On the Integrability for the Jacobian in Anisotropic Sobolev Space

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Abstract

This paper considers the integrability of the Jacobian of orientation-preserving mappings in anisotropic Sobolev space $W^{1, p_1 - \varepsilon}(\mathbb{R}^n, \mathbb{R}^n)$, $p = (p_1, p_2, \ldots, p_n)$, with $1 < p_1 - \varepsilon, p_2 - \varepsilon, \ldots, p_n - \varepsilon < \infty, \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1$ and $0 < \varepsilon < 1$. The main result is a sufficient condition ensuring integrability of the Jacobian, which can be regarded as a refinement of the result of Iwaniec and Sbordone [10].

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1 Introduction and Statement of Result

In recent years, there have been remarkable advances made in the theory of Jacobian of Sobolev mappings, for example [2,5,6,7,9,10 and 11]. Many interesting results of them and their applications in fields such as quasiregular analysis, the geometric theory of measure and integration, the mapping degree theory and nonlinear elasticity have been found. See [8,1] and the references therein.

In order to study space mappings appearing in quasiregular analysis and nonlinear elasticity, it is necessary to integrate the Jacobian. If $f \in W^{1,p}_{loc}(\mathbb{R}^n, \mathbb{R}^n)$, then obviously the Jacobian $J_f(x)$ is locally integrable. But this condition is not necessary to ensure that $J_f(x)$ is locally integrable. It is rather surprising that just one condition, that $J_f(x)$ does not change sign in $\mathbb{R}^n$, implies higher

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integrability of the Jacobian. Stefan Müller [13] was the first to observe this phenomenon, see also [M2]. T.Iwaniec and C.Sbordone gave the minimal hypotheses to ensure the integrability of the Jacobian in [10]. A natural question now arises to under what condition on $f$ is the Jacobian function integrable if $f$ lies in the anisotropic Sobolev space $W^{1,p-\varepsilon}(\mathbb{R}^n, \mathbb{R}^n)$? In this paper, we will solve this problem.

Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \geq 2$. The usual Lebesgue space $L^p(\Omega)$, $1 < p < \infty$, is equipped with the norm

$$\|g\|_{p,\Omega} = \left(\int_\Omega |g(x)|^p dx\right)^{\frac{1}{p}}, \quad \text{for } g \in L^p(\Omega), 1 \leq p < \infty$$

The average of a function $g \in L^1(\Omega)$ is denoted by

$$g_{\Omega} = \frac{1}{|\Omega|} \int_\Omega g(x) dx$$

provided $|\Omega|$ (the Lebesgue measure of $\Omega$) is positive and finite.

The space $L^{p_0}(\Omega)$, consists of all functions $g \in \bigcap_{1 \leq s < p} L^s(\Omega)$ such that

$$\|g\|_{p_0,\Omega} = \sup_{1 \leq s < p} \left[ (p - s) \int_\Omega |g(x)|^s dx \right]^{\frac{1}{p - s}} < \infty$$

(1)

This is a norm in $L^{p_0}(\Omega)$ which makes $L^{p_0}(\Omega)$ a Banach space.

Let $f = (f^1, f^2, \cdots, f^n)$ be a mapping of the anisotropic Sobolev class $W^{1,p-\varepsilon}(\Omega, \mathbb{R}^n)$, where $\mathcal{P} - \varepsilon = (p_1 - \varepsilon, p_2 - \varepsilon, \cdots, p_n - \varepsilon)$ is an $n$-tuple of exponents $p_1 - \varepsilon, p_2 - \varepsilon, \cdots, p_n - \varepsilon \in (1, +\infty)$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1$ and $0 < \varepsilon < 1$. By the condition on $f$ we know that each co-ordinate $f^j, j = 1, 2, \cdots, n$ and its gradient belong to $L^{p_0-\varepsilon}(\Omega)$. The differential and the Jacobian of $f$ are denoted by $Df(x) : \Omega \to \mathbb{R}^n$ and $J_f(x) = \det Df(x)$, respectively. Throughout this paper, unless other stated, we assume that $f$ is an orientation-preserving mapping, that is, the Jacobian function $J_f(x)$ is non-negative almost everywhere. The operator norm of $Df(x)$ is defined by $|Df(x)| = \sup\{|Df(x)\xi| : \xi \in S^{n-1}\}$, where $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$. Since $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1$, then at least one of the exponents $p_k (k = 1, 2, \cdots, n)$ is not greater than $n$. It is no loss of generality to assume that $1 < p_n \leq n$.

We now state the main result of this paper, which is concerned with the determination of the minimal condition on $f$ to ensure integrability of the Jacobian. Precisely, we have

**Theorem** Let $B \subset 3B$ be given concentric balls in $\mathbb{R}^n$ and let $f = (f^1, f^2, \cdots, f^n) : 3B \to \mathbb{R}^n$ be an orientation-preserving mapping. For each $j = 1, 2, \cdots, n$, the co-ordinate $f^j$ lies in the grand space $\bigcap_{1 \leq s < p} W^{1,s}(3B, \mathbb{R}^n)$.
and satisfies
\[ \sup_{1 \leq s < p_j} (p_j - s) \int_{3B} |df_j|^s \, dx < \infty \]

Then, \( J_f(x) \in L^1_{loc}(3B) \) and the following uniform estimate holds:
\[
\int_B J_f(x) \, dx \leq C(n, p_1, p_2, \cdots, p_n) \|df_1\|_{p_1, 3B} \cdots \|df_n\|_{p_n, 3B} \tag{2}
\]

2 A preliminary Lemma

In [4], the authors obtain the following result by using the technique of Hodge decomposition.

**Lemma** Let \( f = (f^1, f^2, \cdots, f^n) : R^n \to R^n \), be a mapping of anisotropic Sobolev class \( W^{1, p^{-\varepsilon}}(R^n, R^n) \), with \( p_1 - \varepsilon, p_2 - \varepsilon, \cdots, p_n - \varepsilon \in (1, \infty), \frac{1}{p_1} + \frac{1}{p_2} + \cdots, + \frac{1}{p_n} = 1, \) and \( 0 < \varepsilon < 1 \). Then
\[
\int_{R^n} |df^1|^{-\tau} J_f(x) \, dx \\
\leq C(n, p_1, p_2, \cdots, p_n) \tau \|df^1\|_{p_1-\varepsilon} \|df^2\|_{p_2-\varepsilon} \cdots \|df^n\|_{p_n-\varepsilon} \tag{3}
\]
where
\[ \tau = 1 - (p_1 - \varepsilon) \left(1 - \sum_{j=2}^n \frac{1}{p_j - \varepsilon}\right) \]

Here and in the sequel, \( C(n, p_1, p_2, \cdots, p_n) \) is always denote some constant depending only on the dimension \( n \) and \( p_1, p_2, \cdots, p_n \). It may take different values in different places even in the same formula.

Some remarks are needed.

**Remark 1** Here we consider the estimate of
\[
\int_{R^n} |df^1|^{-\tau} J_f(x) \, dx \\
\]
for the sake of simplicity. It is easy to see, by using the same method as in [GLZ], that the following estimate holds for each \( k = 1, 2, \cdots, n \),
\[
\int_{R^n} |df^k|^{-\tau_k} J_f(x) \, dx \\
\leq C(n, p_1, p_2, \cdots, p_n) \tau_k \|df^1\|_{p_1-\varepsilon} \cdots \|df^k\|_{p_k-\varepsilon} \cdots \|df^n\|_{p_n-\varepsilon} \\
\]
where
\[ \tau_k = 1 - (p_k - \varepsilon) \left(1 - \sum_{j=1, j\neq k}^n \frac{1}{p_j - \varepsilon}\right) \]
Remark 2  Lemma 3 does not make any assumption on the sign of the Jacobian \( J_f(x) \). The key fact in Lemma 3 is that there is a factor \( \tau \) in the right-hand side of the inequality. Since \( \frac{p - \varepsilon}{1 - \tau}, p_2 - \varepsilon, \ldots, p_n - \varepsilon \) are Hölder conjugate exponents, we have an estimate

\[
\int_{\mathbb{R}^n} |df^1|^{-\tau} J_f(x) dx = \int_{\mathbb{R}^n} |df^1|^{-\tau} |df^2| \cdots |df^n|
\leq \| |df^1|^{1-\tau}\|_{p_1-\varepsilon} \|df^2\|_{p_2-\varepsilon} \cdots \|df^n\|_{p_n-\varepsilon}
\]

Thus, (3) always hold with constant equal to 1 in place of \( C(n)\tau \).

Remark 3  If \( p_1 = p_2 = \cdots = p_n = n \), then \( \tau = \varepsilon \), by Hadamard’s inequality \( |df^k| \leq |Df|, k = 1, 2, \ldots, n \), (3) reduces to

\[
\int_{\mathbb{R}^n} |df^1|^{-\varepsilon} J_f(x) dx = \int_{\mathbb{R}^n} |df^1| |df^2| \cdots |df^n|
\leq C(n)\varepsilon \| df^1 \|_{n-\varepsilon} \| df^2 \|_{n-\varepsilon} \cdots \| df^n \|_{n-\varepsilon}
\leq C(n)\varepsilon \int_{\mathbb{R}^n} |Df(x)|^{n-\varepsilon} dx
\]

which is nothing but [10, Theorem 1].

3 Proof of the Main Result

Let \( B = B(a,r) \subset B(a,2r) = 2B \subset B(a,3r) = 3B \) be given concentric balls in \( \mathbb{R}^n \). Let \( \varphi \in C_0^\infty(2B) \) and \( \psi \in C_0^\infty(3B) \) be cut-off functions such that

(i) \( 0 \leq \varphi \leq 1, \varphi \equiv 1 \) on \( B \), \( |\nabla \varphi| \leq \frac{C(n)}{r} \)

(ii) \( 0 \leq \psi \leq 1, \psi \equiv 1 \) on \( 2B \), \( |\nabla \psi| \leq \frac{C(n)}{r} \)

We shall examine an auxiliary mapping \( F \in W^{1,p-\varepsilon}(\mathbb{R}^n, \mathbb{R}^n) \) with compact support, defined by

\[
F = (\psi f^1, \ldots, \psi f^{n-1}, \varphi f^n)
\]

It is straightforward to see that

\[
\varphi |df^1|^{-\tau} \varphi f^1 \cdots |df^n|^{-\tau} \varphi f^n
= |dF^1|^{-\tau} |dF^1| \cdots |dF^n| - |dF|^1 \cdots |dF^{n-1}| \varphi
\]

where \( \tau \) is defined as in Lemma. Using the language of differential forms we write

\[
J_f(x) dx = df^1 \wedge df^2 \wedge \cdots \wedge df^n
\]

Applying Lemma to the mapping \( F \) we find that

\[
\int_B \varphi |df^1|^{-\tau} J_f(x) dx \leq \int_{3B} |\nabla \varphi| |f^n| |df^1|^{1-\tau}|df^2| \cdots |df^{n-1}| +
+ C(n, p_1, p_2, \ldots, p_n) \tau |dF^1|_{p_1-\varepsilon, 3B} |dF^2|_{p_2-\varepsilon, 3B} \cdots |dF^n|_{p_n-\varepsilon, 3B} \quad (4)
\]
By the definition of $F$ and the conditions (i) and (ii) on $\varphi$ and $\psi$ we can derive that
\[ |dF^j| \leq |df^j| + \frac{C(n)}{r} |f^j|, \quad j = 1, 2, \ldots, n \]
Therefore,
\[ \|dF^j\|_{p_j^{-\varepsilon}, 3B} = \int_{3B} |dF^j|_{p_j^{-\varepsilon}} \, dx \]
\[ \leq C(p_j) \int_{3B} |df^j|_{p_j^{-\varepsilon}} \, dx + C(p_j) r^{\varepsilon-p_j} \int_{3B} |f^j|_{p_j^{-\varepsilon}} \, dx \quad (5) \]
Notice that $df^j$ is not affected when a constant is added to $f^j$; thus we may assume the mean of $f^j$ over the ball $3B$ is equal to zero. This justifies the application of the Poincaré Inequality to the second term of (5). Hence
\[ \|dF^j\|_{p_j^{-\varepsilon}, 3B} \leq C(p_j) \|df^j\|_{p_j^{-\varepsilon}, 3B} \]
Thus, inequality (4) becomes
\[ \int_B \varphi |df^1|^{-r} J_f(x) \, dx \leq \int_{3B} |\nabla \varphi| |f^n| |df^1|^{-1-r} |df^2| \cdots |df^{n-1}| \, dx + \]
\[ + C(n, p_1, p_2, \ldots, p_n) \tau \|df^1\|_{p_1^{-\varepsilon}, 3B} \|df^2\|_{p_2^{-\varepsilon}, 3B} \cdots \|df^n\|_{p_n^{-\varepsilon}, 3B} \quad (6) \]
We first estimate the second term in the right-hand side of (6). Since $\frac{1}{p_1^{-\varepsilon}} + \frac{1}{p_2^{-\varepsilon}} + \cdots + \frac{1}{p_n^{-\varepsilon}} = 1$, then it is easy to see from the definition of $\tau$ and the relationship $\sum_{j=1}^n \frac{1}{p_j} = 1$ that $\tau = O(\varepsilon)(\varepsilon \to 0^+)$. So
\[ C(n, p_1, p_2, \ldots, p_n) \tau \|df^1\|_{p_1^{-\varepsilon}, 3B} \|df^2\|_{p_2^{-\varepsilon}, 3B} \cdots \|df^n\|_{p_n^{-\varepsilon}, 3B} \]
\[ = C(n, p_1, p_2, \ldots, p_n) \varepsilon \left[ \int_{3B} |df^1|_{p_1^{-\varepsilon}} \, dx \right]^{\frac{1}{p_1^{-\varepsilon}}} \cdots \left[ \int_{3B} |df^n|_{p_n^{-\varepsilon}} \, dx \right]^{\frac{1}{p_n^{-\varepsilon}}} \quad (7) \]
The limit of the right-hand side of (7) in finite when $\varepsilon \to 0^+$, for it tends to
\[ C(n, p_1, p_2, \ldots, p_n)|3B|(|df^1|_{p_1}, 3B) \cdots (|df^n|_{p_n}, 3B) \]
The first term in the right-hand side of (6) can be estimated as follows. By Hölder Inequality
\[ \int_{3B} |\nabla \varphi| |f^n| |df^1|^{-1-r} |df^2| \cdots |df^{n-1}| \, dx \]
\[ \leq \frac{C(n)}{r} \int_{3B} |df^1|^{-1-r} |df^2| \cdots |df^{n-1}| |f^n| \, dx \]
\[ \leq \frac{C(n)}{r} \left[ \int_{3B} |df^1|^{-m_1} |df^2| \right]^{\frac{m_1}{m_m}} \cdots \times \left[ \int_{3B} |f^n|^{-m_n} \, dx \right]^{\frac{1}{m_n}} \quad (8) \]
\[ m_2 = \frac{p_2(p_2 - \epsilon)}{p_2 + \sigma}, \ldots, m_{n-1} = \frac{p_{n-1}(p_{n-1} - \epsilon)}{p_{n-1} + \sigma}, m_n = \frac{np_n(p_n - \epsilon)}{n(p_n + \sigma) - p_n(p_n - \epsilon)} \quad (9) \]

\[ \sigma \text{ be a positive number small enough to satisfy} \]

\[ \sigma \sum_{j=2}^{n} \frac{1}{p_j^2} < \frac{1}{n} \quad (10) \]

and \( m_1 \) satisfies \( \sum_{j=1}^{n} \frac{1}{m_j} = 1 \). These conditions ensure that

\[ \lim_{\epsilon \to 0^+} m_1 = \frac{1}{m_1} = \left( \frac{1}{p_1} + \frac{1}{n} + \sigma \sum_{j=2}^{n} \frac{1}{p_j^2} \right)^{-1} < p_1 \]

From the assumption that \( 1 \leq p_n \leq n \), the last term in the right-hand side of (8) can be estimated by using the Poincaré-Sobolev Inequality

\[ \| u - u_{3B} \|_{q,3B} \leq C r^{1+n(\frac{1}{2} - \frac{1}{p})} \| \nabla u \|_{p,3B} \quad 1 < p < n, q \leq \frac{np}{n - p} \]

In fact, if we take \( p = \frac{p_n(p_n - \epsilon)}{p_n + \sigma} < n \) and \( q = m_n \), then \( q = \frac{np}{n - p} \). Since the mean of \( f^n \) over the ball \( 3B \) is equal to zero, then

\[ \left[ \int_{3B} |f^n|^{m_n} \, dx \right]^{\frac{1}{m_n}} = \left[ \int_{3B} |f^n|^{\frac{np_n(p_n - \epsilon)}{n(p_n + \sigma) - p_n(p_n - \epsilon)}} \, dx \right]^{\frac{n(p_n + \sigma) - p_n(p_n - \epsilon)}{np_n(p_n - \epsilon)}} \]

Thus, (8) becomes

\[ \int_{3B} |\nabla \varphi||f^n|^1|df^1|^{1-\tau}|df^2| \cdots |df^{n-1}| \, dx \]

\[ \leq C(n) \left( \int_{3B} |df^1|^{(1-\tau)m_1} \, dx \right)^{\frac{m_1}{m_2}} \left( \int_{3B} |df^2|^{m_2} \, dx \right)^{\frac{m_2}{m_3}} \cdots \left( \int_{3B} |df^n|^{\frac{p_n(p_n - \epsilon)}{p_n + \sigma}} \, dx \right)^{\frac{p_n + \sigma}{p_n(p_n - \epsilon)}} \quad (11) \]

Combining (6) (7) and (11) and divided by \( |B| = \omega_n r^n \) in both sides of the inequality, where \( \omega_n \) is the volume of the unit ball in \( R^n \), we have

\[ \int_B |df^1|^{1-\tau} J_f(x) \, dx \leq C(n, p_1, p_2, \ldots, p_n) \frac{T}{\varepsilon} \left( \int_{3B} |df^1|^{p_1 - \varepsilon} \, dx \right)^{\frac{1}{p_1 - \varepsilon}} \cdots \left( \int_{3B} |df^n|^{p_n - \varepsilon} \, dx \right)^{\frac{1}{p_n - \varepsilon}} \]

\[ + C(n) \left( \int_{3B} |df^1|^{(1-\tau)m_1} \, dx \right)^{\frac{m_1}{m_2}} \left( \int_{3B} |df^2|^{m_2} \, dx \right)^{\frac{m_2}{m_3}} \cdots \left( \int_{3B} |df^n|^{\frac{p_n(p_n - \epsilon)}{p_n + \sigma}} \, dx \right)^{\frac{p_n + \sigma}{p_n(p_n - \epsilon)}} \]

\[ \times \left[ \int_{3B} |df^n|^{m_n - 1} \, dx \right]^{\frac{1}{m_n - 1}} \left[ \int_{3B} |df^n|^{\frac{p_n(p_n - \epsilon)}{p_n + \sigma}} \, dx \right]^{\frac{p_n + \sigma}{p_n(p_n - \epsilon)}} \]
Examining the limit as $\varepsilon$ decreases to zero, by Fatou’s Lemma we find that

$$
\int_B J_f(x) \, dx \leq C(n, p_1, p_2, \ldots, p_n) (|d f^1|)_{p_1, B} \cdots (|d f^n|)_{p_n, B} ^{\frac{p_1 \cdot \ldots \cdot p_n}{p_1 + \ldots + p_n}} \\
+ C(n) \left[ \int_{3B} |d f^1|^{\frac{p_1}{m_1}} \, dx \right] ^\frac{m_1}{p_1} \left[ \int_{3B} |d f^2|^{\frac{p_2}{m_1 + m_2}} \, dx \right] ^\frac{m_2}{p_2} \cdots \left[ \int_{3B} |d f^n|^{\frac{p_n}{m_1 + \ldots + m_n}} \, dx \right] ^\frac{m_n}{p_n} \\
\leq C(n, p_1, p_2, \ldots, p_n) \| |d f^1|\|_{p_1, B} \cdots \| |d f^n|\|_{p_n, B}
$$

This completes the proof of the Theorem.

References


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