Common Fixed Point Theorems for Self Maps of a Generalized Metric Space Satisfying A-Contraction Type Condition

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Abstract

We introduced a general class of contraction maps on a metric space, called $A$-contractions (that includes the contractions originally studied by R. Kannan, M. S. Khan at el, R. Bianchini, and S. Reich), and extended some common fixed point theorems on M. S. Khan’s contractions to general self maps of a metric space satisfying certain $A$-contraction type condition; in this paper, we prove exact analogues of these results in the setting of generalized metric spaces (defined originally by A. Branciari).

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1 Introduction

A general class of metric spaces was introduced by A. Branciari [5]. Various fixed point theorems for self mappings on such generalized metric spaces have been obtained (see [1, 5, 7], for instance). The present authors introduced the class of $A$-contractions [3] that includes classes of contractions originally studied
by R. Kannan [8], M. S. Khan at el [9, 10], R. Bianchini [4] and S. Reich [12] (see [3], for details). In [2], we extended some common fixed point theorems (appeared in [6,11]) from M.S. Khan’s contractions to general self maps of a metric space satisfying certain \( A \)-contraction type condition, namely, the condition \( A \) (see below); in the sequel, a self map of a generalized metric space satisfying the condition \( A \) is called an \( A \)-contraction. We shall prove exact analogues of our above mentioned results for \( A \)-contractions of a generalized metric space.

2 Generalized metric spaces (gms)

We begin with the following definition of generalized metric space as introduced originally by Branciari [5]. Throughout the sequel, \( N \) and \( R^+ \) will symbolize the set of all natural numbers and the set of all non-negative real numbers, respectively.

**Definition:** Let \( X \) be a nonempty set, \( d: X^2 \to R^+ \) a mapping such that for all \( x, y \in X \) and for all distinct points \( z, w \in X-\{x,y\} \), each of them different from \( x, y \) one has

(i) \( d(x, y) = 0 \) if and only if \( x = y \),
(ii) \( d(x, y) = d(y, x) \),
(iii) \( d(x, y) \leq d(x, z) + d(z, w) + d(w, y) \),

then we say that \( (X, d) \) is a generalized metric space (or gms).

Following example shows that there exists a generalized metric which is not a metric:

**Example:** Let \( (X, d) \) be an infinite metric space and \( x, z \in X \) be distinct elements which are fixed throughout the discussion. Define \( d^*: X^2 \to R \) by

\[
d^*(x, z) = d(x, z);
\]

\[
d^*(w, y) = d^*(x, y) = d^*(z, y) = \frac{1}{3}d(x, z), \; \forall y \in X \setminus \{x, z\};
\]

\[
d^*(y, w) = d^*(y, x) = d^*(y, z) = \frac{1}{3}d(x, z), \; \forall y \in X \setminus \{x, z\};
\]

\[
d^*(y, y) = 0, \; \forall y \in X.
\]

We show that \( d^* \) satisfies all the axioms of the generalized metric and hence \( d^* \) is a generalized metric on \( X \). Indeed,

\[
d(x, z) = \left(1 + \frac{1}{3} + \frac{1}{3} \right)d(x, z),
\]

gives

\[
d^*(y, w) = \frac{1}{3}d(x, z) < \frac{1}{3}d(x, z) + \frac{1}{3}d(x, z) + \frac{1}{3}d(x, z), \; \forall y, w \in X \setminus \{x, z\}.
\]
so that \[ d^*(y, w) < d^*(y, x) + \frac{1}{3}d(x, z) + \frac{1}{3}d(x, z), \forall y, w \in X \setminus \{x, z\}. \]

Hence, \[ d^*(y, w) < d^*(y, x) + d^*(x, z) + d^*(z, w), \forall y, w \in X \setminus \{x, z\}. \]

If \( p, q \in X \setminus \{x, z\} \) then \[ d^*(y, w) < d^*(y, p) + d^*(p, q) + d^*(q, w). \]

Thus, in each case, we have
\[
d^*(y, w) < d^*(y, p) + d^*(p, q) + d^*(q, w) \\
\Rightarrow \quad d^*(x, z) < d^*(x, p) + d^*(p, q) + d^*(q, z), \forall p, q \in X
\]

This shows that \( d^* \) is a generalized metric on \( X \). However, \( d^* \) is not a metric on \( X \) since \( d^* \) does not satisfy the triangle inequality because
\[
d(x, z) > \frac{2}{3}d(x, z) = \frac{1}{3}d(x, z) + \frac{1}{3}d(x, z)
\]

\[
\Rightarrow \quad d^*(x, z) > d^*(x, w) + d^*(w, z), w \in X \setminus \{x, z\}.
\]

**Remarks:** (i) If \( (X, d) \) is an infinite metric space, then this admits infinitely many generalized metrics on \( X \). From the above example, observe that \( x, z \) is an arbitrary pair of distinct elements which admits \( d^* \) from \( d \) depending on the choice of \( x, z \). Denoting the generalized metric \( d^* \) corresponding to the pair \( x, z \) by \( d^*_{x, z} \), we obtain infinitely many generalized metrics \( d^*_{x, z} \) on the infinite set \( X \), which are not metrics on \( X \).

(ii) As in the usual metric space settings, a generalized metric space is a topological space with respect to the basis given by \( B = \{B(x, r) : x \in X, r \in R\} \), where \( B(x, r) = \{y \in X : d(x, y) < r\} \) is open ball centered at \( x \).

**Definition:** Let \( (X, d) \) be a gms. A sequence \( \{x_n\} \) in \( X \) is said to be a Cauchy sequence if for any \( \varepsilon > 0 \) there exists \( n_0 \) in \( N \) such that for all \( m, n \leq n_0 \) one has \( d(x_n, x_{n+m}) < \varepsilon \). The space \( (X, d) \) is called complete if every Cauchy sequence in \( X \) is convergent.

### 3 \( A \)-contractions

By \( A \), we would mean the set of all functions \( \alpha: R^3_+ \rightarrow R^+ \) satisfying the conditions:

(i) \( \alpha \) is continuous on the set \( R^3_+ \) of all triplets of non-negative reals (with respect to the Euclidean metric on \( R^3 \));

(ii) \( a \leq k b \) for some \( k \in [0, 1] \) whenever \( a \leq \alpha(a, b, b) \) or \( a \leq \alpha(b, a, b) \) or \( a \leq \alpha(a, b, b) \) for all \( a, b \).

Next, we introduce the following \( A \)-contraction type condition for a pair of self maps of a generalized metric space:
Definition: Two self maps S and T on a generalized metric space X are said to satisfy condition \((\mathcal{A})\) if
\[
d(Sx,TSy) \leq \alpha(d(x,Sx),d(Sy,TSy),d(x,Sy))\]
for all \(x, y \in X\), for some \(\alpha \in A\). \(\ldots \ (\mathcal{A})\)

4 Some common fixed point theorems

Following theorem is a generalization of [3; Theorem 1] in the setting of gms; recall that a point \(x \in X\) is said to be a fixed point of the self-map T of X if \(x = Tx\).

Theorem 1: Let S and T be two self maps defined on a complete generalized metric space X satisfying the condition \((\mathcal{A})\). Then S and T have a unique common fixed point.

Proof: Let \(x_0 \in X\). Define a sequence \(\{x_n\}\) in X by \(x_{2n+1} = Sx_{2n}, \ x_{2n} = Tx_{2n-1}\) for \(n = 1, 2, \ldots\) By condition \((\mathcal{A})\),
\[
d(x_{2n+1},x_{2n+2}) = d(Sx_{2n},TSx_{2n}) \leq \alpha(d(x_{2n},Sx_{2n}),d(Sx_{2n},TSx_{2n}),d(x_{2n},Sx_{2n}))
\]
\[
= \alpha(d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}),d(x_{2n},x_{2n+1}))
\]
\[
\leq k\alpha(d(x_{2n+2},x_{2n+1})) \leq k(k\alpha(d(x_{2n+1},x_{2n})) \leq k^2d(x_{2n-1},x_{2n})
\]
with \(k \in [0,1)\). In general, we have \(d(x_n, x_{n+1}) \leq k^n d(x_0,x_1)\) for all \(n\).

Suppose that \(x_m \neq x_n\) for different \(m, n \in \mathbb{N}\) and consider
\[
d(x_m,x_{m+1}) \leq d(x_m,x_{m+1}) + d(x_{m+1},x_{m+2}) + d(x_{m+2},x_{m+3}) \leq k^n d(x_0,x_1) + k^{n+1} d(x_0,x_1) + k^{n+2} d(x_0,x_1)
\]
\[
= k^n(1+k+k^2)d(x_0,x_1).
\]
Now, for \(m > 2\) and using the supposition that \(x_p \neq x_r\) for \(p, r \in \mathbb{N}\) and \(p \neq r\), for \(p = n, r = n + m\) we have \(d(x_n,x_{n+m}) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{n+m-1},x_{n+m})\)
\[
\leq k^n d(x_0,x_1) + k^{n+1} d(x_0,x_1) + \cdots + k^{n+m-1} d(x_0,x_1) \leq k^n(1+k+k^2+\cdots+k^{m-1}) d(x_0,x_1),
\]
where \(k \in [0,1)\). Thus, \(\{x_n\}\) is a Cauchy sequence in X. Since X is complete gms, \(x_n \to x\) for some \(x\) in X. We claim the point \(x\) is the required common fixed point of S and T. To justify our claim, we proceed as follows: observe that
\[
d(Sx,x) = d(Sx,TSx_{2n-2}) \leq \alpha(d(Sx,x),d(TSx_{2n-2},Sx_{2n-2}),d(x,x))
\]
\[
= \alpha(d(Sx,x),d(x_{2n-2},x_{2n-1}),d(x,x_{2n-1})).
\]
When \(n \to \infty\), \(d(Sx,x) \leq \alpha(d(Sx,x),d(x,x)) = 0\) gives \(Sx = x\).
Similarly, one can show that \(Tx = x\). Thus \(x\) is the common fixed point of S and T. For uniqueness, let \(y\) be another fixed point of S and T. Then \(TSy = y\) so that
\[
d(x,y) = d(Sx,TSy) \leq \alpha(d(Sx,x),d(Sy,TSy),d(x,y)) = \alpha(d(x,x),d(y,y),d(x,y)) \leq 0.
\]
Hence, \(d(x,y) = 0\) giving in turn the uniqueness. \(\square\)

Corollary 1: Let S and T be two self maps of complete generalized metric space X satisfying the condition:
\[
d(Sx,TSy) \leq h(d(x,Sx),d(Sy,TSy))\]
for all \(x, y \in X\) and \(0 \leq h < 1/2\). Then S and T have a unique common fixed point.

Proof: We construct \(\alpha: \mathbb{R}^3 \to \mathbb{R}_+\) by \(\alpha(u, v, w) = h(v + w)\) where \(0 \leq h < 1/2\). Then \(\alpha \in A\). By taking \(u = d(x,y), v = d(x,Sx)\) and \(w = d(Sy,TSy)\) in the above construction of \(\alpha\) and exploiting the above condition \((K)\) of metric d in relation to
S and T, we get \(d(Sx, TSy) \leq \alpha (d(x, Sy), d(x, Sx), d(Sy, TSy))\). Thus the required result follows from Theorem 1. □

**Corollary 2:** Let S and T be two self maps of complete generalized metric space X satisfying the following condition:

\[
d(Sx, TSy) \leq h \sqrt{d(x, Sx)d(Sy, TSy)}
\]

for all \(x, y \in X\) and \(0 \leq h < 1\). Then S and T have a unique common fixed point.

**Proof:** If we define \(\alpha: \mathbb{R}^3 \to \mathbb{R}^+\) by \(\alpha(u, v, w) = h(vw)^{1/2}\) where \(0 \leq h < 1\). So that \(\alpha \in A\). Now, by taking \(u = d(x, Sy), v = d(x, Sx)\) and \(w = d(Sy, TSy)\) in definition of \(\alpha\) and using (M) we get, \(d(Sx, TSy) \leq \alpha (d(x, Sy), d(x, Sx), d(Sy, TSy))\). Thus by Theorem 1, we obtain the required conclusion. □

**Corollary 3:** Let S and T be two self maps of complete generalized metric space X satisfying the following condition:

\[
d(Sx, TSy) \leq \beta \max\{d(Sx, x)+d(TSy, Sy), d(Sx, x)+d(x, Sy), d(TSy, Sy)+d(x, Sy)\}
\]

(O) for all \(x, y \in X\) and some \(\beta \in [0, 1/2)\). Then S and T have a unique common fixed point.

**Proof:** We construct \(\alpha: \mathbb{R}^3 \to \mathbb{R}^+\) by \(\alpha(u, v, w) = \beta \max(v+w, u+v, u+w)\) where \(\beta \in [0, 1/2)\) so that \(\alpha \in A\). Taking \(u = d(x, y), v = d(x, Sx)\) and \(w = d(Sy, TSy)\) in the definition of \(\alpha\) and using condition (O), we get \(d(Sx, TSy) \leq \alpha (d(x, y), d(x, Sx), d(Sy, TSy))\). Thus by Theorem 1, S and T have a unique common fixed point. □

**Corollary 4:** Let S and T be two self maps of complete generalized metric space X satisfying the following condition:

\[
d(Sx, TSy) \leq \beta \max\{d(Sx, x), d(TSy, Sy)\}
\]

for all \(x, y \in X\) and \(\beta \in [0, 1)\). Then S and T have a unique common fixed point.

**Proof:** By constructing \(\alpha: \mathbb{R}^3 \to \mathbb{R}^+\) by \(\alpha (u, v, w) = \beta \max(v, w)\) where \(\beta \in [0, 1)\) and taking \(u, v\) and \(w\) as in the Corollary 3, we see that the condition \((\mathcal{A})\) satisfied. Thus, S and T have a unique common fixed point by Theorem 1. □

**Corollary 5:** Let S and T be two self maps of complete generalized metric space X such that there exist non-negative real numbers \(a, b, c\) with \(a + b + c \leq 1\) satisfying the following condition:

\[
d(Sx, TSy) \leq ad(x, Sy) + bd(Sx, x) + cd(TSy, Sy)
\]

for all \(x, y \in X\). Then S and T have a unique common fixed point.

**Proof:** Define \(\alpha: \mathbb{R}^3 \to \mathbb{R}^+\) by \(\alpha (u, v, w) = au + bv + cw\), where \(a, b\) and \(c\) are non-negative real numbers with \(a + b + c \leq 1\). so that \(\alpha \in A\). By taking suitable \(u, v\) and \(w\) and using condition (R), one can easily get the condition \((\mathcal{A})\) satisfied, which in turn leads to the required conclusion by Theorem 1. □

The following result on sequences of self-maps generalizes [3; Theorem 2]:
Theorem 2: Let \( \{ F_n \} \) and \( \{ G_n \} \) be sequence of self maps in complete generalized metric space \( X \) satisfying condition \( A \) for each \( m, n \). Then \( \{ F_n \} \) and \( \{ G_n \} \) have a unique common fixed point.

Proof: Let \( x_0 \in X \). Define a sequence \( \{ x_n \} \) in \( X \) by \( x_{2n+1} = F_n x_{2n} \) and \( x_{2n} = G_n x_{2n-1} \) for \( n = 1, 2, 3 \ldots \). The case for which \( x_n = x_{2n+1} \) is trivial. Next, suppose \( x_{2n} \neq x_{2n+1} \) for some \( n \). Then we have

\[
d(x_{2n+1}, x_{2n}) = d(F_n x_{2n}, G_n x_{2n-1}) \leq \alpha(d(F_n x_{2n}, x_{2n}), d(G_n x_{2n-1}, x_{2n-1}), d(x_{2n}, x_{2n-1})) \leq k d(x_{2n}, x_{2n-1}) \leq k(kd(x_{2n-1}, x_{2n-2})).
\]

In general, \( d(x_{n+1}, x_n) \leq k^n d(x_1, x_0) \) for some \( k \in [0, 1) \).

Next, consider \( d(x_n, x_{n+3}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) \leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + k^{n+2} d(x_0, x_1) = k^n(1+k+k^2)d(x_0, x_1) \).

Taking limit as \( n \to \infty \), \( d(x_n, x_{n+3}) \to 0 \). This shows that \( \{ x_n \} \) is Cauchy sequence in complete gms \( X \) and hence converges to some \( x \) in \( X \).

Next, for each \( m \), we have that

\[
d(x_n, F_m x) \leq d(x_n, x_{2n-1}) + d(x_{2n-1}, x_{2n}) + d(x_{2n}, F_m x) = d(x_n, x_{2n}) + d(x_{2n-1}, x_{2n}) + d(F_m x, F_m x) = d(x_n, x_{2n}) + d(x_{2n-1}, x_{2n}) + \alpha(d(G_n x_{2n-1}, x_{2n-1}), d(F_m x, x), d(x_{2n-1}, x)) \leq k^n(1+k+k^2)^m d(x_0, x_1), \text{ where } k \in [0, 1).\]

and so by letting \( n \to \infty \), we get

\[
d(x, F_m x) \leq d(x, x) + \alpha(d(x, x), d(F_m x, x), d(x, x)) \leq 0 + 0 + 0 \leq 0.
\]

This gives \( d(x, F_m x) = 0 \) so that \( x = F_m x \) for all \( m \). Similarly \( x = G_m x \) for all \( m \). Thus \( x \) is a common fixed point of \( F_n \) and \( G_n \) for all \( n \). The uniqueness of common fixed point \( x \) can be seen easily. □

The results given below are immediate consequences of Theorem 2:

Corollary 6: Let \( \{ F_n \} \) and \( \{ G_n \} \) be sequence of self maps in complete gms \( X \) satisfying \( d(F_n x, G_n y) \leq h(d(F_n x, x) + d(G_n y, y)) \) for all \( x, y \) in \( X \) and \( 0 \leq h < \frac{1}{2} \). Then \( F_n \) and \( G_n \) have a unique common fixed point.

Corollary 7: Let \( \{ F_n \} \) and \( \{ G_n \} \) be sequence of self maps in complete gms \( X \) satisfying \( d(F_n x, G_n y) \leq h\sqrt{d(x, F_n x)d(y, G_n y)} \) for all \( x, y \) in \( X \) and \( 0 \leq h < 1 \). Then \( F_n \) and \( G_n \) have a unique common fixed point.
Corollary 8: Let \( \{F_n\} \) and \( \{G_n\} \) be sequence of self maps in complete gms \( X \) satisfying 
\[
d(F_n x, G_n y) \leq \beta \max\{d(F_n x, x) + d(G_n y, y), d(F_n x, x) + d(x, y), d(G_n y, y) + d(x, y)\}
\]
for all \( x, y \) in \( X \) and some \( \beta \in [0, 1/2) \). Then \( F_n \) and \( G_n \) have unique common fixed point.

Corollary 9: Let \( \{F_n\} \) and \( \{G_n\} \) be sequence of self maps in complete gms \( X \) satisfying 
\[
d(F_n x, G_n y) \leq \beta \max\{d(F_n x, x), d(G_n y, y)\}
\]
for all \( x, y \) in \( X \) and some \( \beta \in [0, 1) \). Then \( F_n \) and \( G_n \) have a unique common fixed point.

Corollary 10: Let \( \{F_n\} \) and \( \{G_n\} \) be sequence of self maps in complete gms \( X \) such that there exists non-negative real numbers \( a, b, c \) with \( a + b + c \leq 1 \) satisfying 
\[
d(F_n x, G_n y) \leq ad(F_n x, x) + bd(G_n y, y) + cd(x, y)
\]
for all \( x, y \) in \( X \). Then \( F_n \) and \( G_n \) have a unique common fixed point.

References


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