Fractional Integrals Involving General Polynomials, 
\( \tilde{H} \)-Function and Multivariable \( I \)-Function

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Abstract

The aim of this paper is to find a Eulerian integral and a main theorem based on the fractional operator associated with \( \tilde{H} \)-Function [3], general class of polynomial [13] and multivariable \( I \)-Function [4] having general arguments. The special cases of the main theorem (which are also sufficiently general in nature and are of interest in themselves) have also been given.

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1 Introduction

The Riemann-Liouville operator of fractional integration \( R^m f \) of order \( m \) is defined by

\[
_xD_y^{-m}[f(y)] = \frac{1}{\Gamma(m)} \int_x^y (y-t)^{m-1} f(t)dt
\]

(1.1)

for \( \text{Re}(m) > 0 \) and a constant \( x \).

An equivalent form of Beta function is [2, p. 10, eq. (13)]

\[
\int_m^n (t - m)^{a-1}(n - t)^{b-1}dt = (n - m)^{a+b-1}\text{B}(a, b)
\]

(1.2)

where \( m, n \in \mathbb{R}(x < y), \text{Re}(a) > 0, \text{Re}(b) > 0 \).
Making use of [2, p. 62, eq. (15)], we have

\[(pt + q)^\alpha = (xp + q)^\alpha \left[ 1 + \frac{p(t - x)}{xp + q} \right]^\alpha \]

\[= \frac{(xp + q)^\alpha}{\Gamma(-\alpha)} \frac{1}{(2\pi i)} \int_{-\infty}^{+i\infty} \Gamma(-\beta) \Gamma(\beta - \alpha) \left[ \frac{p(t - x)}{xp + q} \right]^\beta d\beta, \quad (1.3)\]

where \(i = \sqrt{-1}; \ p, q, \alpha \in C; \ x, t \in R; \ \arg \left( \frac{p}{xp + q} \right) < \pi \) and the path of integration is necessary in such a manner so as to separate the poles of \(\Gamma(-\beta)\) from those of \(\Gamma(\beta - \alpha)\).

The multivariable \(I\)-function is defined and represented in the following manner [4]:

\[I[z_1, \ldots, z_r] = I_{p_1,q_1,p_2,q_2,\ldots,p_r,q_r}^{0,n_1,0,n_2,\ldots,0,n_r,\cdots,m_1,n_1',\cdots,n_r,n_r'}(z_1, \ldots, z_r, \xi_1, \ldots, \xi_r) \]

\[= \frac{1}{(2\pi i)^r} \int_{L_1} \cdots \int_{L_r} \varphi_1(\xi_1) \cdots \varphi_r(\xi_r) \psi(\xi_1, \ldots, \xi_r) z_1^{\xi_1} \cdots z_r^{\xi_r} d\xi_1 \cdots d\xi_r \]

where \(i = \sqrt{-1}\).

For convergence conditions and other details of multivariable \(I\)-function, see Prasad [4]. The Lauricella function \(F_D^{(h)}\) is defined in the following integral from

\[\frac{\Gamma(a) \Gamma(b_1) \cdots \Gamma(b_h)}{\Gamma(c)} F_D^{(h)}[a, b_1, \ldots, b_h; c, x_1, \ldots, x_h] \]

\[= \frac{1}{(2\pi i)^h} \int_{-\infty}^{+i\infty} \cdots \int_{-\infty}^{+i\infty} \Gamma(a + \xi_1 + \cdots + \xi_h) \Gamma(b_1 + \xi_1) \cdots \Gamma(b_h + \xi_h) \]

\[\cdot \Gamma(-\xi_1) \cdots \Gamma(-\xi_h)(-x_1)^{\xi_1} \cdots (-x_h)^{\xi_r} d\xi_1 \cdots d\xi_h \quad (1.5)\]

where \(\max(|\arg(-x_1)|, \ldots, |\arg(-x_h)|) < \pi; \ c = 0, -1, -2, \ldots \) .
To prove the Eulerian integrals, we use the following formula:

\[
\int_{x}^{y} (t - x)^{a-1} (y - t)^{b-1} (p_1 t + q_1)^{\rho_1} \ldots (p_h t + q_h)^{\rho_h} dt
\]

\[
= (y - x)^{a+b-1} B(a, b)(p_1 x + q_1)^{\rho_1} \ldots (p_h x + q_h)^{\rho_h}
\]

\[
F_{D}^{(h)} \left[ a, -\rho_1, \ldots, -\rho_h; a + b; \frac{(y - x)p_1}{p_1 x + q_1}, \ldots, \frac{(y - x)p_h}{p_h x + q_h} \right] \tag{1.6}
\]

where \(x, y \in \mathbb{R}(x, y); p_j, q_j, \rho_j \in \mathbb{C} (j = 1, \ldots, h);\)

\[
\min[\text{Re}(m), \text{Re}(n)] > 0 \quad \text{and} \quad \max\left[ \left| \frac{(y - x)p_1}{p_1 x + q_1} \right|, \ldots, \left| \frac{(y - x)p_h}{p_h x + q_h} \right| \right] < 1.
\]

Making use of the results (1.2), (1.3) and (1.5), we can prove the formula given in (1.6). For \(h = 1\) and \(h = 2\), we get the known results [5, p. 301 entry (2.2.6.1)] and [11, p. 81, eq. (3.6)], respectively.

In what follows \(h\) is a positive integer and 0, \ldots, 0 would mean \(h\) zero.

The series representation of \(\tilde{H}\)-function [3] is as follows

\[
\tilde{H}_{P,Q}^{M,N}[z] = \tilde{H}_{P,Q}^{M,N} \left[ z \right| \left( e_j, E_j; \alpha_j \right)_{1,N}, (e_j, E_j)_{N+1,P} \left( f_j, F_j; f_j, F_j; \beta_j \right)_{M+1,Q} \right]
\]

\[
= \sum_{g=1}^{M} \sum_{k=0}^{\infty} \frac{(-1)^k \phi(\eta_{g,k})}{k! F_g} z^{\eta_{g,k}} \tag{1.7}
\]

where

\[
\prod_{j=1}^{M} \Gamma(f_j - F_j \eta_{g,k}) \prod_{j=1}^{N} \left\{ \Gamma(1 - e_j + E_j \eta_{g,k}) \right\}^{\alpha_j}
\]

\[
\prod_{j=M+1}^{P} \Gamma(1 - e_j + E_j \eta_{g,k}) \prod_{j=N+1}^{Q} \Gamma(1 - e_j + E_j \eta_{g,k})
\]

and \(\eta_{g,k} = \frac{f_g + k}{F_g}\).

For convergence conditions and other details of the \(\tilde{H}\)-function see Inayat-Hussain [3].

Srivastava [13, p. 75, eq. (1.1)] introduced the general class of polynomials defined in the following manner:

\[
S_{N}^{M}[x] = \sum_{k=0}^{[N/M]} \frac{(-1)^N M_h}{k!} A_{N,k} x^k, \quad N = 0, 1, 2, \ldots \tag{1.8}
\]

where \(M\) is an arbitrary positive integer and the coefficients \(A_{N,k} (N, k > 0)\) are arbitrary constants, real or complex.
# 2 Main Integral

The main integral to be established here is

\[
\int_{m}^{n} (t - m)^{a-1}(n - t)^{b-1} \left\{ \prod_{j=1}^{h} (p_j t + q_j)^{\sigma_j} \right\}
\]

\[
H_{P,Q}^{M,N} \left[ x(t - m)^{\lambda}(n - t)^{\mu} \prod_{j=1}^{h} (p_j t + q_j)^{\sigma_j} \right]
\]

\cdot S_{N'}^{M'} \left[ z(t - m)^{c}(n - t)^{d} \prod_{j=1}^{h} (p_j t + q_j)^{\nu_j} \right]

\cdot I

= G_1 \sum_{g=0}^{M} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^k \varphi(\eta_{g,k}) (-N')_{M'+s}}{k! F_g \, F_s} A_{N',s} x^{\eta_{g,k}} z^s G_2 \cdot G_3

\left[ \begin{array}{l}
R_1 \quad A_1, A_2, A_3, \ldots; (a_{2j}; \alpha_{2j}, \alpha_{2j}', \alpha_{2j}'', \alpha_{2j}''')_{1,p_2}; (a_{3j}; \alpha_{3j}, \alpha_{3j}', \alpha_{3j}'', \alpha_{3j}''')_{1,p_3}; \ldots;
R_2 \quad (b_{2j}; \beta_{2j}, \beta_{2j}', \beta_{2j}'', \beta_{2j}''')_{1,q_2}; (b_{3j}; \beta_{3j}, \beta_{3j}', \beta_{3j}'', \beta_{3j}''')_{1,q_3}; \ldots;
\end{array} \right]

(\alpha_{rj}; \alpha_{rj}'', \alpha_{rj}''', \ldots; 0, \ldots, 0)_{1,p_r}; (a_{j}; \alpha_{j}, \alpha_{j}', \alpha_{j}'', \alpha_{j}''')_{1,p_r}; \ldots;

(\lambda; \mu; \sigma_j, c, d) \in R^+, \rho_j \in R,

(\sigma_j, c, d) \in R^+, \rho_j \in R,

\text{Re} \left[ a + \lambda \frac{f_j}{F_j} + \sum_{i=1}^{r} \frac{b_i^{(i)}}{\beta_j^{(i)}} \right] > 0 \quad (j = 1, \ldots, m^{(i)}),

The following are the conditions of the validity of (2.1):

(i) \( m, n \in R(m < n); \gamma_i; \tau_i; c^{(i)}_j, \gamma_j, \lambda, \mu, \sigma_j, c, d \in R^+, \rho_j \in R, \)

\( p_j, q_j \in R, \quad z_i \in R \quad (i = 1, \ldots, r; j = 1, \ldots, h) \);

(ii) \[ \max_{1 \leq j \leq h} \left\{ \left| \frac{(n - m)p_j}{p_j m + q_j} \right| \right\} < 1 \]

(iii) \[ \text{Re} \left[ a + \lambda \frac{f_j}{F_j} + \sum_{i=1}^{r} \gamma_i \frac{b_i^{(i)}}{\beta_j^{(i)}} \right] > 0 \quad (j = 1, \ldots, m^{(i)}) \]
Fractional integrals involving general polynomials

\[
\text{Re} \left[ b + \mu \frac{f_j}{F_j} + \sum_{i=1}^{r} \tau_i \frac{h_j^{(i)}}{\beta_j^{(i)}} \right] > 0 \quad (j = 1, \ldots, m^{(i)}),
\]

(iv) \[ \left| \arg(z_i) \frac{h}{j=1} (p_j t + q_j)^{-c_j} \right| < \frac{T_i \pi}{2} \quad (m \leq t \leq n; \ i = 1, \ldots, r), \]

where

\[
T_i = \sum_{j=1}^{n^{(i)}} \alpha_j^{(i)} - \sum_{j=n^{(i)}+1}^{p^{(i)}} + \sum_{j=1}^{m^{(i)}} \beta_j^{(i)} - \sum_{j=m^{(i)}+1}^{q^{(i)}} + \sum_{j=n^{(i)}+1}^{p^{(i)}} \alpha_j^{(i)} - \sum_{j=n^{(i)}+1}^{p^{(i)}} + \sum_{j=m^{(i)}+1}^{q^{(i)}} + \ldots
\]

Here

\[
G_1 = (n-m)^{a+b-1} \left\{ \frac{h}{j=1} (p_j m + q_j)^{\rho_j} \right\},
\]

\[
G_2 = (n-m)^{(\lambda+p)\eta_{g,k}} \left\{ \frac{h}{j=1} (p_j m + q_j)^{\sigma_j \eta_{g,k}} \right\},
\]

\[
G_3 = (n-m)^{(c+d)s} \left\{ \frac{h}{j=1} (p_j m + q_j)^{\nu_j s} \right\}
\]

\[ A_1 = (1-a-\lambda \eta_{g,k} - cs) : \gamma_1, \ldots, \gamma_r, 1, \ldots, 1), \]

\[ A_2 = (1-b-\mu \eta_{g,k} - ds) : \tau_1, \ldots, \tau_r, 0, \ldots, 0), \]

\[ A_3 = (1+\rho_j + \sigma_j \eta_{g,k} + \nu_j s) : c'_j, \ldots, c'_j, 0, 1, \ldots, 0)_{1,h} \]

\[ A_4 = (1+\rho_j + \sigma_j \eta_{g,k} + \nu_j s) : c'_j, \ldots, c'_j, 0, 1, \ldots, 0)_{1,h} \]

\[ A_5 = [1-a-b-(\lambda+\mu)\eta_{g,k}-(c+d)s) : (\gamma_1+\tau_1), \ldots, (\gamma_r+\tau_r), 1, \ldots, 1], \]

\[
R_1 = \begin{cases} z_1(n-m)^{c_1+c_1} & \prod_{j=1}^{h} (p_j m + q_j)^{c_j} \\ \vdots \\ z_r(n-m)^{c_r+c_r} & \prod_{j=1}^{h} (p_j m + q_j)^{c_j} \end{cases}
\]
and

\[ R_2 = \begin{cases} 
(n - m)p_1 / (p_1 m + q_1) \\
\vdots \\
(n - m)p_h / (p_h m + q_h) 
\end{cases} \]

**PROOF** In order to prove (2.1), expand the multivariable \( I \)-function in terms of Mellin-Barnes type of contour integral by (1.4) and \( \bar{H} \)-function and general class of polynomials given by (1.7) and (1.8), respectively. Now interchange the order of summation and integration (which is permissible under the conditions of validity stated above). Making use of the results in (1.3), (1.5) and (1.6), we get the desired result.

### 3 Main Theorem

Let

\[ f(t) = (t - m)^{\alpha - 1} \left\{ \prod_{j=1}^{h} (p_j t + q_j)^{\rho_j} \right\} \]

\[ \cdot \bar{H}^{M,N}_{P,Q} \left[ X(t - m)^{\lambda} \prod_{j=1}^{h} (p_j t + q_j)^{\sigma_j} \right] \]

\[ \cdot S^{M'}_{N'} \left[ Z(t - m)^{c} \prod_{j=1}^{h} (p_j t + q_j)^{\epsilon_j} \right] \]

\[ \cdot I \]

\[ W_1(t - m)^{\gamma_1} \prod_{j=1}^{h} (p_j t + q_j)^{-\epsilon_j} \]

\[ \vdots \]

\[ W_r(t - m)^{\gamma_r} \prod_{j=1}^{h} (p_j t + q_j)^{-\epsilon_j} \]

then

\[ mD_y^{-h}(f(y)) = I_1 \sum_{y=0}^{M} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^k \varphi(\eta_y,k)}{k!} \frac{(-N')_{M,s}}{s!} A_{N',s} X^{\eta_y,k} Z^s \cdot I_2 \cdot I_3 \]

\[ Y_1 \begin{bmatrix} D_1, D_2, (a_{2j}, \alpha'_{2j}, \alpha''_{2j}, \alpha'''_{2j})_{1,p_2}; (a_{3j}, \alpha'_{3j}, \alpha''_{3j}, \alpha'''_{3j})_{1,p_3}; \ldots; \\
Y_2 \end{bmatrix} \]

\[ (b_{2j}, \beta'_{2j}, \beta''_{2j})_{1,q_2}; (b_{3j}, \beta'_{3j}, \beta''_{3j})_{1,q_3}; \ldots; (b_{rj}, \beta'_{rj}, \ldots, \beta''_{rj})_{0, q_r}; \]
valid under the same conditions as needed for main integral (2.1), where

\[
I_1 = (y - m)^{a + b - 1} \left\{ \prod_{j=1}^{h} (p_j m + q_j)^{\alpha_j} \right\},
\]

\[
I_2 = (y - m)^{\lambda_{9,k}} \left\{ \prod_{j=1}^{h} (p_j m + q_j)^{\sigma_j \eta_{9,k}} \right\},
\]

\[
I_3 = (y - m)^{cs} \left\{ \prod_{j=1}^{h} (p_j m + q_j)^{\nu_j s} \right\}
\]

\[
D_1 = [1 - a - \lambda \eta_{9,k} - cs : \gamma_1, \ldots, \gamma_r, 1, \ldots, 1],
\]

\[
D_2 = [1 + \rho_j + \sigma_j \eta_{9,k} + \nu_j s : c'_j, \ldots, c_j^{(r)}, 0, \ldots, 1, \ldots, 0]_{1,h}
\]

\[
D_3 = [1 + \rho_j + \sigma_j \eta_{9,k} + \nu_j s : c'_j, \ldots, c_j^{(r)}, 0, \ldots, 0]_{1,h}
\]

\[
D_4 = [1 - a - b - \lambda \eta_{9,k} - cs : \gamma_1, \ldots, \gamma_r, 1, \ldots, 1].
\]

\[
Y_1 = \begin{cases}
W_1(y - m)^{\gamma_1} / \prod_{j=1}^{h} (p_j m + q_j)^{c_j}
\end{cases}
\]

\[
\vdots
\]

\[
W_r(y - m)^{\gamma_r} / \prod_{j=1}^{h} (p_j m + q_j)^{c_j^{(r)}}
\]

and

\[
Y_2 = \begin{cases}
(y - m)p_1 / (p_1 m + q_1)
\end{cases}
\]

\[
\vdots
\]

\[
(y - m)p_h / (p_h m + q_h)
\]

4 Special Cases

(i) If we put \( M' = 1, z = 1 \) and

\[
A_{N', s} = \frac{\alpha + 1)_{N'} (\alpha + \beta + N' + 1)_s}{\alpha + 1)_s N'}
\]

in (3.1),
We get the following corollary involving Jacobi Polynomial [14, p. 677, eq. (4.1)]

**Corollary** Let

\[
f(t) = (t - m)^{a-1}\left\{ \prod_{j=1}^{h} (p_j t + q_j)^{\alpha_j} \right\}
\]

\[
\cdot \bar{H}_{P,Q}^{M,N} \left[ X(t - m)^{\gamma} \prod_{j=1}^{h} (p_j t + q_j)^{\alpha_j} \right]
\]

\[
\cdot P_{N'}^{(\alpha,\beta)} \left[ 1 - 2(t - m)^c \prod_{j=1}^{h} (p_j t + q_j)^{\mu_j} \right]
\]

\[
\cdot I \left[ \begin{array}{c}
W_1(t - m)^{\gamma_1} \prod_{j=1}^{h} (p_j t + q_j)^{-\gamma_j} \\
\vdots \\
W_r(t - m)^{\gamma_r} \prod_{j=1}^{h} (p_j t + q_j)^{-\gamma_j(r)}
\end{array} \right]
\]

then

\[
mD_y^{-b}(f(y))
\]

\[
= I_1 \sum_{g=0}^{M} \sum_{k=0}^{\infty} \sum_{s=0}^{[N']} \frac{(-1)^k \varphi(\eta_{g,k})}{k! F_g} \frac{(-N')_{M's}}{s!} \frac{(\alpha + 1)_{N'}(\alpha + \beta + N' + 1)_s}{(\alpha + 1)_s N'} \cdot X^{\eta_{g,k}} \cdot I_2 \cdot I_3
\]

which holds true under the same conditions as given in (3.1) and where

\[
I_1, I_2, I_3, D_1, D_2, D_3, D_4, Y_1 \text{ and } Y_2 \text{ are the same as in (3.1)}.
\]
(ii) Take $N' = 0$ in (3.1), we get the main theorem obtained by Chaurasia and Kumar [12].

(iii) If we set $n_2 = n_3 = \cdots = n_{r-1} = 0 = p_2 = p_3 = \cdots = p_{r-1} = q_1 = q_2 = \cdots = q_{r-1,N'} = 0$ and $\lambda = 0$, $\mu = 0$, $\sigma_j \to 0$, the results given in (2.1) and (3.1) reduces to the known results obtained by Saigo and Saxena [6].

(iv) For $n_2 = n_3 = \cdots = n_{r-1} = 0 = p_2 = \cdots = p_{r-1}$, $q_1 = q_2 = \cdots = q_{r-1,N'} = 0$ and $\lambda = 0$, $\mu = 0$, $\sigma_j \to 0$, $\gamma_i = 0$ ($i = 1, \ldots, r$) and $h = 1$ in (3.1), then we arrive at the results given by Srivastava and Hussain [11].

(v) On specializing the parameters, we get the results obtained by Chaurasia and Singhal [1].

References


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