A Note on Binomial and Trinomial Operator Representations of Certain Polynomials

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Abstract

Based on the technique used by M.A. Khan and A.K. Shukla [4] here finite series representations of binomial partial differential operators have been used to establish operator representations of various polynomials.

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1. INTRODUCTION:

Recently in 2009, M.A. Khan and A.K. Shukla [4] evolved a new technique to give operator representations of certain polynomials. They gave binomial and trinomial operator representation of certain polynomials. The aim of the present paper is to strengthen the technique evolved by obtaining binomial operator representations of polynomials by taking the functions independent of n.

2. THE DEFINITION, NOTATIONS AND RESULTS USED:

In deriving the operational representations of various polynomials use has been made of the fact that

\[ D^\mu x^\lambda = \frac{\Gamma(1 + \lambda)}{\Gamma(1 + \lambda - \mu)} x^{\lambda - \mu}, \quad D \equiv \frac{d}{dx} \]  

(2.1)

where \( \lambda \) and \( \mu \), \( \lambda \geq \mu \) are arbitrary real numbers. In particular, use has been made of the following results:
\[ D^r e^{-x} = (-1)^r e^{-x} \] (2.2)

\[ D^r x^{-\alpha} = (\alpha)_r (-1)^r x^{-\alpha-r}, \alpha \text{ is not an integer} \] (2.3)

\[ D^r x^{-\alpha-n} = (\alpha + n)_r (-1)^r x^{-\alpha-n-r} \] (2.4)

\[ D^{n-r} x^{\alpha-1+n} = \frac{(\alpha)_n}{(\alpha)_r} x^{\alpha-1+r} \] (2.5)

\[ D^{n-r} x^{-\alpha} = \frac{(\alpha)_n (-1)^n}{(1 - \alpha - n)_r} x^{-\alpha-n+r}, \alpha \text{ is not an integer} \] (2.6)

Where \( n \) and \( r \) are denote positive integers and

\[ (\alpha)_n = \alpha (\alpha + 1) \cdots (\alpha + n-1); \quad (\alpha)_0 = 1 \]

We also need the definitions of the following polynomials in terms of hypergeometric function and also their notations (see [1],[11], [12]).

**LAGUERRE POLYNOMIALS**

It is denoted by the symbol \( L_n^{(\alpha)}(x) \) and is defined as

\[ L_n^{(\alpha)}(x) = \frac{(1 + \alpha)_n}{n!} \, _1F_1 \left[ \begin{array}{c} -n; \\ 1 + \alpha; \\ x \end{array} \right] \] (2.7)

**JACOBI POLYNOMIAL**

It is denoted by the symbol \( P_n^{(\alpha,\beta)}(x) \) and is defined as

\[ P_n^{(\alpha,\beta)}(x) = \frac{(1 + \alpha)_n}{n!} \, _2F_1 \left[ \begin{array}{c} -n, 1 + \alpha + \beta + n; \\ 1 + \alpha; \\ 1 - x \end{array} \right] \] (2.8)

**GAGENBAUER POLYNOMIAL**

The Gagenbauer polynomial \( C_n^v(x) \) is the generalization of Legendre polynomial and is defined as

\[ C_n^v(x) = \frac{(2v)_n}{n!} \, _2F_1 \left[ \begin{array}{c} -n, 2v + n; \\ v + \frac{1}{2}; \\ 1 - x \end{array} \right] \] (2.9)
A JACOBI TYPE GENERALIZATION OF GAGENBAUER POLYNOMIAL:

Consider

\[ C_n^{\alpha, \beta}(x) = \frac{(1 + \alpha + \beta)n}{(1 + \alpha)_n} P_n^{(\alpha, \beta)}(x) \]

then

\[ C_n^{(\alpha, \beta)}(x) = \frac{(1 + \alpha + \beta)n}{(1 + \alpha)_n} 2 F_1 \left[ \begin{array}{c} -n, 1 + \alpha + \beta + n; \\ 1 + \alpha; \frac{1 - x}{2} \end{array} \right] \quad (2.10) \]

for \( \alpha = \beta = \nu - \frac{1}{2} \) we get Gagenbauer polynomials

\[ C_n^{(\nu - \frac{1}{2}, \nu - \frac{1}{2})}(x) = \frac{(2\nu)_n}{n!} 2 F_1 \left[ \begin{array}{c} -n, 1 + \alpha + \beta + n; \\ 1 + \alpha; \frac{1 - x}{2} \end{array} \right] \]

BATEMAN'S \( Z_n(x) \)

Bateman's polynomial \( Z_n(x) \) is defined as

\[ Z_n(x) = 2 F_2 \left[ \begin{array}{c} -n, n + 1; \\ 1, 1; \frac{x}{1} \end{array} \right] \quad (2.11) \]

A Jacobi type generalization of \( Z_n(x) \) may be denoted by the symbol \( Z_n^{(\alpha, \beta)}(x) \) and is defined as

\[ Z_n^{(\alpha, \beta)}(b, x) = 2 F_2 \left[ \begin{array}{c} -n, 1 + \alpha + \beta + n; \\ 1 + \alpha, 1 + b; \frac{x}{1} \end{array} \right] \quad (2.12) \]

RICE POLYNOMIALS

Rice polynomial \( H_n(\xi, p, v) \) is defined as

\[ H_n(\xi, p, v) = 3 F_2 \left[ \begin{array}{c} -n, n + 1; \xi, v; \\ 1, \frac{p}{1} \end{array} \right] \quad (2.13) \]
A Jacobi type generalization of Rice polynomial $H_n(\xi, p, v)$ is due to Khandekar [9] who denoted his generalized polynomial by the symbol $H^{(\alpha, \beta)}_n(\xi, p, v)$ and is defined as
\[
H^{(\alpha, \beta)}_n(\xi, p, v) = 3F_2 \left[ \begin{array}{c} -n, 1 + \alpha + \beta + n ; \\ 1 + \alpha, p ; \end{array} \xi, v \right]
\]  

(2.14)

**BESSEL POLYNOMIALS**

Simple Bessel polynomial $y_n(x)$ is defined as
\[
y_n(x) = 2F_0 \left[ \begin{array}{c} -n, n + 1 ; \\ - \end{array} \frac{-x}{2} \right]
\]  

(2.15)

and the generalized bessel polynomials $y_n(a, b, x)$ and is defined as
\[
y_n(a, b, x) = 2F_0 \left[ \begin{array}{c} -n, a - 1 + n ; \\ - \end{array} \frac{-x}{b} \right]
\]  

(2.16)

**LAGUERRE POLYNOMIALS OF TWO AND THREE VARIABLES:**

S.F. Ragab [10], in 1991 defined Laguerre polynomials of two variables $L^{(\alpha, \beta)}_n(x, y)$ as follows
\[
L^{(\alpha, \beta)}_n(x, y) = \frac{(\alpha + 1)_n(\beta + 1)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}}{(\alpha + 1)_s(\beta + 1)_r} \frac{x^s y^r}{s! r!}
\]  

(2.17)

Similarly Laguerre polynomial of three variables may be denoted by $L^{(\alpha, \beta, \gamma)}_n(x, y, z)$ and be defined as
\[
L^{(\alpha, \beta, \gamma)}_n(x, y, z) = \frac{(\alpha + 1)_n(\beta + 1)_n(\gamma + n)_n}{(n!)^3} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \sum_{k=0}^{n-r-s} \frac{(-n)_{r+s+k}}{(\alpha + 1)_s(\beta + 1)_r(1 + \gamma)_k} \frac{x^k y^s z^r}{k! s! r!}
\]  

(2.18)

**3. OPERATIONAL REPRESENTATIONS:**

If $D_x \equiv \frac{\partial}{\partial x}$ and $D_y \equiv \frac{\partial}{\partial y}$, M.A. Khan and A.K. Shukla [4] wrote the binomial expansion for $(D_x + D_y)^n$ as
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\[(D_x+D_y)^n \equiv \sum_{r=0}^{n} \binom{n}{r} D_x^{n-r} D_y^r \quad (3.1)\]

where

\[\binom{n}{r} = \frac{n!}{r!(n-r)!}\]

By writing the finite series on the right of (3.1) M.A. Khan and A.K. Shukla [4] wrote (3.1) also as

\[(D_x+D_y)^n \equiv \sum_{r=0}^{n} \binom{n}{r} D_x^r D_y^{n-r} \quad (3.2)\]

If \(F(x,y)\) is a function of \(x\) and \(y\), they obtained the following from (3.1) and (3.2)

\[(D_x+D_y)^n F(x, y) \equiv \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} D_x^{n-r} D_y^r F(x, y) \quad (3.3)\]

\[(D_x+D_y)^n F(x, y) \equiv \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} D_x^r D_y^{n-r} F(x, y) \quad (3.4)\]

In particular, if \(F(x,y) = f(x)g(y)\). M.A.Khan and A.K.Shukla [4] wrote (3.3) and (3.4) in the form

\[(D_x+D_y)^n f(x)g(y) \equiv \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} D_x^{n-r} f(x)D_y^r g(y) \quad (3.5)\]

\[(D_x+D_y)^n f(x)g(y) \equiv \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} D_x^r f(x)D_y^{n-r} g(y) \quad (3.6)\]

Now by taking special values of \(f(x)\) and \(g(y)\) in (3.5), we obtain the following partial differential operator representations of the polynomials given above:
\[(D_x+D_y)^n \left\{ x^\alpha e^{-y} \right\} = n! \ x^{\alpha-n} e^{-y} L_n^{\alpha-n}(x) \quad (3.7)\]

\[(D_x+D_y)^n \left\{ x^\alpha y^{-1-\alpha-\beta} \right\} = n! \ x^{\alpha-n} y^{-1-\alpha-\beta} P_n^{(\alpha-n,\beta)} \left( 1 - \frac{2x}{y} \right) \quad (3.8)\]

\[(D_x+D_y)^n \left\{ x^\alpha y^{-1-2\alpha} \right\} = n! \ x^{\alpha-n} y^{-1-2\alpha} P_n^{(\alpha-n,\alpha)} \left( 1 - \frac{2x}{y} \right) \quad (3.9)\]

which is a special case of Jacobi Polynomials.

\[(D_x+D_y)^n \left\{ x^\alpha y^{-1-\alpha-\beta} \right\} = n! (-1)^n \ x^{\alpha-n} y^{-1-\alpha-\beta} C_n^{(\alpha-n,\beta)} \left( 1 - \frac{2x}{y} \right) \quad (3.10)\]

If \( \alpha = \beta = \nu = -\frac{1}{2} \), then

\[(D_x+D_y)^n \left\{ x^\nu y^{-2-\alpha-\beta} \right\} = n! (-1)^n \ x^{\nu-\frac{1}{2}-n} y^{-2-\alpha-\beta} C_n^{(\nu-\frac{1}{2}-n,\nu-\frac{1}{2})} \left( 1 - \frac{2x}{y} \right) \quad (3.11)\]

\[(D_x D_y + D_z)^n \left\{ x^\alpha y^b z^{-1-\alpha-\beta} \right\} = (-\alpha)_n (b)_n x^{\alpha-n} y^{b-n} z^{-1-\alpha-\beta} Z_n^{(\alpha-n,\beta)} \left( b-n, \frac{xy}{z} \right) \quad (3.12)\]

\[(D_w D_x + D_y D_z)^n \left\{ w^\alpha x^{p-1} y^{-1-\alpha-\beta} z^{-\psi} \right\} = n! (-1)^n (1-p)_n \]

\[\times w^{\alpha-n} x^{p-1-n} y^{-1-\alpha-\beta} z^{-\psi} H_n^{(\alpha-n,\beta)} \left( \psi, p-n, -\frac{wx}{yz} \right) \quad (3.13)\]

\[(D_v D_w + D_x D_y D_z)^n \left\{ v^\alpha w^{p-1-\alpha-\beta} y^{-\psi} e^{-z} \right\} = n! (-1)^n (1-p)_n \]

\[\times v^{\alpha-n} w^{p-1-n} x^{-1-\alpha-\beta} y^{-\psi} e^{-z} H_n^{(\alpha-n,\beta)} \left( \psi, p-n, \frac{vw}{xy} \right) \quad (3.14)\]

\[(D_b + D_y)^n \left\{ \left( b \right)^{-(\alpha+\beta+1)} e^{-y} \right\} = \left( b \right)^{-(\alpha-\beta-1)} e^{-y} y_n (\alpha, \beta-n, b, x) \quad (3.15)\]
4. TRINOMIAL OPERATOR REPRESENTATIONS:

If \( D_x \equiv \frac{\partial}{\partial x}, \ D_y \equiv \frac{\partial}{\partial y}, \ D_z \equiv \frac{\partial}{\partial z} \) M.A. Khan and A.K. Shukla [4] wrote the trinomial expansion for \((D_x + D_y + D_z)^n\) as

\[
(D_x + D_y + D_z)^n \equiv \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)^{r+s}(-1)^{r+s}}{r!s!} D_x^{n-r-s} D_y^r D_z^s
\]

(4.1)

operating (4.1) on \(F(x, y, z)\), M.A. Khan and A.K. Shukla [4] got

\[
(D_x + D_y + D_z)^n \ F(x, y, z) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)^{r+s}(-1)^{r+s}}{r!s!} D_x^{n-r-s} D_y^r D_z^s F(x, y, z)
\]

(4.2)

in particular, if \(F(x, y, z) = f(x)g(y)h(z)\), then (4.2) gives

\[
(D_x + D_y + D_z)^n \ \{f(x)g(y)h(z)\} = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)^{r+s}(-1)^{r+s}}{r!s!} D_x^{n-r-s} f(x) D_y^r g(y) D_z^s h(z)
\]

(4.3)

Similarly

\[
(D_x D_y + D_x D_z + D_y D_z)^n \ \{f(x)g(y)h(z)\}
= \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)^{r+s}(-1)^{r+s}}{r!s!} D_x^{n-s} f(x) D_y^r D_z^s h(z)
\]

(4.4)

choosing special values of \(f(x), g(y)\) and \(h(z)\) in (4.4), we obtain the following trinomial operator representations:

\[
(D_x D_y + D_x D_z + D_y D_z)^n \ \{x^\alpha y^\beta e^{-z}\}
= \frac{(n!)^2(-\alpha)_n(-\beta)_n x^{\alpha-n} y^{\beta-n} e^{-y}}{\alpha + 1 - n \ \beta + 1 - n}_n L_n^{(\alpha-n, \beta-n)}(x, y)
\]

(4.5)

\[
(D_y D_x + D_x D_y D_w + D_x D_z D_w)^n \ \{x^\alpha y^\beta z^\gamma e^{-w}\}
= \frac{(n!)^3(-1)^n(-\alpha)_n(-\gamma)_n x^{\alpha-n} y^{\beta-n} z^{\gamma-n} e^{-w}}{\alpha + 1 - n \ \beta + 1 - n \ \gamma + 1 - n}_n L_n^{(\alpha-n, \beta-n, \gamma-n)}(x, y, z)
\]

(4.6)
REFERENCES


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