A Study of Some Incomplete Elliptic Integrals and Associated Definite Integrals Involving Single and Multivariable Polynomials and Functions

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Abstract

In this paper, we establish two new theorems involving incomplete elliptic integral and generalized zeta functions. Besides, the first theorem involves the product of general class of polynomial and the $H$-function and the second product of multivariable polynomial and multivariable $H$-function. Next, as an application of our theorems we obtain eight new and interesting finite integrals involving several special functions notably confluent form of Appell function, Miller-Ross function, Mittag-Leffer function, Lorenzo-Hartley R-function, reduced Green function, generalized Wright Bessel function. The integrals established in this paper may find applications in certain engineering problems.

Keywords: Incomplete elliptic integral, general class of polynomial, multivariable polynomial, the $H$-function and multivariable $H$-function

1. Introduction

Incomplete elliptic integral and associated definite integrals play an important part in several diverse physical problems. Motivated by this idea, we evaluate certain incomplete integrals and associated general definite integrals in this paper. Bushell [3] gave a generalization of Barton’s integral in terms of the case when $\gamma \geq 0$ of the following unification of the complete elliptic integrals $K(k)$ and $E(k)$ of the first and second kind [3, p.2,eq.(2.2)],[11,p.1177, eq.(1.5) & (1.6)]:

$$H(k;\gamma) := \int_0^1 \frac{(1 - k^2t^2)^{\gamma - \frac{1}{2}}}{\sqrt{(1 - t^2)}} dt \quad (|k^2| < 1; \gamma \in C) \quad (1.1)$$

So that, obviously,
\[
H(k; 0) = K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \left( |k^2| < 1 \right) \quad (1.2)
\]

and

\[
H(k; 1) = E(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}} \, dt \left( |k^2| < 1 \right) \quad (1.3)
\]

Now we give definitions of functions and polynomials occurring in this paper.

The $H$-function occurring in the present paper was introduced by Inayat Hussain [10] and studied by Buschman and Srivastava [2] and others. It will be defined and represented in the following manner:

\[
\overline{H}_{P,Q}^{M,N}[z] = \overline{H}_{P,Q}^{M,N} \left( z \left| \begin{array}{c} (e_j, E_j)_1, N, (e_j, E_j)_{N+1}, P, (f_j, F_j)_1, M, (f_j, F_j; \ell_j)_{M+1}, Q \end{array} \right. \right) \quad (1.4)
\]

\[
= \frac{1}{2\pi i} \int_L \overline{\phi}(\xi) \, z^\xi d\xi \quad (z \neq 0)
\]

where

\[
\overline{\phi}(\xi) = \frac{\prod_{j=1}^M \Gamma(f_j - F_j \xi) \prod_{j=1}^N \{\Gamma(1 - e_j + E_j \xi)\} \prod_{j=M+1}^P \{\Gamma(1 - f_j + F_j \xi)\} \prod_{j=M+1}^P \{\Gamma(e_j - E_j \xi)\}}{\prod_{j=M+1}^Q \{\Gamma(1 - f_j + F_j \xi)\} \prod_{j=M+1}^P \{\Gamma(e_j - E_j \xi)\}} \quad (1.5)
\]

The nature of contour $L$ in (1.4) and various conditions on its parameters can be seen in the paper by Gupta, Jain and Agrawal [6].

The multivariable $H$-function used in the paper is defined and represented in the following form [16, p.251-252, eq. (C.1-C.3)]:

\[
H(z_1, \ldots, z_r) = H_{p,q; p_1,q_1; \ldots; p_r,q_r}^{0,n; m_1,n_1; \ldots; m_r,n_r} \left( \begin{array}{c} z_1 \\ \vdots \\ z_r \\ \end{array} \right| \begin{array}{c} (a_j^{(1)}, \ldots, a_j^{(r)})_{1,p}, (c_j^{(1)}, \mathbb{Z}_j^{(1)})_{1,p}; \ldots; (c_j^{(r)}, \mathbb{Z}_j^{(r)})_{1,p} \\ (b_j^{(1)}, \ldots, b_j^{(r)})_{1,q}, (d_j^{(1)}, \mathbb{Z}_j^{(1)})_{1,q}; \ldots; (d_j^{(r)}, \mathbb{Z}_j^{(r)})_{1,q} \end{array} \right) \quad (1.6)
\]

\[
= \frac{1}{(2\pi \omega)^r} \int_{L_1} \cdots \int_{L_r} \phi(\xi_1 \cdots \xi_r) \prod_{i=1}^r \left\{ \theta_i(\xi_i) z_i^{(\xi_i)} \right\} d\xi_1 \cdots d\xi_r \quad (1.6)
\]

where
\[ \omega = \sqrt{-1} \]

\[ \phi (\xi_1, \ldots, \xi_r) = \frac{\prod_{j=1}^{n} \Gamma (1 - a_j + \sum_{j=1}^{r} \alpha_j (\xi_i))}{\prod_{j=1}^{q} \Gamma (1 - b_j + \sum_{j=1}^{r} \beta_j (\xi_i)) \prod_{j=n+1}^{p} \Gamma (a_j - \sum_{j=1}^{r} \alpha_j (\xi_i))} (1.7) \]

\[ \theta_i (\xi_i) = \frac{\prod_{j=1}^{m_i} \Gamma (d_j (i) - \delta_j (i) \xi_i) \prod_{j=1}^{n_i} \Gamma (1 - c_j (i) + \gamma_j (i) \xi_i)}{\prod_{j=n_i+1}^{q_i} \Gamma (c_j (i) - \gamma_j (i) \xi_i) \prod_{j=1}^{p_i} \Gamma (1 - d_j (i) + \delta_j (i) \xi_i)} , \ (i = 1, 2, \ldots r) \ (1.8) \]

throughout this paper it is assumed that this function satisfies its appropriate conditions of existence and convergence [16, p. 252-253, eq. (C.4-C.6)].

The General class of polynomials, introduced by Srivastava [14] is given by

\[ S_U^V [x] = \sum_{R=0}^{V/U} (-V)_{U,R} A(V,R) \frac{x^R}{R!} , \ V = 0,1,2 \ldots \ (1.9) \]

where \( U \) is an arbitrary positive integer and the coefficient \( A(V,R) \) are arbitrary constant, real or complex. \( S_U^V [x] \) yields a number of known polynomials as its special cases. These include among others the Jacobi polynomials, the Laguerre polynomials, the Gould Hopper polynomials, the Hermite polynomials, Cesaro Polynomial, Konhauser Polynomial and several others [4,15].

The general multivariable Polynomial, \( S_{U_1, \ldots, U_k}^V [x_1, \ldots, x_k] \) introduced by Srivastava and Garg [15, p. 686, Eq. (1.4)] is defined in the following manner:

\[ S_{U_1, \ldots, U_k}^V [x_1, \ldots, x_k] = \sum_{R_1, \ldots, R_k=0}^{\sum_{i=1}^{k} U_i, R_i \leq V} (-V)_{\sum_{i=1}^{k} U_i, R_i} A(V,R_1,\ldots,R_k) \frac{x_1^{R_1}}{R_1!} \cdots x_k^{R_k} \frac{x_1^{R_k}}{R_k!} , \ (1.10) \]

where \( U_1, \ldots, U_k \) are arbitrary positive integers, \( V=0,1,2,\ldots \) and the coefficients \( A(V,R_1,\ldots,R_k) \) are arbitrary constants, real or complex.

2. Main Theorem

Theorem 1: If

\[ \varphi (z, \tau, \eta) = \sum_{a=0}^{\infty} \frac{z^a}{(\eta + a)!} , \ (|z| < 1) , \ (\eta \neq 0, -1, -2, \ldots) \ (2.1) \]

then

\[ \int_{0}^{1} k^\rho L^\sigma \varphi (zkL, \tau, \eta) S_S^U [zk^\rho L^\sigma \eta] H^M,N [z_1 (k^\rho L^\sigma)]^2 H (\zeta L; \gamma) dk \]
\[
\begin{aligned}
\int \frac{A(V, R)}{R!} \left( \zeta_0^R \frac{z^a}{\eta + a} \right) \frac{1}{(1 + r)^{1/2}} \zeta^{2r} \\
\int_{P+2,Q+1}^{M,N+2} z_1 \left[ \left( \frac{1 - \rho - \rho \zeta_0^R}{2}, \rho_1; 1 \right) , \left( -\frac{\sigma + \alpha + \rho + \rho \zeta_0^R}{2}, \sigma_1; 1 \right) , (e_j, E_j; e_j)_{1,N} , (e_j, E_j)_{N+1,P} \right] \\
(f_j, F_j)_{1,M} , \left( -\frac{\sigma + \alpha + \rho + \rho \zeta_0^R}{2}, \rho_1 + \sigma_1; 1 \right) , (f_j, F_j; e_j)_{M+1,Q} \\
\end{aligned}
\]

where

\[
L = \sqrt{1 - k^2}
\]

Provided that

\[
\text{Re} (\rho) + \min_{1 \leq j \leq M} \left\{ \text{Re} (2\rho_j) \left( \frac{f_j}{F_j} \right) \right\} + 1 > 0, \text{Re} \left( \frac{\sigma}{2} \right) + \min_{1 \leq j \leq M} \left\{ \text{Re} (\sigma_j) \left( \frac{f_j}{F_j} \right) \right\} + 1 > 0, |\zeta| < 1
\]

**Theorem 2**: If

\[
\varphi (z, \tau, \eta) = \sum_{a=0}^{\infty} \frac{z^a}{(\eta + a)^{\sigma}} \quad (|z| < 1), \quad (\eta \neq 0, -1, -2, \ldots)
\]

then

\[
\int_0^1 k^\rho L^\sigma \varphi (zk, \tau, \eta) S_{V,1,...,V_s}^U \left( z_0^{(1)} k^\rho_0^{(1)} L^{\sigma_0}^{(1)}, \ldots, z_s^{(s)} k^\rho_s^{(s)} L^{\sigma_0}^{(s)} \right) \\
H \left[ z_1 (k^\rho_1 L^{\sigma_1})^2, \ldots, z_r (k^\rho_r L^{\sigma_r})^2 \right] H (\zeta L; \gamma) \, dk
\]

\[
= \frac{\sqrt{\pi}}{4} \sum_{a=0}^{\infty} \sum_{R_1,...,R_s=0}^V (-V) \sum_{U_1 R_1,...,U_s R_s=0}^s \left( z_1 \frac{R_1}{R_1!} \right) \left( \frac{z^a}{(\eta + a)^{\sigma}} \right) \frac{1}{\Gamma \left( \frac{1}{2} - \gamma \right)}
\]

\[
H_{0,n+2,2}^{0,2+q+1,1;} \\
H_{p+2,q+1,2}^{p+2,2+1,1;} \\
\begin{bmatrix}
\zeta_1 \\
\vdots \\
\zeta_r \\
-\zeta^2
\end{bmatrix}
\]

\[
A^* : \left( \begin{array}{c}
(\zeta_1^{(1)}, \delta_j^{(1)})_{1,p_1} ; \\
\vdots \\
(\zeta_r^{(r)}, \delta_j^{(r)})_{1,p_r}
\end{array} \right) ; \left( \begin{array}{c}
(0,1) ; (0,1)
\end{array} \right)
\]

\[
B^* : \left( \begin{array}{c}
(\zeta_1^{(1)}, \delta_j^{(1)})_{1,q_1} ; \\
\vdots \\
(\zeta_r^{(r)}, \delta_j^{(r)})_{1,q_r}
\end{array} \right) ; \left( \begin{array}{c}
(0,1) ; (0,1)
\end{array} \right)
\]

where

\[
L = \sqrt{1 - k^2}
\]

\[
A^* = \left( -\frac{1}{2} \left[ \rho + a + \sum_{i=1}^s \rho_i^{(i)} R_i - 1 \right] ; \rho_1, \ldots, \rho_r, 0 \right) , \left( -\frac{1}{2} \left[ \sigma + \sum_{i=1}^s \sigma_i^{(i)} R_i \right] ; \sigma_1, \ldots, \sigma_r, 1 \right)
\]

\[
(a_j ; \alpha_1, \ldots, \alpha_r, 0)_{1,p}
\]
Provided that
\[ \frac{1}{2} \left[ \rho + \sigma + a + \sum_{i=1}^{s} \left( \rho_{0}^{(i)} + \sigma_{0}^{(i)} \right) R_{i} + 1 \right] ; \rho + \sigma_{1}, ..., \rho + \sigma_{r}, 1 \].

Proof: To prove the Theorem 1, we express \( \varphi(z, \tau, \eta) \), \( S_{V}^{U} [x] \) and \( \overline{\Pi} \)-function in terms of series and contour integral with the help of (2.1), (1.9) and (1.4) respectively. Further \( H(\zeta L; \gamma) \) express in terms of series of \( 2_{F}^{1} \) with the help of (1.1) & [9, p.286], now we interchange the order of summation and \( \xi, k^- \) integrals. We have the following form (say \( \Delta \)) for the left hand side of (2.2):
\[
\Delta = \sum_{a=0}^{\infty} \sum_{r=0}^{\infty} \sum_{R=0}^{\infty} (-V)_{UR}^R \frac{A(V, R)}{R!} (z_0)^R \frac{z^a}{(\eta + a)} \left( \frac{1}{2} - \gamma \right)_{x} \left( \frac{1}{2} \right)_{\gamma} (\zeta)^{2r} \times \int_{L} \mathcal{D}(\xi) (z_1)^{\xi} \int_{0}^{1} k^{\rho + \rho_0 R + a + 2\rho_1 \xi} \left( \sqrt{1 - k^2} \right)^{\sigma + \sigma_0 R + a + 2r + 2\sigma_1 \xi} \frac{z^a}{(\eta + a)}
\]
evaluating the above well known \( k \)-integral and reinterpret the result thus obtained in terms of \( \overline{\Pi} \)-function, we easily arrive at the Theorem 1, after a little simplification.

Proof of the Theorem 2 can be developed by proceeding on the lines similar to the proof of Theorem 1, by making use of (1.10) and (1.6).

3. Applications

FIRST INTEGRAL

Now if in the Theorem 1, we reduce \( S_{V}^{U} [x] \) to Gottlieb polynomial \( I_{V} \) [1, p.163, eq. (A.23)] and \( \overline{\Pi} \)-function to Generalized Wright Bessel function [7, p.271, eq. (8)], after a little simplification, we obtain the following integral valid under the conditions derived from the conditions stated with (2.2):

\[
\int_{0}^{1} k^\rho L^\sigma \varphi(zkL, \tau, \eta) (1 - k^{\rho_0})^{V} I_{V}(y; \log(1 - k^{\rho_0})) J_{V}^{\mu}(z_1 k^{2\rho_1}) H(\zeta L; \gamma) dk
\]
\[
= \frac{\pi}{4} \sum_{a=0}^{\infty} \sum_{r=0}^{\infty} \sum_{R=0}^{\infty} (-V)_{UR}^R \frac{(-y)_{R}}{R! R!} (\zeta)^{2r} \Gamma \left( 1 + r + \frac{\sigma + a}{2} \right) \frac{1}{r!} \frac{1}{\eta + a} \frac{z^a}{(\eta + a)}
\]

\( \overline{\Pi}_{1,3}^{1,1} \left[ \begin{array}{c}
\left. \frac{1}{2} \rho - a - \rho_0 R \right|_{0, 1}^{1, 1} \\
(0, 1) \left( \frac{1}{2} \rho + \sigma + 2a + \rho_0 R + 2r + 1, \rho_1 ; 1 \right)
\end{array} \right] \). (3.1)
SECOND INTEGRAL
Again we reduce $S_{V}^{U} [x]$ to Laguerre polynomial and $\mathcal{H}$-function to polylogarithm of order $p$ with the help of [1,p.158,eq.(A.8)] and [4,p.30,eq.(14); 5,p.315,eq.(1.9)] respectively in the Theorem 1, we can easily arrive at the following interesting integral, after a little simplification:

$$
\int_{0}^{1} k^\rho L^\sigma \varphi (zkL, \tau, \eta) L_{V}^{(\alpha)} (k^{\rho_0}) \ F (z_1 k, \omega) \ H (\zeta L; \gamma) \ dk = -\frac{\pi}{4} \sum_{a=0}^{\infty} \sum_{r=0}^{\infty} \sum_{R=0}^{\infty} (-V)_R \left( \frac{V + \alpha}{V} \right) R! \left( \frac{1}{(\alpha + 1)_R} \right) \frac{z^a}{(\eta + a) R!} \left( (\zeta)^{2r} \left( \frac{1}{2} - \gamma \right) \right)
$$

$$
\Gamma \left( 1 + r + \frac{\sigma + a}{2} \right) \mathcal{H}^{1,2}_{2,3} \left[ -z_1, (1,1; \omega + 1), R \left( 1, 1 \right), \left( \frac{1-\rho-a-\rho_0 R}{2}, 1, 1 \right), R \left( 1, 0, 1; \omega \right), R \left( 1, 1, 1 \right) \right]
$$

provided that the conditions easily obtainable from the existing conditions of (2.2) are satisfied.

THIRD INTEGRAL
On reducing the multivariable polynomial to general class of polynomials and the multivariable H-function to Fox’s H-function in the Theorem 2, we obtain the following integral valid under the conditions derived from the conditions stated with (2.4):

$$
\int_{0}^{1} k^\rho L^\sigma \varphi (zk, \tau, \eta) S_{V}^{U} (z_0 k^{\rho_0} L^{\sigma_0}) \ H \left[ z_1 (k^{\rho_1} L^{\sigma_1})^2 \right] \ H (\zeta L; \gamma) \ dk = \frac{\sqrt{\pi}}{4} \sum_{a=0}^{\infty} \sum_{R=0}^{(V/U)} (-V)_{UR} \left( \frac{A(V, R)}{R!} \right) \left( z_0 \right)^{R} \frac{z^a}{(\eta + a) R!} \left( (\frac{1}{2} - \gamma) \right)
$$

$$
\mathcal{H}^{0,1;m,n+1;1,2}_{1,1;\rho+1,q;2,2} \left[ z_1, \left( -\frac{\sigma+\sigma_0}{2}; \sigma_1, 1 \right); \left( \frac{1-\rho-a-\rho_0 R}{2}, \rho_1 \right), (a_j, \alpha_j)_{1,p}; \left( \frac{1}{2} + \gamma, 1 \right), \left( \frac{1}{2}, 1 \right) \right]
$$

$$
\left( -\frac{\rho+\sigma+a+(\rho_0+\sigma_0) R+1}{2}; \rho_1 + \sigma_1, 1 \right); \left( b_j, \beta_j \right)_{1,q}; (0, 1), (0, 1) \right]
$$

FOURTH INTEGRAL
On specializing the $S_{V}^{U_1,\ldots,U_s}$ to multivariable Bessel polynomial [1,p.167,eq(A.31)] and multivariable H-function to confluent form of Appell function [16,p.89,eq.(6.4.8)] in the Theorem 2, it takes the following interesting integral, after a little simplification:
Provided that the conditions easily obtainable from the existing conditions of (2.4) are satisfied.

Fifth integral

In the theorem 2, if we reduce $S_{V_1, \ldots, U_s}$ to multivariable Jacobi polynomial [1,p.166.eq.(A.29)], multivariable H-function to Miller-Ross Function [12,p.eq.(101);13,p.314.eq.(3.1)] and $H(\zeta L; \gamma)$ to $K(\zeta L)[11,p.1177, eq. (3.5)]$, we arrive at the following integral, after a little simplification:

\[
\int_0^1 k^\theta L^\alpha \varphi(zk, \tau, \eta) \mathcal{Y}_{V_{\alpha_1, \ldots, \alpha_s}}^{(2\theta_1, \ldots, 2\theta_s)} (-2k^{\theta_1} L_0^{(1)}, \ldots, -2k^{\theta_s} L_0^{(s)}) \phi(d, c ; b; z_1 L_{\alpha_1}, z_2 L_{\alpha_2}) H(\zeta L; \gamma) \, dk
\]

\[
= \frac{\pi}{4} \sum_{a=0}^\infty \sum_{R_1, \ldots, R_s=0}^\infty (-V) \sum_{i=1}^{R_i} \frac{(1 + \alpha_1 + V)_{R_i}}{R_i!} \prod_{i=2}^s \frac{(1 + \alpha_i + n_i)_{R_i}}{R_i!} \left(\frac{z^a}{(\eta + a)^\gamma}\right) \Gamma^{1/2} \left(\rho + a + \sum_{i=1}^s \rho_0^{(i)} R_i + 2\right)
\]

\[
\frac{\Gamma^{1/2} \left(\rho + \sigma + a + \sum_{i=1}^s \rho_0^{(i)} R_i + 3\right)}{\Gamma^{1/2} \left(\rho + \sigma + a + \sum_{i=1}^s \rho_0^{(i)} R_i + 2\right)} F_{2;0;1}^{2;1;2} \left(\left[\frac{z_1}{\zeta^2} A^{**} : (c, 1) ; \sigma, 1; 1, 0\right], \frac{1}{2} \left[\rho + \sigma + a + \sum_{i=1}^s \rho_0^{(i)} R_i + 3\right] ; \sigma, 1, 1, 0\right)
\]

provided that the conditions easily obtainable from the existing conditions of (2.4) are satisfied.

Sixth integral

Incomplete elliptic integrals

\[
\int_0^1 k^\theta L^\alpha \varphi(zk, \tau, \eta) \mathcal{Y}_{V_{\alpha_1, \ldots, \alpha_s}}^{(2\theta_1, \ldots, 2\theta_s)} (-2k^{\theta_1} L_0^{(1)}, \ldots, -2k^{\theta_s} L_0^{(s)}) \phi(d, c ; b; z_1 L_{\alpha_1}, z_2 L_{\alpha_2}) H(\zeta L; \gamma) \, dk
\]

\[
= \frac{\pi}{4} \sum_{a=0}^\infty \sum_{R_1, \ldots, R_s=0}^\infty (-V) \sum_{i=1}^{R_i} \frac{(1 + \alpha_1 + V)_{R_i}}{R_i!} \prod_{i=2}^s \frac{(1 + \alpha_i + n_i)_{R_i}}{R_i!} \left(\frac{z^a}{(\eta + a)^\gamma}\right) \Gamma^{1/2} \left(\rho + a + \sum_{i=1}^s \rho_0^{(i)} R_i + 2\right)
\]

\[
\frac{\Gamma^{1/2} \left(\rho + \sigma + a + \sum_{i=1}^s \rho_0^{(i)} R_i + 3\right)}{\Gamma^{1/2} \left(\rho + \sigma + a + \sum_{i=1}^s \rho_0^{(i)} R_i + 2\right)} F_{2;0;1}^{2;1;2} \left(\left[\frac{z_1}{\zeta^2} A^{**} : (c, 1) ; \sigma, 1; 1, 0\right], \frac{1}{2} \left[\rho + \sigma + a + \sum_{i=1}^s \rho_0^{(i)} R_i + 3\right] ; \sigma, 1, 1, 0\right)
\]

provided that the conditions easily obtainable from the existing conditions of (2.4) are satisfied.
Again, by taking \( s = 1 \) and reducing \( S_{V_{1}, \ldots, V_{s}}^{U_{1}, \ldots, U_{s}} \) to Bateman polynomial [1, p.161, eq. (A.16)], multivariable H-function to Mittag-Leffler function [8, p.161, eq. (12)] and \( H(\zeta \mathbf{L}; \gamma) \) to \( E(\zeta \mathbf{L}) \) [11, p.1177, eq. (1.6)] in the Theorem 2, we obtain the following interesting integral valid under the conditions derived from the conditions stated with (2.4):

\[
\int_{0}^{1} k^{\rho} L^{\sigma}_{v} \varphi(zk, \tau, \eta) \quad Z_{V}(k^{\rho_{0}}) \quad E_{x, y}(z_{1} k^{2\rho_{1}}) \quad E(\zeta \mathbf{L}) \quad dk
\]

\[
= \frac{\pi}{4} \sum_{a=0}^{\infty} \left[ (\eta + a)^{-1} \frac{\Gamma(\rho + a + \rho_{0} R + 1) \Gamma(1 + \frac{\sigma}{2})}{\Gamma(\rho + \sigma + a + \rho_{0} R + 3) \Gamma(\mu)} \right] \sum_{R=0}^{\infty} (-V)^{R} \frac{(1 + V)^{R}}{(R!)^{3}} \frac{z^{a}}{(\eta + a)}
\]

\[
to \quad F_{1:1:1}^{0:2:3} \quad \left\{ \begin{array}{c}
\frac{z_{1}}{\zeta^{2}} \\
\sigma_{0}^{(i)} R_{i} \\
\end{array} \right\} \quad \left\{ \begin{array}{c}
(\rho + a + \rho_{0} R + 1, \rho_{1}) \\
(0, 1); (-\frac{1}{2}, 1); (\frac{1}{2}, 1); (1 + \frac{\sigma}{2}, 1) \\
\end{array} \right\} \quad ; \quad (1, 1)
\]

SEVENTH INTEGRAL

If, we reduce \( S_{V_{1}, \ldots, V_{s}}^{U_{1}, \ldots, U_{s}} \) to multivariable Hermit polynomial [1, p.168, eq. (A.33)] and multivariable H-function to Lorenzo-Hartley R-function [8, p.162, eq. (14)] in the Theorem 2, we obtain the following integral which is believed to be new, after a little simplification:

\[
\int_{0}^{1} k^{\rho} L^{\sigma_{-\sigma_{1}+\nu+1+\frac{s}{2}\sigma_{0}^{(i)}}} \varphi(zk, \tau, \eta) \quad H_{V}(X_{1}, \ldots, X_{s}) \quad R_{\sigma_{1}, \nu}(z_{1}, L^{\sigma_{1}}) \quad H(\zeta \mathbf{L}; \gamma) \quad dk
\]

\[
= \frac{\pi}{4} \sum_{a=0}^{\infty} \sum_{R_{1}, \ldots, R_{s}=0}^{2R_{1}\leq V} (-V)^{s} \sum_{R_{1}, \ldots, R_{s}=0}^{2R_{1}\leq V} (-1)^{R_{1}+\cdots+R_{s}} \prod_{i=1}^{s} \left( \frac{1}{R_{i}} \right) \quad \frac{z^{a}}{(\eta + a)} \quad \frac{\Gamma(\rho + a + \rho_{0} R_{i} + 1) \Gamma(1 + \frac{\sigma}{2})}{\Gamma(\rho + \sigma + a + \rho_{0} R_{i} + 3) \Gamma(1 + \rho_{1})}
\]

\[
to \quad F_{1:1:1}^{1:1:1} \quad \left\{ \begin{array}{c}
\frac{z_{1}}{\zeta^{2}} \\
\sigma_{0}^{(i)} R_{i} \\
\end{array} \right\} \quad \left\{ \begin{array}{c}
(\frac{1}{2} \left( \sigma + \sum_{i=1}^{s} \sigma_{0}^{(i)} R_{i} \right) + 1; \frac{\sigma}{2}, 1) \\
(0, 1); (-\frac{\sigma}{2} - \gamma, 1); (\frac{1}{2}, 1) \\
\end{array} \right\} \quad ; \quad (1, 1)
\]

provided that the conditions easily obtainable from the existing conditions of (2.4) are satisfied. where

\[
X_{1} = \frac{1}{2 \sqrt{\left( L^{\sigma_{0}^{(i)}} \right)}}
\]

\[
X_{j} = \frac{L^{\sigma_{0}^{(j)}}}{L^{\sigma_{0}^{(j)}}} \quad (j = 2, \ldots, s)
\]
EIGHTH INTEGRAL

In the Theorem 2, multivariable polynomial reduce to unity, multivariable H-function reduce to Reduced Green- function \[8,p.13,eq.(13)\] and \( H(\zeta L; \gamma) \) to \( K(\zeta L) \) \[11,p.1177,eq.(1.5)\], we obtain a useful integral after a little simplification:

\[
\int_0^1 k^{\rho+1} L^{\sigma+1} \phi(zk, \tau, \eta) \ K_{\alpha, \beta}^\theta(z_1kL) K(\zeta L) \, dk
\]

\[
= \frac{1}{4} \sum_{a=0}^{\infty} \frac{z^a}{(a+a)^\gamma} \frac{1}{\alpha z_1}
\]

\[
\begin{bmatrix}
\begin{array}{c}
\zeta^1 \\
-\zeta^2
\end{array}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
(-\frac{\rho}{2}, 1, 1) \\
(-\frac{1}{2} (\rho + a - 1), 1, \omega)
\end{array}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
(1, \frac{\beta}{\alpha}) \\
(1, 1, 1)
\end{array}
\end{bmatrix}
\]

\[
H^{0,1;2,2;1,2}_{1,1;4,3;2,2} ( \begin{array}{c}
\zeta \\
1
\end{array} ) \begin{bmatrix}
\begin{array}{c}
(-\frac{1}{2} (\rho + a + 1), 2, 1) \\
(1, \frac{\beta}{\alpha})
\end{array}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
(1, 1, 1, 1) \\
(0, 1, 0)
\end{array}
\end{bmatrix}
\]

\]

provided that the conditions easily obtainable from the existing conditions of (2.4) are satisfied where \( \omega = \frac{a-\theta}{2a} \).

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References


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