\phi_1(M_1(RSy, RSTx, ct), M_1(x, RSy, t), M_1(x, RSTx, t), M_1(Sy, STx, t), ) & \geq 0 \\
\phi_2(M_2(TRz, TRSy, ct), M_2(y, TRz, t), M_2(y, TRSy, t), M_2(z, Sy, t), M_2(Rz, RSy, t)) & \geq 0 \\
\phi_3(M_3(STx, STRz, ct), M_3(z, STx, t), M_3(z, STRz, t), M_1(x, Rz, t), M_2(Tx, TRz, t)) & \geq 0
\end{align*}

(1) (2) (3)

for all $x \in X,$ $y \in Y,$ $z \in Z,$ $t > 0$ where $c \in (0,1)$ and $\phi_1, \phi_2, \phi_3 \in \Phi.$

If one of the maps $T, S, R$ is continuous, then $RST$ has a unique fixed point $\alpha \in X,$ $TRS$ has a unique fixed point $\beta \in Y$ and $STR$ has a unique fixed point $\gamma \in Z.$ Moreover, $T\alpha = \beta,$ $S\beta = \gamma$ and $R\gamma = \alpha.$

Proof: Let $x_0$ be an arbitrary point in $X.$ Construct the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in $X,$ $Y$ and $Z,$ respectively, as follows:

$x_n = (RST)^n x_0,$ $y_n = T x_{n-1},$ $z_n = S y_n,$ $n \in N.$

We will show that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences.

Denote:

$d_n(t) = M_1(x_n, x_{n+1}, t), \rho_n(t) = M_2(y_n, y_{n+1}, t), \sigma_n(t) = M_3(z_n, z_{n+1}, t)$

By the inequality (2) for $z = z_{n-1}$ and $y = y_n$ we get:

\begin{align*}
\phi_2(M_2(y_n, y_{n+1}, ct), M_2(y_n, y_{n+1}, t), M_2(y_n, y_{n+1}, t), M_3(z_{n-1}, z_n, t), M_1(x_{n-1}, x_n, t) = \\
= \phi_2(\rho_n(ct), 1, \rho_n(t), \sigma_n(t), d_{n-1}(t)) & \geq 0
\end{align*}

(4)

Next, from (4), after the application of property (b) of $\phi_2$ we have

$\rho_n(ct) \geq \min\{\sigma_n(t), d_{n-1}(t)\}, \forall n \in N$

(5)

By the inequality (3) for $x = x_{n-1}$ and $z = z_n$ we get:

\begin{align*}
\phi_3(M_3(x_{n-1}, x_n, ct), M_3(x_{n-1}, x_n, t), M_3(x_{n-1}, x_n, t), M_1(z_{n-1}, z_n, t), M_2(x_{n-1}, x_n, t) = \\
= \phi_3(\rho_n(ct), 1, \rho_n(t), \sigma_n(t), d_{n-1}(t)) & \geq 0
\end{align*}

(6)
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\( \varphi_3(M_3(z_n, z_{n+1}, c), M_3(z_n, z_{n}, t), M_3(z_n, z_{n+1}, t), M_3(x_{n-1}, x_n, t), M_3(y_n, y_{n+1}, t)) = \varphi_3(\sigma_n(t), 1, \sigma_n(t), d_{n-1}(t), \rho_n(t)) \geq 0 \)

(6)

From (6) and the property (b) of \( \varphi_3 \) we get \( \sigma_n(ct) \geq \min\{d_{n-1}(t), \rho_n(t)\} \)

From the last inequality, by the fact that \( \rho_n(t) \geq \rho_n(ct) \) and inequality (5), we get:

\[ \sigma_n(ct) \geq \min\{\sigma_{n-1}(t), d_{n-1}(t)\}, \quad \forall n \in N \]

(7)

In similar way, by (1) for \( y = y_n \) and \( x = x_n \) we find:

\[ \varphi_1(M_1(x_n, x_{n+1}, ct), M_1(x_n, x_n, t), M_1(x_n, x_{n+1}, t), M_1(y_n, y_{n+1}, t), M_1(z_n, z_{n+1}, t)) = \varphi_1(d_n(ct), 1, d_n(t), \rho_n(t), \sigma_n(t)) \geq 0 \]

And from (b) we have: \( d_n(ct) \geq \min\{\rho_n(t), \sigma_n(t)\} \)

By this inequality and by (5), (7) it follows:

\[ d_n(ct) \geq \min\{\sigma_{n-1}(t), d_{n-1}(t)\}, \quad \forall n \in N \]

(8)

Applying (8) and (7), considering as \( t \) the number \( \frac{t}{c} \), we obtain

\[ d_{n-1}(t) = d_{n-1}(c \frac{t}{c}) \geq \min\{\sigma_{n-2}(\frac{t}{c}), d_{n-2}(\frac{t}{c})\} \]

and

\[ \sigma_{n-1}(t) = \sigma_{n-1}(c \frac{t}{c}) \geq \min\{\sigma_{n-2}(\frac{t}{c}), d_{n-2}(\frac{t}{c})\} . \]

By induction we have

\[ d_n(ct) \geq \min\{\sigma_{n-1}(t), d_{n-1}(t)\} \geq \min\{\sigma_{n-2}(\frac{t}{c}), d_{n-2}(\frac{t}{c})\} \geq \ldots \geq \min\{\sigma_{1}(\frac{t}{c^{n-2}}), d_{1}(\frac{t}{c^{n-2}})\} \]

or

\[ d_n(t) \geq \min\{\sigma_{1}(\frac{t}{c^{n-2}}), d_{1}(\frac{t}{c^{n-2}})\} \]

In the same way

\[ \rho_n(t) \geq \min\{\sigma_{1}(\frac{t}{c^{n-2}}), d_{1}(\frac{t}{c^{n-2}})\} \]

and

\[ \sigma_n(t) \geq \min\{\sigma_{1}(\frac{t}{c^{n-2}}), d_{1}(\frac{t}{c^{n-2}})\} . \]

Thus, for all \( n \in N \) and \( t > 0 \) we have

\[ M_1(x_n, x_{n+1}, t) \geq \min\{M_3(z_1, z_2, \frac{t}{k^{n-1}}), M_1(x_1, x_2, \frac{t}{k^{n-1}})\} \]

\[ M_2(y_n, y_{n+1}, t) \geq \min\{M_3(z_1, z_2, \frac{t}{k^{n-1}}), M_1(x_1, x_2, \frac{t}{k^{n-1}})\} \]
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\[ M_1(z_n, z_{n+1}, t) \geq \min \{ M_3(z_1, z_2, \frac{t}{K_{n-1}}), M_1(x_1, x_2, \frac{t}{K_{n-1}}) \} \]

But \( \lim_{n \to \infty} \frac{t}{C^{n-1}} = \infty \) because \( c \in (0,1) \) and applying (FM-6) we get:

\[ \lim_{n \to \infty} \sigma_1(\frac{t}{C^{n-1}}) = \lim_{n \to \infty} M_3(z_1, z_2, \frac{t}{C^{n-1}}) = 1 \]

and

\[ \lim_{n \to \infty} d_1(\frac{t}{C^{n-1}}) = \lim_{n \to \infty} M_1(x_1, x_2, \frac{t}{C^{n-1}}) = 1 \]

Consequently,

\[ \lim_{n \to \infty} M_1(x_n, x_{n+1}, t) = \lim_{n \to \infty} M_2(y_n, y_{n+1}, t) = \lim_{n \to \infty} M_3(z_n, z_{n+1}, t) = 1. \]

Now, for all \( n \) and \( p \), using the Definition 2.3 (FM-4), we get:

\[ M_1(x_n, x_{n+p}, t) \geq M_1(x_n, x_{n+1}, t) * M_1(x_{n+1}, x_{n+2}, \frac{t}{p}) * \ldots * M_1(x_{n+p-1}, x_{n+p}, t) \]

When \( n \) tends to infinity, we have

\[ \lim_{n \to \infty} M_i(x_{n+p}, x_n, t) \geq 1 \]

This shows that \( \{x_n\} \) is a Cauchy sequence in \( X \). We can show in the same way that \( \{y_n\} \) and \( \{z_n\} \) are also Cauchy sequences in \( Y \) and \( Z \), respectively. That is,

\[ \lim_{n \to \infty} x_n = \alpha \in X, \lim_{n \to \infty} y_n = \beta \in Y, \lim_{n \to \infty} z_n = \gamma \in Z. \]

Suppose that \( S \) is continuous. Then, since \( z_n = S y_n \), taking the limit we have

\[ S \beta = \gamma \]

Applying (1) we get

\[ \phi \left( \frac{M_i(R S \beta, x_{n+1}, ct), M_i(x_n, R S \beta, t), M_i(x_n, x_{n+1}, t), M_i(\beta, y_{n+1}, t)}{M_3(S \beta, z_{n+1}, t)} \right) \geq 0 \]

Now, when \( n \) tends to infinity, using (9), we have

\[ \phi(M_i(R S \beta, \alpha, ct), M_i(\alpha, R S \beta, t), 1, 1, 1) \geq 0 \]

After the application of property (b) of \( \phi_1 \) we have

\[ M_i(R S \beta, \alpha, ct) = 1 \]

This means (Lemma 2.10) that

\[ R S \beta = \alpha \] (10)

And from (9) we get

\[ R \gamma = \alpha \] (11)

Using (10) and (2), we obtain

\[ \phi_2(M_2(T \alpha, y_{n+1}, ct), M_2(y_n, T \alpha, t), M_2(y_n, y_{n+1}, t)M_3(S \beta, S y_n, t)M_i(R S \beta, x_n, t)) \geq 0 \]

Letting \( n \) tend to infinity we take
Thus,
\[ T\alpha = \beta \] (12)

Next, from (9), (11) and (12), we have
\[ \text{TRS} \beta = TR\gamma = T\alpha = \beta , \text{ STR}\gamma = ST\alpha = S\beta = \gamma , \text{ RST} \alpha = RS\beta = R\gamma = \alpha . \]

So, \( \alpha \) is a fixed point for \( \text{RST} \), \( \beta \) is a fixed point for \( \text{TRS} \) and \( \gamma \) is a fixed point for \( \text{STR} \).

To prove the uniqueness, we suppose that \( \alpha' \) is another fixed point of \( \text{RST} \).

Applying (1) for \( y = T\alpha \) and \( x = \alpha' \), we have
\[ \phi_1 \left( M_1(\text{RST} \alpha, \text{RST} \alpha', c_t), M_1(\alpha', \text{RST} \alpha, t), M_1(\alpha', \text{RST} \alpha', t) \right) = \phi_1(M_1(\alpha, \alpha', c_t), M_1(\alpha, \alpha', t), 1, M_1(T\alpha, T\alpha', t), M_3(ST\alpha, ST\alpha', t)) \geq 0 \]

Applying now the property (b) for \( \phi_1 \), we obtain
\[ M_1(\alpha, \alpha', c_t) \geq \min \{ M_2(T\alpha, T\alpha', t), M_3(ST\alpha, ST\alpha', t) \} \] (13)

Next, from (2) it follows that
\[ \phi_2 \left( M_2(\text{TRST} \alpha, \text{TRST} \alpha', c_t), M_2(T\alpha', \text{TRST} \alpha, t), M_2(T\alpha', T\alpha', t), M_3(ST\alpha, ST\alpha', t) \right) = \phi_2(M_2(\alpha, \alpha', c_t), M_2(T\alpha', T\alpha', t), 1, M_1(ST\alpha, ST\alpha', t), M_3(\alpha, \alpha', t)) \geq 0 \]

Thus, we have
\[ M_2(T\alpha, T\alpha', c_t) \geq \min \{ M_3(ST\alpha, ST\alpha', t), M_1(\alpha, \alpha', t) \} \] (14)

Now, from (13) and (14) and from the fact that \( M_2(T\alpha, T\alpha', t) \geq M_2(T\alpha, T\alpha', c_t) \), we have
\[ M_1(\alpha, \alpha', c_t) \geq M_1(ST\alpha, ST\alpha', t) \] (15)

Finally, from (3), it follows that
\[ \phi_3 \left( M_3(\text{STRST} \alpha, \text{STRST} \alpha', c_t), M_3(ST\alpha', ST\alpha, t), M_3(ST\alpha', ST\alpha', t), \right) = \phi_3(M_3(ST\alpha, ST\alpha', c_t), M_3(ST\alpha', ST\alpha, t), 1, M_1(\alpha, \alpha', t), M_2(T\alpha, T\alpha', t)) \geq 0 \]

Hence,
\[ M_3(ST\alpha, ST\alpha', c_t) \geq \min \{ M_1(\alpha, \alpha', t), M_2(T\alpha, T\alpha', t) \} \] (16)

Again, from (14), (15), (16) and from the fact that
\[ M_3(ST\alpha, ST\alpha', t) \geq M_3(ST\alpha, ST\alpha', c_t) \]

we have
\[ M_1(\alpha, \alpha', c_t) \geq M_3(ST\alpha, ST\alpha', t) \geq M_3(ST\alpha, ST\alpha', c_t) \geq \min \{ M_1(\alpha, \alpha', t), M_2(T\alpha, T\alpha', t) \} = M_2(T\alpha, T\alpha', t) \geq M_2(T\alpha, T\alpha', c_t) \geq \min \{ M_3(ST\alpha, ST\alpha', t), M_1(\alpha, \alpha', t) \} = M_3(ST\alpha, ST\alpha', t) \]
From the inequalities \( M_1(\alpha, \alpha', ct) \geq M_2(ST\alpha, ST\alpha', ct) \geq M_3(ST\alpha, ST\alpha', t) \) it follows that
\[
ST\alpha = ST\alpha', M_3(ST\alpha, ST\alpha', ct) = 1 \quad \text{and} \quad M_1(\alpha, \alpha', ct) \geq 1.
\]
So,
\[
\alpha = \alpha'.
\]
Thus, \( \alpha \) is the unique fixed point for \( RST \). In the same way we show that \( \beta \) is the unique fixed point for \( TRS \) and \( \gamma \) the unique fixed point for \( STR \). This completes the proof.

4. Corollaries

The theorem 3.1 extends in fuzzy metric spaces the theorems of Nung [13], Jain et al [8], Kikina [9], etc.

**Corollary 4.1** Let \((X, M_1, *)\), \((Y, M_2, *)\) and \((Z, M_3, *)\) be three complete fuzzy metric spaces and \( T : X \to Y \), \( S : Y \to X \), \( R : Z \to X \) three maps which satisfy the conditions:
\[
M_1(RSTx, RSy, ct) \geq f_1(M_1(x, RSy, t), M_1(x, RSTx, t), M_2(y, Tx, t), M_3(Sy, STx, t)) +
+ F_1(g_1(M_1(x, RSy, t), M_1(x, RSTx, t), M_2(y, Tx, t), M_3(Sy, STx, t))
\]
\[
M_2(TRSy, TRz, ct) \geq f_2(M_2(y, TRz, t), M_2(y, TRSy, t), M_3(z, Sy, t), M_1(Rz, RSy, t)) +
+ F_2(g_2(M_2(y, TRz, t), M_2(y, TRSy, t), M_3(z, Sy, t), M_1(Rz, RSy, t))
\]
\[
M_3(STRz, STx, ct) \geq f_3(M_3(z, STx, t), M_3(z, STRz, t), M_1(x, Rz, t), M_2(Tx, TRz, t)) +
+ F_3(g_3(M_3(z, STx, t), M_3(z, STRz, t), M_1(x, Rz, t), M_2(Tx, TRz, t))
\]
for all \( x \in X, y \in Y, z \in Z, t > 0 \) where \( c \in (0,1) \) and \( f_1, f_2, f_3 \in F_4; g_1, g_2, g_3 \in G_4; F_i, F_2, F_3 \in F \). If one of the maps \( T, S, R \) is continuous, then \( RST \) has a unique fixed point \( \alpha \in X \), \( TRS \) has a unique fixed point \( \beta \in Y \) and \( STR \) has a unique fixed point \( \gamma \in Z \). Moreover, \( T\alpha = \beta, S\beta = \gamma \) and \( R\gamma = \alpha \).

**Proof:** The proof follows by theorem 3.1 in the case \( \varphi_i \in \Phi_5, f_i \in F_4, g_i \in G_4, F_i \in F \) such that
\[
\varphi_i(t_1, t_2, t_3, t_4, t_5) = t_1 - f_i(t_2, t_3, t_4, t_5) - F_i(g_i(t_2, t_3, t_4, t_5)) \quad \text{for} \quad i = 1, 2, 3.
\]

**Corollary 4.2** If in theorem 3.1 we take
\[
\varphi_i(t_1, t_2, t_3, t_4, t_5) = t_1 - \min\{t_2^p, t_3^p, t_4^p, t_5^p\} - F_i(\max\{t_2^p, t_3^p, t_4^p, t_5^p\}) \quad \text{for} \quad i = 1, 2, 3,
\]
then we obtain the fuzzy version of Theorem Nešić [12] extended in three metric spaces.

**Corollary 4.3** If in theorem 3.1 we take
\[
\varphi_i(t_1, t_2, t_3, t_4, t_5) = t_1 - \min\{t_2^p, t_3^p, t_4^p, t_5^p\} - F_i(\max\{t_2^p, t_3^p, t_4^p, t_5^p\}) \quad \text{for} \quad i = 1, 2, 3.
\]
we obtain the proposition which fuzzyfies the theorem of Kikina (Theorem 2.1[9]) for metric space.

Corollary 4.4 If in theorem 3.1 we take $\phi = \phi_1 = \phi_2 = \phi_3 \in \Phi$, where $\phi(t_1, t_2, t_3, t_4, t_5) = t_1 - \min\{t_2, t_3, t_4, t_5\}$, we obtain the theorem which fuzzyfies the result of Nung [13] for metric spaces.

Corollary 4.5 If in theorem 3.1 we take $\phi = \phi_1 = \phi_2 = \phi_3 \in \Phi$, where $\phi(t_1, t_2, t_3, t_4, t_5) = t_1 - \sqrt{\min\{t_2, t_3, t_4, t_5\}}$, we take the proposition which fuzzyfies the theorem of Jain, Shrivastava and Fisher (Theorem 3 [8]).

Remark. As corollaries of these results we can obtain other propositions determined by the form of implicit functions.

References


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