A Common Fixed Point Theorem in Non-Archimedean Menger Probabilistic Metric Spaces Using R-Weakly Commuting Maps

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Abstract. In this paper we define the concept of R-weakly commuting mappings in non-Archimedean Menger space and obtain a common fixed point Theorem which generalizes the result of Alamgir Khan and Sumitra [1].

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1. INTRODUCTION


The aim of this paper is to define the concept of R-weakly commuting maps and prove a common fixed point theorem in non-Archimedean Menger spaces.

Definition 1.1 (B. Schweizer and A. Sklar, [5]):
A function $F: \mathbb{R} \rightarrow [0, 1]$ is called a distribution function if $F$ is non-decreasing, left continuous and $\inf_{x \in \mathbb{R}} F(x) = \sup_{x \in \mathbb{R}} F(x) = 1$.
$
\mathcal{D}
$

$\mathcal{D}$ denotes the set of distribution functions and $\mathcal{D}^+ = \{F: F \in \mathcal{D} \text{ and } F(0) = 0\}$.
The Heaviside function is a distribution function defined by

$$
H(t) = \begin{cases} 
1 & \text{if } t > 0 \\
0 & \text{if } t < 0 
\end{cases}
$$

Definition 1.2 (B. Schweizer and A. Sklar, [5]):
Let $X$ be a non-empty set and let $F: X \times X \rightarrow \mathcal{D}^+$. For $p, q \in X$, we denote the image of the pair $(p, q)$ by $F_{p, q}$ which is a distribution function so that $F_{p, q}(x) \in [0, 1]$, for every real $x$. Suppose $F$ satisfies:

a) $F_{p, q}(x) = 1$ for all $x > 0$ if and only if $p = q$

b) $F_{p, q}(0) = 0$

c) $F_{p, q}(x) = F_{q, p}(x)$

d) If $F_{p, q}(x) = 1$ and $F_{q, r}(y) = 1$ then $F_{p, r}(x + y) = 1$ where $p, q, r \in X$.

Then $(X, F)$ is called a probabilistic metric space.

Definition 1.3 (B. Schweizer and A. Sklar, [5]):
A triangular norm $*: [0,1] \times [0,1] \rightarrow [0,1]$ is a function satisfying the following conditions

(i) $a * 1 = a \quad \forall a \in [0,1]$

(ii) $a * b = b * a \quad \forall a, b \in [0,1]$

(iii) $c * d \geq a * b \quad \forall a, b, c, d \in [0,1] \text{ with } c \geq a \text{ and } d \geq b$

(iv) $(a * b) * c = a * (b * c) \quad \forall a, b, c \in [0,1]$

A triangular norm is also denoted by t-norm.
Common fixed point theorem

Note: If \( * \) is defined by 
\[ * (a, b) = \min\{a, b\} \quad \forall \ a, b \in [0,1], \]
then \( * \) is a t-norm called the min t-norm.

**Definition 1.4 (B. Schweizer and A. Sklar, [5]):**
Let \( X \) be a non empty set, \( * \) be a t-norm and \( F: X \times X \to \mathbb{D}^+ \) be a function satisfying
1. \( F_{p, q}(0) = 0 \quad \forall \ p, q \in X \)
2. \( F_{p, q}(x) = 1 \) for all \( x > 0 \) if and only if \( p = q \)
3. \( F_{p, q}(x) = F_{q, p}(x) \quad \forall \ p, q \in X \)
4. \( F_{p, r}(x + y) \geq F_{p, q}(x) * F_{q, r}(y) \) for all \( x, y \geq 0 \) and \( p, q, r \in X \).
Then the triplet \( (X, F, *) \) is called a Menger space.

**Definition 1.5 (B. Schweizer and A. Sklar and E. Thorp, [6]):**
(i) Let \( (X, F, *) \) be a Menger space and \( p \in X. \)
\( \epsilon > 0, 0 < \lambda < 1, \) the \( (\epsilon, \lambda) \)-neighborhood of \( p \) is defined as 
\[ U_p(\epsilon, \lambda) = \{ q \in X: F_{p, q}(\epsilon) > 1 - \lambda \} \].
It may be observed that, if \( * \) is continuous then the topology induced by the family
\( \{U_p(\epsilon, \lambda): p \in X, \epsilon > 0, 0 < \lambda < 1\} \) is a Hausdorff topology on \( X \) and is known as the \( (\epsilon, \lambda) \)-topology.
(ii) A sequence \( \{x_n\} \) in \( X \) is said to converge to \( p \in X \) in the \( (\epsilon, \lambda) \)-topology, if for any \( \epsilon > 0 \) and \( 0 < \lambda < 1 \) there exists a positive integer \( N = N(\epsilon, \lambda) \) such that \( F_{x_n p}(\epsilon) > 1 - \lambda \) where \( n > N. \)
(iii) A sequence \( \{x_n\} \) in \( X \) is said to be a Cauchy sequence in the \( (\epsilon, \lambda) \)-topology, if for \( \epsilon > 0 \) and \( 0 < \lambda < 1 \) there exists a positive integer \( N = N(\epsilon, \lambda) \) such that \( F_{x_m x_n}(\epsilon) > 1 - \lambda \) for all \( m, n > N. \)
(iv) We observe that a sequence \( \{x_n\} \) is not Cauchy in \( X \), then there exist \( \epsilon_0 > 0, t_0 > 0 \) and two sequences \( \{m_i\}, \{n_i\} \) of positive integers such that
\[ m_i > n_i + 1 \text{ and } n_i \to \infty \text{ as } t \to \infty \]
\[ F_{y_{m_i}, y_{n_i}}(t_0) < 1 - \epsilon_0 \text{ and } F_{y_{m_i-t}, y_{n_i}}(t_0) \geq 1 - \epsilon_0 \text{ for } i = 1,2,3 \ldots \]
(v) A Menger space \( (X, F, *) \) where \( * \) is continuous, is said to be complete if every Cauchy sequence in \( X \) is convergent in \( (\epsilon, \lambda) \)-topology.

**Definition 1.6 (S.S. Chang, [3]):**
A Probabilistic metric space \( (X, F) \) is called non-Archimedean if 
\[ F_{x,y}(t_1) = 1, F_{y,z}(t_2) = 1 \text{ then } F_{x,z}(\max \{t_1, t_2\}) = 1 \quad \forall \ x, y, z \in X, t_1, t_2 \geq 0. \]

**Definition 1.7 (S.S. Chang, [2], [3]):**
A Menger PM space \( (X, F, *) \) is called non-Archimedean if 
\[ F_{x,z}(\max \{t_1, t_2\}) \geq F_{x,y}(t_1) * F_{y,z}(t_2) \quad \forall \ x, y, z \in X, t_1, t_2 \geq 0. \]
Note: We observe that \( (X, F, *) \) is non Archimedean if and only if
\[ F_{x,z}(t) \geq F_{x,y}(t) * F_{y,z}(t) \quad \forall \ x, y, z \in X \text{ and } t \geq 0. \]

**Definition 1.8:** If \((X, F, *)\) is not non-Archimedean, then we say that \((X, F, *)\) is Archimedean.

Note: We observe that \((X, F, *)\) is Archimedean if and only if there exists \(\forall t \geq 0\), \(\exists a, b, c \in X\)

\[ F_{x,y}(t) \leq F_{x,a}(t) * F_{a,y}(t) \quad \forall t \geq 0. \]

**Example 1.9:** Let \((X, d)\) be a metric space with \(d\) defined as \(d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}\) and \(*\) be any t-norm. If the distribution function is defined as

\[ F_{x,y}(t) = t \quad \forall t > 0, \text{ then } (X, F, *) \text{ is a non Archimedean Menger space.} \]

**Example 1.10:** Let \((X, d)\) be a metric space and \(*\) be as in example 1.9. If the distribution function defined as

\[ F_{x,y}(t) = r(t - d(x, y)) \quad \forall t > 0, \text{ then } (X, F, *) \text{ is an Archimedean Menger space.} \]

**Notations:**

(i) \(\Omega = \{g / g : [0,1] \rightarrow [0, \infty) \text{ is continuous, strictly decreasing and } g(1) = 0\}\)

(ii) \(\Phi = \{\phi / \phi : [0, \infty) \rightarrow [0, \infty), \phi \text{ is continuous and } \phi(t) < t, \forall t > 0\}\).

We observe that \(\lim_{n \rightarrow \infty} \phi^n(t) = 0 \quad \forall t > 0.\)

**Definition 1.11 (B. Schweizer and A. Sklar [5]):**

A Probabilistic metric space \((X, F)\) is said to be of type \((C)g\) if there exists \(g \in \Omega\) such that

\[ g \left( F_{x,y}(t) \right) \leq g \left( F_{x,x}(t) \right) * g \left( F_{y,z}(t) \right) \quad \forall x, y, z \in X, \ t \geq 0. \]

**Definition 1.12 (B. Schweizer and A. Sklar, [5]):**

A non-Archimedean Menger probabilistic metric space \((X, F, *)\) is said to be of type \((D)g\), if there exists \(g \in \Omega\) such that

\[ g(s * t) \leq g(s) + g(t) \quad \forall s, t \in [0,1]. \]

We observe that \(* = \min \text{ t-norm of type } (D)g, \ g \in \Omega.\)

**Definition 1.13:** Two maps \(A\) and \(S\) of a non-Archimedean PM space \((X, F, *)\) into itself are said to be \(R\)-weakly commuting if there exists some \(R > 0\) such that

\[ F_{ASx, SAx}(t) \geq F_{ASx, Sx} \left( \frac{t}{R} \right) \quad \forall x \in R and \ t > 0. \]

The following are results proved by M. Alamgir Khan and Sumitra [1].
Theorem 1.14 (M. Alamgir Khan and Sumitra, [1]): Let $S$ and $T$ be two continuous self-maps of a complete non-Archimedean Menger space $(X, F, *)$, where $*$ is continuous and strictly increasing t-norm. Let $A$ be self map of $X$ satisfying

(i) $\{A, S\}$ and $\{A, T\}$ are point wise R-weakly commuting
and $A(X) \subseteq S(X) \cap T(X)$

(ii) $g(F_{Ax, Ay}(t)) \leq \varphi[\max \{g(F_{Sx, Ty}(t)), g(F_{Sx, Ax}(t)), g(F_{Sx, Ay}(t))
\bigg]\}

for all $x, y \in X$ and $\varphi: [0, \infty) \rightarrow [0, \infty)$ is upper semi-continuous from the right. Then $A, S$ and $T$ have a unique common fixed point in $X$.

Corollary 1.15: Let $(X, F, *)$ be a complete non-Archimedean Menger space and $S$ be a continuous self mapping of $X$. Let $A$ be another self mapping of $X$ satisfying that $\{A, S\}$ is R-weakly commuting of type with $A(X) \subseteq S(X)$ and

$g(F_{Ax, Ay}(t)) \leq \varphi[\max \{g(F_{Sx, Sy}(t)), g(F_{Sx, Ax}(t)), g(F_{Sx, Ay}(t)), g(F_{Sy, Ay}(t))\}]

for all $x, y \in X$ and $\varphi: [0, \infty) \rightarrow [0, \infty)$ is upper semi-continuous from the right. Then $A$ and $S$ have a unique common fixed point in $X$.

2. Main results

The following theorem is an extension of Theorem 1.14 with $\varphi \in \Phi$.

Theorem 2.1: Let $S$ and $T$ be two continuous self-maps of a complete non-Archimedean Menger space $(X, F, *)$, where $*$ as continuous t-norm. Let $A$ be self map on $X$ satisfying

(i) $A(X) \subseteq S(X) \cap T(X)$

(ii) $g(F_{Ax, Ay}(t)) \leq \varphi[\max \{g(F_{Ax, Sx}(t)), g(F_{Ax, Sy}(t)), g(F_{Ax, Ty}(t))
\bigg]\}

for all $x, y \in X, t > 0, for some g \in \Omega$ and $\varphi \in \Phi$.

Let $x_0 \in X$. Define sequences $\{x_n\}$ and $\{y_n\}$ by $y_n = Ax_n = Sx_{n+1} = Tx_{n+2}$ for all $n = 0, 1, 2 \ldots$. Suppose $\lim_{n \to \infty} F_{y_n y_{n+1}}(t) = 1 \forall t > 0$. 

\[ F_{y_n y_{n+1}}(t) = \lim_{n \to \infty} F_{y_n y_{n+1}}(t) = 1 \forall t > 0. \]
Then \( \{y_n\} \) is a Cauchy sequence in \( X \).

Proof: Suppose \( \{y_n\} \) is not a Cauchy sequence in \( X \).
Then there exist \( \epsilon_0 \in (0,1) \), \( t_0 > 0 \) and two sequences \( \{m_i\}, \{n_i\} \) of positive integers such that \( m_i > n_i + 1 \) and \( n_i \to \infty \) as \( i \to \infty \)
\[
F_{y_{m_i}y_{n_i}}(t_0) < 1 - \epsilon_0 \quad \text{and} \quad F_{y_{m_i-1}y_{n_i}}(t_0) \geq 1 - \epsilon_0 \quad \text{for} \ i = 1,2,3 \ldots
\]
By taking \( x = x_{m_i}, y = x_{n_i+1} \) condition (ii), we get
\[
g(F_{Ax_{m_i}Ax_{n_i+1}}(t)) \leq \varphi[\max\{g(F_{Ax_{m_i}x_{m_i}}(t)), g(F_{Ax_{m_i+1}x_{n_i+1}}(t)),
 g(F_{Ax_{n_i+1}x_{n_i+1}}(t)), g(F_{x_{m_i}x_{n_i+1}}(t)), g(F_{x_{m_i}x_{n_i+1}}(t))\}]
\]
\[
g(F_{y_{n_i+1}y_{n_i}}(t)) \leq \varphi[g(\min\{F_{y_{m_i}y_{m_i-1}}(t), F_{y_{m_i}y_{n_i}}(t), F_{y_{m_i-1}y_{n_i-1}}(t), F_{y_{n_i-1}y_{n_i-1}}(t)\})]\]

\[
\text{Since } X \text{ is non-Archimedean, we have}
\]
\[
1 - \epsilon_0 > F_{y_{m_i}y_{n_i}}(t) \geq F_{y_{m_i}y_{m_i-1}}(t) * F_{y_{m_i-1}y_{n_i}}(t) > F_{y_{m_i}y_{m_i-1}}(t) * (1 - \epsilon_0)
\]
\[
\to 1 * (1 - \epsilon_0) = (1 - \epsilon_0) \quad \text{as} \ i \to \infty
\]
so it follows that

I) \( \lim_{i \to \infty} F_{y_{m_i}y_{n_i}}(t) = 1 - \epsilon_0 \) and II) \( \lim_{i \to \infty} F_{y_{m_i-1}y_{n_i}}(t) = 1 - \epsilon_0 \)

Similarly we can obtain the following results

III) \( \lim_{i \to \infty} F_{y_{m_i-1}y_{n_i-1}}(t) = 1 - \epsilon_0 \), IV) \( \lim_{i \to \infty} F_{y_{m_i}y_{n_i-1}}(t) = 1 - \epsilon_0 \).

V) \( \lim_{i \to \infty} F_{y_{m_i-1}y_{n_i-1}}(t) = 1 - \epsilon_0 \) and VI) \( \lim_{i \to \infty} F_{y_{m_i}y_{n_i+1}}(t) = 1 - \epsilon_0 \).

On letting \( i \to \infty \), by using results I, II, III, IV, V and VI, and continuity of \( g \), the inequality (2.1.1) become
\[
g(1 - \epsilon_0) \leq \varphi(g(\min\{1,1 - \epsilon_0, 1 - \epsilon_0, 1 - \epsilon_0, 1 - \epsilon_0, 1 - \epsilon_0, 1\}))
\]
\[
\Rightarrow g(1 - \epsilon_0) \leq \varphi(g(1 - \epsilon_0)) < g(1 - \epsilon_0), \text{ which is a contradiction.}
\]
Therefore \( \{y_n\} \) is a Cauchy sequence in \( X \).

We prove the next theorem by Lemma 2.2 with * as min t-norm.
**Lemma 2.2:** If * is a min t-norm and $\alpha, \beta, \gamma \in [0,1]$ be such that $\alpha * \beta \leq \gamma$, $\beta * \gamma \leq \alpha$, and $\gamma * \alpha \leq \beta$ then $\alpha > \beta$ implies $\beta = \gamma$.

**Theorem 2.3:** Let $S$ and $T$ be two continuous self-maps of a complete non-Archimedean Menger space $(X, F, *)$, where * is min t-norm. Let $A$ be self map on $X$ satisfying

(i) $\{A, S\}$ and $\{A, T\}$ are point wise R-weakly commuting and $A(X) \subseteq S(X) \cap T(X)$

(ii) $g(F_{AX,AY}(t)) \leq \varphi \max \{g(F_{AX,SS}(t)), g(F_{AX,SY}(t)), g(F_{AX,TY}(t)) \}$

for all $x, y \in X$, $t > 0$, for some $g \in \Omega$ and $\varphi \in \Phi$.

Then $A$, $S$ and $T$ have unique common fixed point in $X$. In fact for any $x_0 \in X$, the sequence $\{y_n\}$ defined as $y_n = A x_n = S x_{n+1} = T x_{n+2}$ for $n = 0, 1, 2, \ldots$ then $\{y_n\}$ converges to the unique common fixed point of $A$, $S$ and $T$ in $X$.

Proof: First we show that $\lim_{n \to \infty} F_{yn,y_{n+1}}(t) = 1$ $\forall t > 0$.

By taking $x = x_n$, $y = x_{n+1}$ in condition (ii), we get

$g(F_{AXn,AXn+1}(t)) \leq \varphi \max \{g(F_{AXn,SSn}(t)), g(F_{AXn,SYn}(t)), g(F_{AXn,TYn}(t)) \}$

$g(F_{AXn+1,SSn}(t)), g(F_{AXn+1,SYn}(t)), g(F_{AXn+1,SYn+1}(t)), g(F_{SXn,SYn}(t)), g(F_{SXn,SYn+1}(t))$

$\Rightarrow g(F_{yn,yn+1}(t)) \leq \varphi [g(\min \{F_{yn,yn-1}(t), F_{yn,yn}(t), F_{yn,yn-1}(t), F_{yn+1,yn-1}(t), F_{yn+1,yn}(t), F_{yn+1,yn-1}(t), F_{yn,yn-1}(t)) \}$

$\Rightarrow g(F_{yn,yn+1}(t)) \leq \varphi [g(\min \{F_{yn,yn-1}(t), F_{yn+1,yn-1}(t), F_{yn+1,yn}(t)) \}] \cdots (2.3.1)$

Case I: Suppose for some $n$, $F_{yn,yn+1}(t) = 1$.

Then from (2.3.1), we can write

$g(F_{yn+1,yn+2}(t)) \leq \varphi [g(\min \{F_{yn,yn+1}(t), F_{yn+1,yn+2}(t), F_{yn,yn+2}(t) \}) \cdots (2.3.2)$

Write $\alpha = F_{yn,yn+1}(t), \beta = F_{yn+1,yn+2}(t), \gamma = F_{yn,yn+2}(t) \cdots (2.3.4)$
Then, since $X$ is non-Archimedean and $\alpha = 1$, equation (2.3.4) satisfies the hypothesis of Lemma 2.2, we get
\[ g(\beta) \leq \varphi(g(\beta)) < g(\beta) \text{ if } g(\beta) > 0, \text{ a contradiction.} \]
Hence $g(\beta) = 0$
so that $g\left(F_{y_{n+1},y_{n+2}}(t)\right) = 0$.
Consequently $F_{y_{n+1},y_{n+2}}(t) = 1 \forall \ t > 0$.
Hence by induction $F_{y_{m},y_{m+1}}(t) = 1$, for $m \geq n$.
Therefore $\lim_{m \to \infty} F_{y_{m},y_{m+1}}(t) = 1$, for every $t > 0$.
Case II: Suppose $F_{y_{n+1},y_{n+2}}(t) < 1 \forall \ n$.
Write $\alpha = F_{y_{n-1},y_{n}}(t), \beta = F_{y_{n},y_{n+1}}(t), \gamma = F_{y_{n-1},y_{n+1}}(t) \ldots (2.3.4)$
If $\alpha \geq \min \{\alpha, \beta, \gamma\}$, then from Lemma 2.2, we get $\beta = \gamma = \min \{\alpha, \beta, \gamma\}$.
Then from inequality (2.3.1), we have
\[ g(\beta) \leq \varphi(g(\beta)) < g(\beta) \text{ if } g(\beta) > 0, \text{ a contradiction.} \]
Therefore $\alpha = \min \{\alpha, \beta, \gamma\}$.
This implies $g(\beta) \leq \varphi(g(\alpha))$.
Hence $g\left(F_{y_{n},y_{n+1}}(t)\right) \leq \varphi\left(g\left(F_{y_{n-1},y_{n}}(t)\right)\right) \leq \cdots \leq \varphi^n\left(g\left(F_{y_0,y_1}(t)\right)\right) 
\rightarrow \infty \text{ as } i \rightarrow \infty$.
Consequently $F_{y_{n},y_{n+1}}(t) = 1, for \ every \ t > 0$.
Therefore from both cases (I) and (II), we obtain
\[ \lim_{m \to \infty} F_{y_{m},y_{m+1}}(t) = 1, \text{ for every } t > 0. \]
Hence $\{y_n\}$ is a Cauchy sequence in $X$.
Since $(X, F, \ast)$ is complete, the sequence $\{y_n\}$ converge to a point $z$, say in $X$.
Now by definition of the sequence $\{y_n\}$, we have
$y_n \to z \Rightarrow Ax_n \to z,Sx_n \to z$ and $Tx_n \to z$ as $n \to \infty$
Since $S$ and $T$ are continuous $\Rightarrow Sy_n \to Sz$ and $Ty_n \to Tz,SAx_n \to Sz$
and $TAx_n \to Tz$.
Since $\{A, S\}$ and $\{A, T\}$ are $R$-weakly commuting,
Hence $F_{ASx_n,SAx_n}(t) \geq F_{Ax_n,Sx_n}\left(\frac{t}{R}\right), R > 0$
and $F_{ATx_n,Tx_n}(t) \geq F_{Ax_n,Tx_n}\left(\frac{t}{R}\right), R > 0 \ldots (2.3.5)$
from (2.3.5), we can write
\[ \lim_{n \to \infty} F_{ASx_n,SAx_n}(t) = \lim_{n \to \infty} F_{A_{y_n}Sy_n}(t) = \lim_{n \to \infty} F_{A_{y_n}Sz}(t) \]
\[
\geq \lim_{n \to \infty} F_{A x_n, S x_n}(t) = \lim_{n \to \infty} F_{z, z}(t) = 1
\]
i.e. \( A y_n \to S z \) as \( n \to \infty \) \( \ldots \) (2.3.6)

Similarly \( \lim_{n \to \infty} F_{A T x_n, T A x_n}(t) \geq 1 \)

Now \( 1 \geq \lim_{n \to \infty} F_{A y_n, T z}(t) \geq 1 \)

\( \Rightarrow \lim_{n \to \infty} F_{A y_n, T z}(t) = 1 \)

\( \Rightarrow A y_n \to T z \) \( \ldots \) (2.3.7)

From (2.3.6) and (2.3.7), we get \( S z = T z \).

Now, by taking \( x = S x_n \) and \( y = z \) in condition (ii), we get \( S z = A z \).

Hence \( A z = S z = T z \).

By taking \( x = x_{n+1} \) and \( y = z \) in condition (ii), we get \( z = A z \).

Hence \( z = A z = S z = T z \).

Therefore \( z \) is a common fixed point of \( A, S \) and \( T \).

To prove uniqueness, let \( x \) be a common fixed point of \( A, S \) and \( T \).

From condition (ii), we have

\[
g(F_{A x, A z}(t)) \leq \varphi[\max \{ g(F_{A x, S x}(t)), g(F_{A x, S z}(t)), g(F_{A x, T z}(t)) \}
\]

\( \geq g(F_{A x, S x}(t)), g(F_{A x, S z}(t)), g(F_{A x, T z}(t)), g(F_{S x, S z}(t)), g(F_{S x, T z}(t)), g(F_{S z, T z}(t))]) \]

\( \Rightarrow g(F_{x, x}(t)) \leq \varphi[\max \{ g(F_{x, x}(t)), g(F_{x, z}(t)), g(F_{x, x}(t)) \}
\]

\( \geq g(F_{x, x}(t)), g(F_{x, x}(t)), g(F_{x, z}(t)), g(F_{x, x}(t)), g(F_{x, x}(t)), g(F_{x, z}(t))]) \]

\( \Rightarrow g(F_{x, x}(t)) \leq \varphi[\max \{ 0, g(F_{x, x}(t)) \}] \)

\( \Rightarrow g(F_{x, x}(t)) = 0 \ \forall \ t > 0 \)

\( \Rightarrow F_{x, x}(t) = 1 \)

\( \Rightarrow x = z \)
Therefore $z$ is a unique common fixed point of $A$, $S$ and $T$.

### 3. Results without $g, g \in \Omega$

In this section we prove theorems similar to those of section 2 without using the role of $g, g \in \Omega$.

**Notation:** Write $\Psi = \{\psi : (0,1) \to (0,1) : \psi$ is continuous and $\psi(t) > t\}$. We observe that $\lim_{n \to \infty} \psi^n(t) = 1$.

**Example 3.1:** Define $\psi : (0,1) \to (0,1)$ as $\psi(t) = \frac{t+1}{2}$ for all $t \in (0,1)$, then $\psi \in \Psi$.

**Theorem 3.2:** Let $S$ and $T$ be two continuous self-maps of a complete non-Archimedean Menger space $(X, F, \ast)$, where $\ast$ as continuous t-norm. Let $A$ be self map of $X$ satisfying

(i) $A(X) \subseteq S(X) \cap T(X)$

(ii) $g(F_{AX,AY}(t)) \geq \psi[\min\{F_{AX,SX}(t), F_{AX,SY}(t), F_{AX,TY}(t), F_{AY,SX}(t), F_{AY,SY}(t), F_{AY,TY}(t)\}]$


\[
F_{AX,AY}(t) \geq \psi[\min\{F_{AX,SX}(t), F_{AX,SY}(t), F_{AX,TY}(t), F_{AY,SX}(t), F_{AY,SY}(t), F_{AY,TY}(t)\}]
\]

for all $x, y \in X, t > 0$ and for some $\psi \in \Psi$.

Let $x_0 \in X$. Define sequence $\{x_n\}$ and $\{y_n\}$ by $y_n = Ax_n = Sx_{n+1} = Tx_{n+2}$

for all $= 0,1,2 \ldots$. Suppose $\lim_{n \to \infty} F_{y_n,y_{n+1}}(t) = 1$ for all $t > 0$.

Then $\{y_n\}$ is a Cauchy sequence in $X$.

**Proof:** Suppose $\{y_n\}$ is not a Cauchy sequence in $X$.

Then there exist $\varepsilon_0 > 0, \lambda \in (0,1)$ and two sequences $\{m_i\}, \{n_i\}$ of positive integers such that $m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$

$F_{y_{m_i},y_{n_i}}(t_0) < 1 - \varepsilon_0$ and $F_{y_{m_i-1},y_{n_i}}(t_0) \geq 1 - \varepsilon_0$ for $i = 1,2,3 \ldots$

By taking $x = x_{m_i}, y = x_{n_i+1}$ condition (ii), we get

$F_{Ax_{m_i},Ax_{m_i+1}}(t) \geq \psi[\min\{F_{AX,m_i,Sx_{m_i}}(t), F_{AX,m_i,SY_{m_i+1}}(t), F_{AX,m_i,Tx_{n_i+1}}(t), F_{AX_{n_i+1},Sx_{n_i+1}}(t), F_{AX_{n_i+1},TY_{n_i+1}}(t), F_{SS,m_i,SX_{n_i+1}}(t), F_{SS_{m_i},TX_{n_i+1}}(t), F_{SS_{n_i+1},TX_{n_i+1}}(t)\}])$

$F_{y_{m_i},y_{n_i+1}}(t) \geq \psi[\min\{F_{y_{m_i},y_{m_i-1}}(t), F_{y_{m_i},y_{n_i}}(t), F_{y_{m_i},y_{n_i-1}}(t), F_{y_{n_i+1},y_{m_i-1}}(t)\}]$
Common fixed point theorem

\[ F_{y_{n+1}, y_{n+1}}(t), F_{y_{n+1}, y_{n-1}}(t), F_{y_m, y_{n-1}}(t), F_{y_m, y_{n-1}}(t), F_{y_n, y_{n-1}}(t) \] \ (3.2.1)

On letting \( t \to \infty \), by using results I, II, III, IV, V and VI in Theorem (2.1), we get
\[ 1 - \varepsilon_0 \geq \psi(\min \{1, 1 - \varepsilon_0, 1 - \varepsilon_0, 1, 1, 1 - \varepsilon_0, 1 - \varepsilon_0, 1\}) \]
\[ \Rightarrow 1 - \varepsilon_0 \geq \psi(1 - \varepsilon_0), \] which is a contradiction.

Therefore \( \{y_n\} \) is a Cauchy sequence in \( X \).

**Theorem 3.3:** Let \( (X, F, *) \) be a complete non-Archimedean Menger space, with * as min t-norm. \( A, S \) and \( T \) be self map on \( X \) such that

(i) \( S \) and \( T \) are continuous
(ii) \( A(X) \subseteq S(X) \cap T(X) \)
(iii) \( \{A, S\} \) and \( \{A, T\} \) are R-weakly commuting in \( X \)
(iv) \( F_{AX, AY}(t) \geq \psi[\min \{F_{AX, SX}(t), F_{AX, SY}(t), F_{AX, TY}(t), F_{AY, SX}(t)
\]
\[ F_{AY, SY}(t), F_{AY, TY}(t), F_{SX, SY}(t), F_{SX, TY}(t), F_{SY, TY}(t)\}]] \)

for all \( x, y \in X, t > 0 \) and for some \( \psi \in \Psi \).

Then \( A, S \) and \( T \) have unique common fixed point in \( X \). In fact for any \( x_0 \in X \) and sequence \( \{y_n\} \) defined as \( y_n = Ax_n = Sx_{n+1} = Tx_{n+2} \) for all \( n = 0, 1, 2 \ldots \).

Then \( \{y_n\} \) converges to unique common fixed point of \( A, S \) and \( T \) in \( X \).

Proof: First we show that \( \lim_{n \to \infty} F_{y_n, y_{n+1}}(t) = 1 \ \forall \ t > 0 \)

By taking \( x = x_n, y = x_{n+1} \) in condition (ii), we get
\[ F_{AX, AX}(t) \geq \psi(\min \{F_{AX, SX}(t), F_{AX, SY}(t), F_{AX, TY}(t), F_{AX, SX}(t), F_{AX, SX}(t), F_{AX, SX}(t), F_{AX, SX}(t), F_{AX, SX}(t)\}) \]
\[ \Rightarrow F_{y_n, y_{n+1}}(t) \geq \psi(\min \{F_{y_n, y_{n-1}}(t), F_{y_n, y_n}(t), F_{y_n, y_{n+1}}(t), F_{y_n, y_{n-1}}(t), F_{y_n, y_{n+1}}(t)\} \]
\[ F_{y_n, y_{n+1}}(t) \geq \psi(\min \{F_{y_n, y_{n-1}}(t), F_{y_n, y_{n+1}}(t), F_{y_n, y_{n-1}}(t), F_{y_n, y_{n+1}}(t)\}) \] \ (3.3.1)

Case I: Suppose for some \( n, F_{y_n, y_{n+1}}(t) = 1 \).

Then from (3.3.1), we can write
\[ F_{y_{n+1}, y_{n+2}}(t) \geq \psi(\min \{F_{y_n, y_{n+1}}(t), F_{y_n, y_{n+2}}(t), F_{y_n, y_{n+2}}(t)\}) \]

Write \( \alpha = F_{y_n, y_{n+1}}(t), \beta = F_{y_n, y_{n+2}}(t), \gamma = F_{y_n, y_{n+2}}(t) \) \ (3.3.2)
Then, since $X$ is non Archimedean and $\alpha = 1$, by using Lemma 2.2, we get $\beta \geq \psi(\beta)$, a contradiction.

Hence $\beta = 1$

Consequently $F_{y_{n+1},y_{n+2}}(t) = 1 \ \forall \ t > 0$.

Hence by induction $F_{y_m,y_{m+1}}(t) = 1$, for every $t > 0$.

Case I: Suppose $F_{y_n,y_{n+1}}(t) < 1 \ \forall \ t > 0$.

Write $\alpha = F_{y_{n-1},y_n}(t)$, $\beta = F_{y_n,y_{n+1}}(t)$, $\gamma = F_{y_{n-1},y_{n+1}}(t)$

If $\alpha > \min \{\alpha, \beta, \gamma\}$, then from Lemma 2.2, we get $\beta = \gamma = \min \{\alpha, \beta, \gamma\}$.

Then from inequality (3.3.1), we have $\beta \geq \psi(\beta)$, a contradiction.

Therefore $\alpha = \min \{\alpha, \beta, \gamma\}$.

Hence $\beta = F_{y_n,y_{n+1}}(t) \geq \psi \left( F_{y_{n-1},y_n}(t) \right) \geq \ldots \geq \psi^n \left( F_{y_0,y_1}(t) \right) \rightarrow 1$ as $n \rightarrow \infty$

Consequently $\lim_{n \rightarrow \infty} F_{y_n,y_{n+1}}(t) = 1$, for every $t > 0$.

Hence by induction $F_{y_m,y_{m+1}}(t) = 1$, for $m \geq n$.

Therefore from both cases (I) and (II), we obtain $\lim_{n \rightarrow \infty} F_{y_n,y_{n+1}}(t) = 1$, for every $t > 0$.

Hence $\{y_n\}$ is a Cauchy sequence in $X$.

Since $(X, F, \ast)$ is complete, the sequence $\{y_n\}$ converge to a point $z$, say in $X$.

Now by definition of the sequence $\{y_n\}$, we have $y_n \rightarrow z \Rightarrow Ax_n \rightarrow z, Sx_n \rightarrow z$ and $Tz_n \rightarrow z$ as $n \rightarrow \infty$

Also $S, T$ are continuous $\Rightarrow Sy_n \rightarrow Sz$ and $Ty_n \rightarrow Tz$, $SAX_n \rightarrow Sz$ and $TAX_n \rightarrow Tz$.

Since $\{A, S\}$ and $\{A, T\}$ are $R$-weakly commuting,

Hence $F_{Ax_n,Sx_n}(t) \geq F_{Ax_n,Sx_n} \left( \frac{t}{R} \right), R > 0$

and $F_{ATx_n,Tx_n}(t) \geq F_{Ax_n,Tx_n} \left( \frac{t}{R} \right), R > 0 \ldots (3.3.5)$

from (3.3.5), we can write

$$\lim_{n \rightarrow \infty} F_{Ax_n,Sx_n}(t) = \lim_{n \rightarrow \infty} F_{Ay_n,Sy_n}(t) = \lim_{n \rightarrow \infty} F_{Ay_n,Sz}(t) \geq \lim_{n \rightarrow \infty} F_{Ax_n,Sx_n}(t) = \lim_{n \rightarrow \infty} F_{Az}(t) = 1$$

i.e. $Ay_n \rightarrow Sz$ as $n \rightarrow \infty$ .... (3.3.6)

Similarly $\lim_{n \rightarrow \infty} F_{ATx_n,Tx_n}(t) \geq 1$
Now $1 \geq \lim_{n \to \infty} F_{Ay_n,Tz}(t) \geq 1$
\[ \Rightarrow \lim_{n \to \infty} F_{Ay_n,Tz}(t) = 1 \]
\[ \Rightarrow Ay_n \to Tz \quad \ldots \quad (3.3.7) \]
From (3.3.6) and (3.3.7), we get $Sz = Tz$.

Now, by taking $x = Sx_n$ and $y = z$ in condition (ii), we get $Sz = Az$.

Hence $Az = Sz = Tz$.

By taking $x = x_{n+1}$ and $y = z$ in condition (ii), we get $z = Az$.

Hence $z = Az = Sz = Tz$.

Therefore $z$ is a common fixed point of $A$, $S$, and $T$.

To prove uniqueness, let $x$ be a common fixed point of $A$, $S$, and $T$.

From condition (ii), we have

\[ F_{Ax,Az}(t) \geq \psi(\min\{F_{Ax,Sx}(t), F_{Ax,Sz}(t), F_{Ax,Tz}(t), F_{Az,Sx}(t), F_{Az,Sz}(t), F_{Az,Tz}(t), F_{Sx,Sz}(t), F_{Sx,Tz}(t), F_{Sz,Tz}(t))\}) \]
\[ \Rightarrow F_{x,x}(t) \geq \psi(\min\{F_{x,x}(t), F_{x,z}(t), F_{x,z}(t), F_{z,z}(t), F_{z,x}(t), F_{z,x}(t), F_{z,z}(t), F_{z,z}(t), F_{z,z}(t)\} \]
\[ \Rightarrow F_{x,x}(t) \geq \psi(F_{x,x}(t)) \]
\[ \Rightarrow F_{x,x}(t) = 1 \]
\[ \Rightarrow x = z \]

Therefore $z$ is a unique common fixed point of $A$, $S$, and $T$.

**Example 3.4**: Let $X = \{0, 1\}$ with metric $d$ is defined as $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$ and $*$ is min t-norm and $F_{xy}(t) = \frac{t}{t+d(x,y)}$.

Define $g(t) = 1 - t$ and $\varphi(t) = \eta t$, $0 < \eta < 1$. Let $A$, $S$, and $T$ be self-maps on $X$ such that $A(x) = x_0 \\forall x \in X, x_0 \in X$, $S = T = I$. Then $(X, F, *)$ is complete non-Archimedean Menger space, $g \in \Omega$ and $\varphi \in \Phi$. $A$, $S$, and $T$ satisfies all conditions.
of Theorem 2.2 and Theorem 3.3. $x_0$ is the unique common fixed point of A, S and T in X.

References


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