On $\omega$-Confluent Mappings

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Abstract

In this work, we introduce the notion of $\omega$-continuum sets and obtain properties of this class of sets. By using these sets, we introduce a new class of mappings, called $\omega$-confluent maps, and we study the relation between these mappings and the classical confluent mappings. Also, we study some operations such as factorizations, pullbacks, compositions and products of $\omega$-confluent mappings.

Keywords: continuum sets, confluent maps, $\omega$-continuum sets, $\omega$-confluent maps

1 Introduction

From a topological space $(X, \tau)$ (or for brevity $X$), there are many ways one can generalize the notion of open sets and then derive many interesting results using this new notion. One notion in particular that has received much attention lately is the so-called $\omega$-open sets which can be characterized as follows [1]: A subset $W$ of a space $(X, \tau)$ is an $\omega$-open set if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. It is well known that the family of all $\omega$-open subsets of a space $(X, \tau)$, denoted by $\tau_w$, forms a topology on $X$ finer than $\tau$. Using this notion of $\omega$-open sets one can then define notions such as $\omega$-compact and $\omega$-connected sets whose definitions follow closely the definitions of the related classical notions. For example, a space $X$ is called $\omega$-connected provided that $X$ is not the union of two disjoint nonempty $\omega$-open sets. And $X$ is said to be $\omega$-compact if every $\omega$-open cover of $X$ has a finite subcover. For more information regarding these notions and some recent related results, see [2], [3], [4].

Recall that a subset $K$ of a space $X$ is said to be a continuum if $K$ is connected and compact. Using this idea of a continuum, Charatonik introduced
and studied the idea of a confluent mapping between topological spaces [5] (see later for definitions).

In this paper we are interested in the natural generalizations of the work of Charatonik in the context of \(\omega\)-open sets. In particular, we start off by using the idea of \(\omega\)-open sets to extend the notion of continuum sets. Thus, in particular we introduce and study the notion of \(\omega\)-continuum sets by using the notions of \(\omega\)-compact and \(\omega\)-connected sets. Then motivated by the work of Charatonik on confluent maps, we shall then define the notion of an \(\omega\)-confluent map and study its relation with the classical confluent maps. We also study operations like compositions, pullback, factorizations and products of \(\omega\)-confluent mappings.

\section{\(\omega\)-Confluent Mappings}

In this section, we introduce the main focus of this paper which is the \(\omega\)-confluent mappings. Thus, we recall the basic definitions needed in this work. Throughout this paper all mappings are assumed to be continuous.

\textbf{Definition 2.1.} [2] A space \(X\) is called \(\omega\)-connected provided that \(X\) is not the union of two disjoint nonempty \(\omega\)-open sets.

\textbf{Definition 2.2.} [2] A space \(X\) is said to be \(\omega\)-compact if every \(\omega\)-open cover of \(X\) has a finite subcover.

Now, we define the following notion.

\textbf{Definition 2.3.} A space \(X\) is said to be an \(\omega\)-continuum if, it is \(\omega\)-connected and \(\omega\)-compact at the same time. A subset \(K\) of a space \(X\) is said to be \(\omega\)-continuum if, \(K\) is \(\omega\)-connected and \(\omega\)-compact as a subspace of \(X\).

\textbf{Theorem 2.4.} Let \(X\) be a topological space. Then:

(1) Every \(\omega\)-connected subset \(K\) of \(X\) is connected;

(2) Every \(\omega\)-compact subset \(K\) of \(X\) is compact;

(3) Every \(\omega\)-continuum subset \(K\) of \(X\) is a continuum.

\textbf{Proof.} (1) Suppose that \(K\) is not a connected subset of \(X\). Then there exist nonempty disjoint open sets \(V\) and \(U\) such that \(K = U \cup V\). Since every open set is an \(\omega\)-open then \(K\) is not \(\omega\)-connected. This is a contradiction. This means that \(K\) is connected.

The proof of (2) and (3) are in the same manner of (1). \hfill \blacksquare
On \( \omega \)-confluent mappings

The converse of the above theorem is not true in general as the following example shows:

**Example 2.5.** Let \( X = \mathbb{Z} \) be the integer number with topologies:

- \( \tau = \{ \emptyset, X, \mathbb{Z}^+, \mathbb{Z}^- \} \);
- \( \sigma = \{ \emptyset, X, \mathbb{Z} - \{0, 1\} \} \);
- \( \gamma = \{ \emptyset, \mathbb{Z}, \mathbb{Z} - \{0\}, \mathbb{Z} - \{1\}, \mathbb{Z} - \{0, 1\} \} \).

One can deduce that \((X, \sigma)\) is connected but not \( \omega \)-connected, \((X, \tau)\) is compact but not \( \omega \)-compact, and \((X, \gamma)\) is a continuum but not an \( \omega \)-continuum.

The following is due to Charatonik.

**Definition 2.6.** [5] A mapping \( f : X \rightarrow Y \) is said to be confluent provided that for each continuum \( K \) of \( Y \) and for each component \( C \) of \( f^{-1}(K) \) we have \( f(C) = K \).

Now, we can introduce the following definition.

**Definition 2.7.** A mapping \( f : X \rightarrow Y \) is said to be \( \omega \)-confluent provided that for each \( \omega \)-continuum \( K \) of \( Y \) and for each component \( C \) of \( f^{-1}(K) \) we have \( f(C) = K \).

The following proposition shows the relation between a confluent mapping and \( \omega \)-confluent mapping.

**Proposition 2.8.** Every confluent mapping \( f : X \rightarrow Y \) is an \( \omega \)-confluent mapping.

Proof. Let \( K \) be any \( \omega \)-continuum in \( Y \), and let \( C \) be any component in \( f^{-1}(K) \). Then \( K \) is a continuum subset of \( Y \) by Theorem 2.4. Since \( f \) is confluent mapping, we have \( f(C) = K \). Thus, \( f \) is \( \omega \)-confluent mapping. \( \square \)

**Remark 2.9.** The \( \omega \)-component of \( p \) in \( X \) is the largest \( \omega \)-connected subset of \( X \) containing \( p \) and we denote it by \( C_\omega(X, p) \). Then from Theorem 2.4 \( C_\omega(X, p) \) is a connected subset of \( X \). Since the component of \( p \) in \( X \), \( C(X, p) \), is the largest connected subset of \( X \) containing \( p \) we then have

\[
C_\omega(X, p) \subseteq C(X, p).
\]

In general we cannot expect the \( \omega \)-components to be equal to the (usual) components of \( X \) (see the example below). Of course when equality holds using \( \omega \)-components instead of just components in Definition 2.7 will also make Proposition 2.8 holds true.
Example 2.10. Let $X = \mathbb{Q} \cap [0, 1]$ with topologies: $\tau_1 = \{\phi, X, X \cap \mathbb{Z}\}$, and $\tau_2 = \{\phi, X, X \cap [0, \frac{1}{2}), X \cap [\frac{1}{2}, 1]\}$. Then $X$ has only one component $C(X) = X$ and the $\omega$-components of $X$ is $C_\omega(X) = \{x\}$ for each $x \in X$, with respect to $\tau_1$.

Also the components of $X$ are $C_1 = X \cap [0, \frac{1}{2})$, and $C_2 = X \cap [\frac{1}{2}, 1]$ and the $\omega$-components of $X$ is $C_\omega((X, \tau_1)) = \{x\}$ for each $x \in X$, with respect to $\tau_2$.

Remark 2.11. A mapping which is $\omega$-confluent is not necessarily confluent as shown by the following examples:

Example 2.11.1. Let $\mathbb{R}$ be the real number with the lower limit topology and $X = \{0, 1\}$, with topology $\tau = \{\phi, X, \{0\}\}$.

Define $f : \mathbb{R} \to X$ as follows:

$$f(x) = \begin{cases} 
1, & \text{if } x \geq 0; \\
0, & \text{if } x < 0.
\end{cases}$$

The mapping $f$ is $\omega$-confluent but not confluent. Since if we take the continuum $K = \{0, 1\}$ then any components of $f^{-1}(K) = \mathbb{R}$, are the one-point sets. If $x \in [0, \infty)$, then $C(\mathbb{R}, x) = \{x\}$. Thus $f(C) = \{1\} \neq K$. Also, if $x \in (-\infty, 0)$, then $C(\mathbb{R}, x) = \{x\}$ and $f(C) = \{0\} \neq K$.

Example 2.11.2. Let $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and $Y = \{1, 2, 3\}$ with $\sigma = \{\phi, Y, \{1\}, \{2\}, \{1, 2\}\}$. Define a mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = 1$, $f(b) = 2$, $f(c) = 3$. The mapping $f$ is $\omega$-confluent but not confluent. Since, if we take the continuum $K = \{1, 3\}$.

Then the components of the inverse image $f^{-1}(K)$ are $C = \{a\}$, $C' = \{c\}$ and $f(C) = 1 \neq K$, $f(C') = 3 \neq K$. Therefore $f$ is not confluent.

Now, we recall the following definition that will be needed in the following theorem.

Definition 2.12. [2] A space $(X, \tau)$ is said to be $\omega$-space if every $\omega$-open set is open in $X$.

The following theorem shows that under certain condition, $\omega$-confluent mappings will be confluent.

Theorem 2.13. Every $\omega$-confluent mapping $f : X \to Y$ of space $X$ into $\omega$-space $Y$ is confluent.

Proof. Let $K$ be any continuum in $Y$, and let $C$ be any component of $f^{-1}(K)$. Since $Y$ is an $\omega$-space, then every $\omega$-open set is open in $X$, by Definition 2.12. So, $\tau_\omega \subseteq \tau$, but for any space $(X, \tau)$, we have $\tau \subseteq \tau_\omega$. Thus $\tau_\omega = \tau$. Then the
continuum subsets of $X$ are coincide with $\omega$-continuum subsets of $X$. So, we have $f(C) = K$. Thus $f$ is confluent. ■

Before we get to the next theorem, we need to state the following definition and lemma.

**Definition 2.14.** [4] A space $(X, \tau)$ is called locally countable, if each point $x \in X$ has a countable open neighborhood.

**Lemma 2.15.** [4] If $X$ is a locally countable space, then $\tau_\omega$ is the discrete topology on $X$.

**Theorem 2.16.** If $Y$ is a locally countable space, then any mapping $f : X \rightarrow Y$ of space $X$ into space $Y$ is $\omega$-confluent.

Proof. Suppose that $Y$ is a locally countable space. Then $\tau_\omega = P(X)$ where $P(X)$ is the power set of $X$, by Lemma 2.15. Thus the $\omega$-continuum sets of $Y$ are only the singletons sets $K = \{x\}$ for each $x \in X$. So, for any component $C$ of $f^{-1}(K)$, we have $f(C) = K = \{x\}$. Hence $f$ is an $\omega$-confluent mapping. ■

**Theorem 2.17.** For any topological space $X$, let $R_C$ be the equivalence relation determined by its components: $x \sim x'$ if and only if $x$ and $x'$ lie in a common component. Let $X/R_C$ with identification topology determined by the projection $p : X \rightarrow X/R_C$. Then the following statements are equivalent:

1. The projection $P$ is confluent mapping;
2. The projection $P$ is an $\omega$-confluent mapping.

Proof. (1)$\Rightarrow$(2): Obvious.
(2)$\Rightarrow$(1): Since $X/R_C$ is the equivalence relation determined by the components of $X$, then the quotient space $X/R_C$ is totally disconnected. Then the components of $X/R_C$ is $C(X/R_C, y) = y$ for each $y \in X/R_C$. Therefore the class of all $\omega$-continuum sets in $X/R_C$ is coincide with class of all continuum sets in $X/R_C$. Thus $P$ is an $\omega$-confluent mapping. ■

### 3 Composition and Factorization of $\omega$-Confluent Mappings

In this section, we study the composition of $\omega$-confluent mappings. Thus, we need the following lemma.

**Lemma 3.1.** Let $f : X \rightarrow Y$ be a surjective mapping and $g : Y \rightarrow Z$ be any mapping, such that $f$ is $\omega$-confluent. If $h = gof$, then for each
ω-continuum $K$ in $Z$, and for each component $C$ of $h^{-1}(K)$, we have $f(C)$ is a component of $g^{-1}(K)$.

Proof. Let $K$ be any ω-continuum in $Z$ and let $C$ be any component of $h^{-1}(K)$. Then $f(C) \subseteq g^{-1}(K)$. Let $S$ be a component of $g^{-1}(K)$. Then, We have to show that $f(C) = S$. Since $f(C)$ is connected in $g^{-1}(K)$ by the continuity of $f$. So, $f(C) \subseteq S$. Then $C \subseteq f^{-1}(S)$. Thus,

$$ C \subseteq f^{-1}(S) \subseteq h^{-1}(K). $$

Then $C$ will be a component of $f^{-1}(S)$. Let $Q \subseteq S$ be any ω-continuum. Then $f^{-1}(Q) \subseteq f^{-1}(S)$. But $C$ is the component of $f^{-1}(S)$. So, that $C \cap f^{-1}(Q)$ is a component of $f^{-1}(Q)$, and since $f$ is an ω-confluent, then we get $f(C \cap f^{-1}(Q)) = Q$. So, $f(C \cap f^{-1}(Q)) = f(C) \cap Q$. This implies $Q = f(C) \cap Q$. Thus, $Q \subseteq f(C)$. But $Q \subseteq S$. Then we get that $f(C) = S$. ■

Now, we are ready to state and prove the following theorem.

**Theorem 3.2.** Let $f : X \to Y$ and $g : Y \to Z$ are two ω-confluent mappings, where $f$ is a surjective. Then $h = gof$ is an ω-confluent mapping.

Proof. Let $K$ be any ω-continuum in $Z$, and let $C$ be any component of $f^{-1}(K)$. Since $f$ is an ω-confluent, then $f(C)$ is a component of $g^{-1}(K)$ by Lemma 3.1. Since $g$ is an ω-confluent mapping, then we get

$$ gof(C) = g(f(C)) = K. $$

Therefore $h = gof$ is an ω-confluent mapping. ■

**Theorem 3.3.** Let $f : X \to Y$ and $g : Y \to Z$ are two mappings, where $f$ is a surjective. If $h = gof$ is an ω-confluent mapping, then $g$ is also ω-confluent mapping.

Proof. Let $K$ be any ω-continuum in $Z$, let $C$ be any component of $g^{-1}(K)$, and let $S$ be any component of $h^{-1}(K)$. Then $f^{-1}(C) \subseteq h^{-1}(K)$. Let $S \subseteq f^{-1}(C)$. Since $h$ is an ω-confluent, then $h(S) = K$. Since $f(S) \subseteq C$. So, that $K \subseteq g(C)$ by the continuity of $g$. But we have $g(C) \subseteq K$. Thus, $g(C) = K$. Therefore $g$ is also ω-confluent. ■

**Remark 3.4.** In the above theorem $f$ is not necessarily ω-confluent mapping as shown by the following example:

**Example 3.4.1.** Let $X = Y = \mathbb{R}$, and $Z = \{a, b\}$ with topologies: the upper limit topology on $X = \mathbb{R}$, the standard topology on $Y = \mathbb{R}$, and $\tau_x = \{\phi, Z, \{a\}\}$ on $Z$. Let $f : X \to Y$ be a mapping defined by $f(x) = x$, $\forall x \in X$, and let $g : Y \to Z$ be a mapping defined by:

$$ g(x) = \begin{cases} a, & \text{if } x \in (0, \infty); \\
 b, & \text{if } x \in (-\infty, 0]. \end{cases} $$

Then $h = gof$ is ω-confluent, but $f$ is not ω-confluent.
Now, we show that under certain conditions, the mapping $f$ will be $\omega$-confluent.

Let $f : X \to Y$ be a mapping. A set $U \subseteq X$ is said to be an inverse set if and only if, $U = f^{-1}(f(U))$, [5].

**Theorem 3.5.** If $Z$ is an $\omega$-space, and if $f : X \to Y$ and $g : Y \to Z$ are two mappings, where $f$ is a surjective, then if $h = gof$ is an $\omega$-confluent mapping, then $f$ is an $\omega$-confluent, such that every set in $Y$ is an inverse set.

**Proof.** Let $K$ be any $\omega$-continuum in $Y$, and let $C$ be any component in $f^{-1}(K)$. Since, $Z$ is an $\omega$-space, then the continuum and $\omega$-continuum sets in $Z$ are the same. Thus, $g(K)$ is $\omega$-continuum(or continuum) in $Z$ by the continuity of $g$. Since $g^{-1}(g(K))$ is a subset of $Y$, then from assumption $g^{-1}(g(K)) = K$. Thus, $h^{-1}(g(K)) = f^{-1}og^{-1}(g(K)) = f^{-1}(K)$.

This implies $C$ is a component of $h^{-1}(g(K))$. Since $h$ is an $\omega$-confluent mapping, then $h(C) = g(K)$. But, we have $h^{-1}(g(K)) = f^{-1}(K)$. So that, $f(C) = K$. Then, $f$ is an $\omega$-confluent mapping. ■

Now, we study a direct consequence of Whyburn’s factorization theorem for $\omega$-confluent mappings. First, we recall the definition of a factorable mapping.

**Definition 3.6.** [6]. If $f : X \to Z$ is a mapping, any representation of $f$ in the form $f = f_2of_1$ where $f_1 : X \to Y$ and $f_2 : Y \to Z$ are two mappings and $Y$ is a certain space, will said to be factorization of $f$. Here $f$ is said to be a factorable mapping and $Y$ is a middle space.

Now, we can get the factorization of a $\omega$-confluent mapping in the following result.

**Theorem 3.7.** If $f : X \to Z$ is an $\omega$-confluent of strongly connected compact space $X$ into Hausdorff space $Z$, then there exists a unique factorization for $f$ into two $\omega$-confluent mappings:

$$f(x) = f_2of_1(x), \forall x \in X,$$

such that $f_1$ is confluent mapping.

**Proof.** First, we have to show that $f$ has factorizations into $\omega$-confluent maps. Let $f = f_2of_1$, where $f_1 : X \to Y$ and $f_2 : Y \to Z$ are two mappings and $Y$ is a certain space. Since $Z$ is Hausdorff space, then for each $\omega$-continuum $K$ in $Z$ is closed set. So, that $f_2^{-1}(K)$ is closed in $Y$. But we have $Y$ is strongly connected compact by the continuity of $f_1$. Therefore, $f_2^{-1}(K)$ has only one component $C = f_2^{-1}(K)$. So, $f_2(C) = K$. Thus $f_2$ is an $\omega$-confluent mapping.
Since $f_1$ is confluent mapping. Then by the Proposition 2.8, $f_1$ is also $\omega$-confluent. Therefore $f$ has a factorization into two $\omega$-confluent mappings.

Finally, we prove the uniqueness. Suppose that $f = f_2 \circ f_1 = goh$, where $g$ and $h$ be any mappings. Then $f(x) = f_2 \circ f_1(x) = goh(x)$, $\forall x \in X$. So, that $f_2(f_1(x)) = g(h(x))$, $\forall x \in X$. This implies $f_1(x) = h(x)$, $\forall x \in X$. Thus, $f_1 = h$ which implies $f_2 = g$. ■

Now, we state the following theorem which is needed to prove the next proposition.

**Theorem 3.8.** [5] If $X$ is compact and $f : X \rightarrow Y$ is a confluent mapping of $X$ into $Y$, then there exists a unique factorization for $f$ into two confluent mappings in the form $h = gof$.

**Proposition 3.9.** Let $f : X \rightarrow Y$ be an $\omega$-confluent mapping of compact space $X$ into space $Y$. If $Y$ is totally disconnected, then $f$ has unique factorization in the form $h = gof$, into two confluent mappings $f$, and $g$.

Proof. Since $Y$ is totally disconnected, then the component $C(Y, y) = y$ for each $y \in Y$. So, that that the class of all continuum set in $Y$ coincide with class of all $\omega$-continuum set. Thus $f$ is also confluent mapping. Therefore $f$ has unique factorization in the form $h = gof$, into two confluent mappings $f$ and $g$ by Theorem 3.8. ■

4 Product of $\omega$-confluent Mappings

Let $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ be any two families of topological spaces. The product space of $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ is denoted by $\prod_{i \in I} X_i$ and $\prod_{i \in I} Y_i$, respectively. Let $f_i : X_i \rightarrow Y_i$ be a mapping for each $i \in I$. Let $f : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$ be the product mappings as follows: $f((x_i)) = (f_i(x_i))$ for each $(x_i) \in \prod_{i \in I} X_i$. The projection of $\prod_{i \in I} X_i$ and $\prod_{i \in I} Y_i$ onto $X_i$ and $Y_i$, respectively is denoted by $p_i$ and $q_i$.

Before we prove the main result in this section, we state and prove the following result:

**Lemma 4.1.** [7] (a) If $A$ is an $\omega$-open subset of a space $(X, \tau)$, then $A - C$ is $\omega$-open for every countable subset $C$ of $X$.

(b) The open image of an $\omega$-open set is $\omega$-open.
Theorem 4.2. Let \((X_i, \tau_i)_{i \in I}\) be an indexed family of topological spaces. Then

\[
\left( \prod_{i \in I} \tau_i \right)_\omega \subseteq \prod_{i \in I} (\tau_i)_\omega, \forall i \in I.
\]

Proof. Let \(W \in \left( \prod_{i \in I} \tau_i \right)_\omega\). Then for each \(x = (x_i)_{i \in I} \in W\), there exists an open set \(A\) in \(\prod_{i \in I} X_i\) containing \(x\). Thus, there exists an open set \(\prod_{i \in I} B_i\) with \(x \in \prod_{i=1}^n B_i \times \prod_{i \neq j} X_j \subseteq A\). So \(A = \bigcup (\prod_{i=1}^n B_i \times \prod_{i \neq j} X_j), \) such that

\[
\bigcup_{i=1}^n (\prod_{i \neq j} X_j) - W, \text{ is countable.}
\]

Where \(B_i\) is open in \(X_i\). Put

\[
W_i = (B_i \cap p_i(W)) - (P_i(A - W) - \{x_i\}),
\]

where \((B_i \cap p_i(W))\) is an \(\omega\)-open in \(X_i\) and \((P_i(A - W) - \{x_i\})\) is countable, and \(p_i : \prod_{j \in I} X_j \to X_i\) is the natural projection for each \(i \in I\). Then \(W_i \in (\tau_i)_\omega\), by using the (Lemma 4.1), and \(x = (x_i) \in \prod_{i \in I} W_i \subseteq W\). Thus, \(W \in \prod_{i \in I} (\tau_i)_\omega\). ■

The following example shows the \(\omega\)-product topology is, in general, a proper subset of the product of \(\omega\)-topologies.

Example 4.3. Let \(X = \mathbb{R}\) with the usual topology \(\tau_u\) and \(Y = \mathbb{Z}\) with cofinite topology \(\tau_{cof}\). Then the product topology is \((X \times Y, \tau_u \times \tau_{cof})\). Take \(A = \{1\}\) be an \(\omega\)-open in \(Y = \mathbb{Z}\) and \(B = \mathbb{R} - \mathbb{Q}\) be an \(\omega\)-open in \(X = \mathbb{R}\). Then \(A \times B \in (\tau_u)_\omega \times (\tau_{cof})_\omega\), but \(A \times B \not\in (\tau_u \times \tau_{cof})_\omega\). Thus

\[
(\tau_u \times \tau_{cof})_\omega \subset (\tau_u)_\omega \times (\tau_{cof})_\omega.
\]

Theorem 4.4. Let \(f_i : X_i \to Y_i\) be an \(\omega\)-confluent mapping, for each \(i \in I\) of space \(X_i\) into Hausdorff space \(Y_i\). Then

\[
f : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i
\]

is an \(\omega\)-confluent mapping if the following equality hold,

\[
\prod_{i \in I} \tau_i \subseteq \prod_{i \in I} (\tau_i)_\omega, \forall i \in I.
\]
Proof. Let $K$ be any $\omega$-continuum in $\prod_{i \in I} Y_i$ and $C$ be any component of $f^{-1}(K)$. Since $Y_i$ is a Hausdorff then $\prod_{i \in I} Y_i$ is Hausdorff. Thus, $K$ is closed. Suppose that $K = \prod_{i \in I} S_i$ for $S_i \subseteq Y_i$. Then $\prod_{i \in I} S_i$ is an $\omega$-continuum in $(\prod_{i \in I} Y_i, (\prod_{i \in I} \tau_i)\omega)$. Since $(\prod_{i \in I} \tau_i)\omega = \prod_{i \in I} (\tau_i)\omega$, $\forall i \in I$, then $K = \prod_{i \in I} S_i$ is an $\omega$-continuum in $(\prod_{i \in I} Y_i, \prod_{i \in I} (\tau_i)\omega)$. Thus, $S_i$ is an $\omega$-continuum in $Y_i$ for $i \in I$. But, we have $K = \prod_{i \in I} S_i$. So, $f^{-1}(K) = \prod_{i \in I} f^{-1}(S_i)$. Let $x = (x_i)_{i \in I}$ be any point in $K = \prod_{i \in I} S_i$, then $C(f^{-1}(K), x) = \prod_{i \in I} C(f^{-1}(S_i), x_i)$. Since $f_i$ is an $\omega$-confluent, then $f_i(C(f_i^{-1}(S_i), x_i)) = S_i$, for each $i \in I$. Therefore, $f(C(f^{-1}(K), x)) = \prod_{i \in I} f_i(C(f_i^{-1}(S_i), x_i)) = \prod_{i \in I} S_i = K$. Hence, $f$ is an $\omega$-confluent mapping.

**Theorem 4.5.** Let $f_i : X_i \rightarrow Y_i$ be an $\omega$-confluent mapping, for each $i \in I$ of space $X_i$ into Hausdorff $\omega$-space $Y_i$. Then the product mapping

$$f : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i,$$

is $\omega$-confluent.

Proof. The Proof is similar to the Proof of the Theorem 4.4.

**Theorem 4.6.** Let $\{X_i\}_{i \in I}$ be any families of strongly connected compact Hausdorff spaces for each $i \in I$. Then any projection $p_j : \prod X_i \rightarrow X_j$ is an $\omega$-confluent.

Proof. Let $K_j$ be any $\omega$-continuum in $X_j$, and let $C_j$ be any component of $p_j^{-1}(K_j)$. Then $K_j$ is a continuum by Theorem 2.4. Since $X_j$ Hausdorff space, then $K_j$ is closed in $X_j$. Also $p_j^{-1}(K_j)$ is closed in $\prod X_i$. Since the product of an arbitrary family of strongly connected compact spaces is strongly connected compact. So, that $\prod X_i$ is strongly connected compact space. Thus $p_j^{-1}(K_j)$ is connected. Therefore, $C = p_j^{-1}(K_j)$. Thus, $p_j(K_j) = K_j$. There on, $p_j$ is an $\omega$-confluent mapping, for each $j \in X$.

## 5 Pullback of $\omega$-Confluent Mappings

In this section, we study the pullback of $\omega$-confluent mappings. So, we recall the following definitions:

**Definition 5.1.** [8] A fiber structure is a triple $(X, f, Y)$ consisting of two spaces $X$ and $Y$, and a mapping $f : X \rightarrow Y$. The space $X$ is said to be the fibered (or, total) space, $f$ is termed the projection, and $Y$ is the base space.

Next, we recall the definition of the pullback.
**Definition 5.2.** [8] Let \((X, f, Y)\) be a fiber structure. Let \(Z\) be any space and let \(g : Z \to Y\) be any mapping into the base \(Y\). Let \(E_f\) be a subspace of the cartesian product \(X \times Z\) where \(E_f = \{(x, z) : f(x) = g(z)\}\), and let \(p : E_f \to Z\) be the projection of \(E_f\) onto \(Z\) such that \(p(x, z) = z\), \(\forall (x, z) \in E_f\). The fiber structure \((E_f, p, Z)\) is said to be the fiber structure over \(Z\) induced by the mapping \(g\) and the projection \(p\) is said to be the pullback of \(f\) by \(g\).

Now, let \(\gamma : E_f \to X\) be the projection such that \(\lambda(x, z) = x\), \(\forall (x, z) \in E_f\). We observe that the following diagram is commutative.

\[
\begin{array}{ccc}
E_f & \xrightarrow{\lambda} & X \\
p \downarrow & & \downarrow f \\
\quad Z & \quad \xrightarrow{g} & \quad Y
\end{array}
\]

Before we prove the main results in this section, we state and prove the following lemma.

**Lemma 5.3.** Let \(f : X \to Y\) be a mapping, let \(Z\) be any space, and let \(g : Z \to Y\) be any mapping, if \(K \subseteq Z\), then \(p^{-1}(K) = f^{-1}(g(K)) \times K\), where \(p\) is the pullback of \(f\) by \(g\).

**Proof.** Let \((x, z) \in p^{-1}(K)\). Then, \(p(x, z) \in K\). So, \(z \in K\), and we get \(g(z) \in g(K)\). Then \(f^{-1}(g(z)) \subseteq f^{-1}(g(K))\). Since \(f(x) = g(z)\), then \(x \in f^{-1}(g(z))\). Thus, \(x \in f^{-1}(g(K))\). That is, \((x, z) \in f^{-1}(g(K)) \times K\). This implies \(p^{-1}(K) \subseteq f^{-1}(g(K)) \times K\). Conversely, let \((x, z) \in f^{-1}(g(K)) \times K\). Then \(x \in f^{-1}(g(K))\) and \(z \in K\). Thus \(f(x) \in g(K)\) and \(g(z) \in g(K)\). But \(f(x) = g(z)\). So, \(p(x, z) \in K\). Therefore \((x, z) \in p^{-1}(K)\). This means \(f^{-1}(g(K)) \times K \subseteq p^{-1}(K)\). Then, \(f^{-1}(g(K)) \times K = p^{-1}(K)\). 

**Theorem 5.4.** The pullback of confluent mapping is an \(\omega\)-confluent mapping.

**Proof.** Let \(f : X \to Y\) be a confluent mapping, and let \(Z\) be any space and let \(g : Z \to Y\) be an mapping. Let \(K \subseteq Z\) be any \(\omega\)-continuum, and let \(C\) be an arbitrary component of \(p^{-1}(K)\). Then \(p^{-1}(K) = f^{-1}(g(K)) \times K\) by Lemma 5.3. Then \(K\) is continuum in \(Z\) by Theorem 2.4. Then \(g(K)\) is continuum in \(Y\) by the continuity of \(g\). Since, \(f\) is confluent mapping, then \(f(S) = g(K)\), for each \(S\) component of \(f^{-1}(g(K))\). Then

\[
K = p(f^{-1}(g(K) \times K) = p(S \times K),
\]

such that \(C = S \times K\) for some component \(S\) of \(f^{-1}(g(K))\).
So, that,

\[ p(C) = p(S \times K) = K. \]

Therefore, \( p \) is an \( \omega \)-confluent mapping. \( \blacksquare \)

**Remark 5.5.** If \( f : X \rightarrow Y \) is an \( \omega \)-confluent mapping then the pullback \( p \) of \( f \) by \( g \) is not necessary \( \omega \)-confluent mapping as shown by the following example:

**Example 5.5.1.** Let \( X = \mathbb{R} \), \( Y = \{1, 0\} \), and \( Z = \mathbb{R} \), with topologies: The standard topology on \( X = \mathbb{R} \), \( \tau_Y = \{\phi, Y, \{1\}\} \), and \( Z = \mathbb{R} \) with cofinite topology. Let \( f : X \rightarrow Y \) a mapping defined by:

\[
    f(x) = \begin{cases} 
    1, & \text{if } x > 0; \\
    0, & \text{if } x \leq 0. 
    \end{cases}
\]

and let \( g : Z \rightarrow Y \) be a mapping defined by:

\[
    g(z) = \begin{cases} 
    1, & \text{if } z \in \mathbb{R} \setminus \{1, 2, 3\}; \\
    0, & \text{if } z \in \{1, 2, 3\}. 
    \end{cases}
\]

Note that \( f \) is \( \omega \)-confluent.

Let \( E_f \) be a subspace of the cartesian product \( X \times Z \) where

\[
    E_f = \{(x, z) : f(x) = g(z)\}.
\]

Then the pullback of \( f \) by \( g \) is the projection \( p : E_f \rightarrow Z \) which is defined by

\[
    p(x, z) = z, \forall (x, z) \in E_f.
\]

The projection \( p \) is not \( \omega \)-confluent mapping. Since, if we take the \( \omega \)-continuum \( K = [-3, \infty) \subset Z \), then by the lemma 5.3 we get \( p^{-1}(K) = f^{-1}(g(K)) \times K \). But, \( g(K) = \{0, 1\} \) is not \( \omega \)-continuum in \( Y \).

Under certain condition, the pullback \( p \) of \( \omega \)-confluent mapping \( f \) is \( \omega \)-confluent, as shown by the following theorem.

**Theorem 5.6.** Every pullback of an \( \omega \)-confluent mapping \( f : X \rightarrow Y \) of space \( X \) into \( \omega \)-space \( Y \) is \( \omega \)-confluent.

Proof. Let \( Z \) be any space and let \( g : Z \rightarrow Y \) be any mapping. Let \( K \) be any \( \omega \)-continuum in \( Z \), and \( C \) be any component of \( p^{-1}(K) \). Then \( K \) is continuum in \( Z \) by Theorem 2.4, and \( g(K) \) is continuum in \( Y \). Since \( Y \) is an \( \omega \)-space, then by the theorem 2.13 \( f \) is confluent mapping. Then \( p \) is an \( \omega \)-confluent by Theorem 5.4. \( \blacksquare \)
References


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