Characterization of Set of k-g-Inverses

P. Jenita
Department of Mathematics
KPR Institute of Engineering and Technology
KPR knowledge city, Arasur, Coimbatore, India

A. R. Meenakshi
Department of Mathematics
Karpagam college of Engineering
Coimbatore - 641 032, India

Abstract

In this paper, we have obtained the characterization of the set of all k-g-inverses of a fuzzy matrix and characterized the set of various k-g-inverses associated with a k-regular fuzzy matrix.

Mathematics Subject Classification: 15A57; 15A09

Keywords: Fuzzy matrices, k-regular fuzzy matrices, k-g-inverses.

1. Introduction

A matrix over \( F = [0,1] \) is called a fuzzy matrix with operations \((+,-)\) defined as \(a+b=\max\{a,b\}\) and \(a\cdot b=\min\{a,b\}\) for all \(a, b \in F\). Let \( F_n \) be the set of all \( n \times n \) fuzzy matrices over \( F \). \( A^T \) denotes the transpose of \( A \). If a solution exists for the matrix equation \( AXA=A \), then \( A \) is called a regular fuzzy matrix and such a solution called a generalized (g-) inverse of \( A \) and is denoted as \( A^{-1} \). \( A\{1\} \) denotes the set of all g-inverses of a regular matrix \( A \). Recently, Meenakshi and Jenita [4] have introduced the concept of k-regular fuzzy matrix analogous to that of generalized inverse of a complex matrix [1] and as a generalization of a regular fuzzy matrix [2,3].
Definition 1.1[4]:
A matrix $A \in \mathbb{F}_n$ is said to be right k-regular if there exists a matrix $X \in \mathbb{F}_n$ such that $A^kXA = A^k$, for some positive integer $k$. $X$ is called a right k-g-inverse of $A$. Let $A\{1^k\} = \{X/ A^kXA = A^k \}.$

Definition 1.2[4]:
A matrix $A \in \mathbb{F}_n$, is said to be left k-regular if there exists a matrix $Y \in \mathbb{F}_n$ such that $AYk = A^k$, for some positive integer $k$. $Y$ is called a left k-g-inverse of $A$. Let $A\{1^k\} = \{Y/ AYA^k = A^k \}.$

In general, right k-regular is different from left k-regular. Hence a right k-g-inverse need not be a left k-g-inverse [4]. Hence forth we call a right k-regular (or) left k-regular matrix as a k-regular matrix. Let $A\{1^k\} = A\{1^k\} \cup A\{1^k\}.$

Definition 1.3[5]:
A matrix $A \in \mathbb{F}_n$, is said to have a $\{3^k\}$ inverse if there exists a matrix $X \in \mathbb{F}_n$ such that $(A^kX)^T = A^kX$, for some positive integer $k$. $X$ is called the $\{3^k\}$ inverse of $A$. Let $A\{3^k\} = \{X/ (A^kX)^T = A^kX \}.$

Definition 1.4[5]:
A matrix $A \in \mathbb{F}_n$, is said to have a $\{4^k\}$ inverse if there exists a matrix $X \in \mathbb{F}_n$ such that $(XA^k)^T = XA^k$, for some positive integer $k$. $X$ is called the $\{4^k\}$ inverse of $A$. Let $A\{4^k\} = \{X/ (XA^k)^T = XA^k \}.$

Theorem 1.1[5]:
For $A \in \mathbb{F}_n$ and for any $G \in \mathbb{F}_n$, if $A^kX = A^kG$, where $X$ is a $\{1^k, 3^k\}$ inverse of $A$ then, $G$ is a $\{1^k, 3^k\}$ inverse of $A$.

Theorem 1.2[5]:
For $A \in \mathbb{F}_n$, $X$ is a $\{1^k, 3^k\}$ inverse of $A$ and $G$ is a $\{1^k, 3^k\}$ inverse of $A$ then, $A^kX = A^kG$.

Theorem 1.3[5]:
For $A \in \mathbb{F}_n$ and for any $G \in \mathbb{F}_n$, if $XA^k = G A^k$, where $X$ is a $\{1^k, 4^k\}$ inverse of $A$ then, $G$ is a $\{1^k, 4^k\}$ inverse of $A$. 
Theorem 1.4[5]:
For \(A \in \mathcal{F}_n\), \(X\) is a \(\{1^k,4^k\}\) inverse of \(A\) and \(G\) is a \(\{1^k,4\}\) inverse of \(A\) then, \(XA^k = GA^k\).
In particular for \(k=1\), Theorems (1.1) to (1.4) reduces to the following:

Theorem 1.5[3]:
For \(A \in \mathcal{F}_{mn}\), the set \(A\{1,3\}\) consists of all solutions for \(X\) of \(AX=AG\), where \(G\) is a \(\{1,3\}\) inverse of \(A\) and the set \(A\{1,4\}\) consists of all solutions for \(X\) of \(XA=GA\), where \(G\) is a \(\{1,4\}\) inverse of \(A\).

2. Characterization of set of k-g-inverses

Lemma 2.1:
For \(A \in \mathcal{F}_n\), if \(G\) and \(G^*\) are right k-g-inverses of \(A\) such that \(G^* \geq G\), then \(G+H\) is a right k-g-inverse of \(A\) for some \(H \in \mathcal{F}_n\) such that \(G^* \geq G+H \geq G\).

Proof:
Since \(G\) and \(G^*\) are right k-g-inverses of \(A\) with \(G^* \geq G\), let \(G^* - G = H\). Then \(G^* \geq H\) and \(G^* \geq G + H \geq G\) \(\implies\) \(A^k(G+H)A \geq A^kGA\). Thus \(G+H\) is a k-g-inverse of \(A\).

Lemma 2.2:
For \(A \in \mathcal{F}_n\), if \(G\) and \(G^*\) are left k-g-inverses of \(A\) such that \(G^* \geq G\), then \(G+K\) is a left k-g-inverse of \(A\) for some \(K \in \mathcal{F}_n\) such that \(G^* \geq G+K \geq G\).

Proof:
Proof is similar to Lemma (2.1) and hence omitted.

Theorem 2.1:
Let \(A \in \mathcal{F}_n\) and \(G\) be a particular right k-g-inverse of \(A\).
Then \(A_G \{1^k\} = \{G + H/\text{ for all } H \in \mathcal{F}_n\text{ such that } A^k \geq H \geq A^kGA\}\) \(\implies\) (2.2) is the set of all right k-g-inverses of \(A\) dominating \(G\).

Proof:
Let \(\varnothing\) denote the set on the R.H.S of (2.2).
Suppose \(G^* \in A_G \{1^k\}\), then \(G^* \geq G\). Let \(G^*-G=H\).
By Lemma (2.1), $G^*\geq G+H\geq G$ and $G+H$ is a right $k$-g-inverse of $A$ dominating $G$. Then, $A^k(G+H)A = A^k \Rightarrow A^kGA + A^kHA = A^k$ 
$\Rightarrow A^k + A^kHA = A^k$ 
$\Rightarrow A^k \geq A^kHA$.

Hence $G+H \in \varphi$. Thus for each $G^* \in A_{\varphi} \{1,2\}^k$, there exists a unique element in $\varphi$.

Conversely, for any $G^* \in \varphi$, $G^* = G+H \geq G$ with $A^k \geq A^kHA$, then

$A^kG^*A = A^k(G+H)A$
$= A^kGA + A^kHA$
$= A^k + A^kHA$
$= A^k$.

Thus $G^* \in A_{\varphi} \{1,2\}^k$. Hence the theorem.

**Theorem 2.2:**

Let $A \in \mathbb{F}_n$ and $G$ be a particular left $k$-g-inverse of $A$. Then $A_{\varphi} \{1\}^k = \{G + K/\text{for all } K \in \mathbb{F}_n \text{ such that } A^k \geq AKA^k\}$ is the set of all left $k$-g-inverses of $A$ dominating $G$.

**Proof:**

This can be proved in the same manner as in Theorem (2.1) and hence omitted.

**Theorem 2.3:**

For $A \in \mathbb{F}_n$ and $G \in A \{1,3\} \{1,3\}$,

$A_{\varphi} \{1,3\}^k = \{G + H/\text{for all fuzzy matrix } H \in \mathbb{F}_n \text{ such that } A^kG \geq A^kH\}$ is the set of all $\{1,3\}^k$ inverses of $A$ dominating $G$.

**Proof:**

Let $\varphi$ denote the set on the R.H.S of (2.3).

Suppose $G^* \in A_{\varphi} \{1,3\} \{1,3\}$, then $G^* \geq G$. Let $G^* = G = H$.

Since $A_{\varphi} \{1,3\} \leq A_{\varphi} \{1\}$, by Lemma (2.1), $G^* \geq G + H \geq G$ implies $A^kG^* = A^k(G+H) \geq A^kG$.

By Theorem (1.2), $G^* \in A_{\varphi} \{1,3\} \{1,3\}$ and $G \in A \{1,3\} \Rightarrow A^kG^* = A^kG$

$\Rightarrow A^k(G + H) = A^kG$

$\Rightarrow A^kG + A^kH = A^kG$

$\Rightarrow A^kG \geq A^kH$.

Hence $G+H \in \varphi$. Thus for each $G^* \in A_{\varphi} \{1,3\} \{1,3\}$, there exists a unique element in $\varphi$.

Conversely, for any $G^* \in \varphi$, $G^* = G + H \geq G$ with $A^kG \geq A^kH$, then
Characterization of set of $k$-g-inverses

$A^kG^* = A^k(G + H) = A^kG + A^kH = A^kG$. Since $G \in A\{l^k, 3\} \Rightarrow G \in A\{l^k, 3^k\}$. Therefore, by Theorem (1.1), $G^* \in A_G\{l^k, 3^k\}$. Hence the theorem.

**Theorem 2.4:**

For $A \in \mathcal{F}_n$ and $G \in A\{l^k, 4\}$,

$A_G\{l^k, 4^k\} = \{G + K/ \text{for all fuzzy matrix } K \in \mathcal{F}_n \text{ such that } GA^k \geq KA^k\}$ \hspace{1cm} (2.4)

is the set of all $\{l^k, 4^k\}$ inverses of $A$ dominating $G$.

**Proof:**

Let $\varphi$ denote the set on the R.H.S of (2.4).

Suppose $G^* \in A_G\{l^k, 4^k\}$, then $G^* \geq G$. Let $G^* - G = K$.

Since $A_G\{l^k, 4^k\} \subseteq A_G\{l^k\}$, by Lemma (2.2), $G^* \geq G + K \geq G \Rightarrow G^*A^k = (G + K)A^k \geq GA^k$.

By Theorem (1.4), $G^* \in A_G\{l^k, 4^k\}$ and $G \in A\{l^k, 4\}$ \Rightarrow $G^*A^k = GA^k$

\begin{align*}
&\Rightarrow (G + K)A^k = GA^k \\
&\Rightarrow GA^k + KA^k = GA^k \\
&\Rightarrow GA^k \geq KA^k.
\end{align*}

Hence $G + K \in \varphi$. Thus for each $G^* \in A_G\{l^k, 4^k\}$, there exists a unique element in $\varphi$.

Conversely, for any $G^* \in \varphi$, $G^* = G + K \geq G$ with $GA^k \geq KA^k$, then

$G^*A^k = (G + K)A^k = GA^k + KA^k = GA^k$. Since $G \in A\{l^k, 4\} \Rightarrow G \in A\{l^k, 4^k\}$.

Therefore, by Theorem (1.3), $G^* \in A_G\{l^k, 4^k\}$. Hence the theorem.

**References**


Received: October, 2010