On Fuzzy Biclosure Spaces

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Abstract

In this paper we introduce the concept of fuzzy biclosure spaces and study some of their properties.

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1. Introduction

Closure spaces were introduced by E.Čech [3]. The notion of closure system and closure operators are very useful tools in several areas of classical mathematics. They play an important role in topological spaces, Boolean algebra, convex sets etc. This led several authors to investigate the closure operator in the frame work of fuzzy set theory. Gerla et al. [1] studied fuzzy closure operator and fuzzy closure system as extension of closure operator and closure system. Fuzzy closure spaces were first studied by A.S. Mashhour and M.H Ghanim [5]. Recently, Chawalit Boonpok [2] introduced the notion of biclosure spaces. Such spaces are equipped with two arbitrary closure operators. He extended some of the standard results of separation axioms in closure space to biclosure space. Thereafter a large number of papers have been written to generalize the concept of closure space to biclosure space. In this paper we define and study the concept of fuzzy biclosure spaces.
2. Preliminaries

Let X be an arbitrary set, \( I = [0,1] \) and \( I^X \) be a family of all fuzzy sets of \( X \). For a fuzzy set \( A \) of \( X \), \( \text{cl}(A) \), \( \text{int}(A) \) and \( 1-A \) will denote the closure of \( A \), the interior of \( A \) and the complement of \( A \) respectively whereas the constant fuzzy sets taking on the values 0 and 1 on \( X \) are denoted by \( 0_X \) and \( 1_X \) respectively.

**Definition 2.1** [5,7]: A function \( u : I^X \rightarrow I^X \) defined on the family \( I^X \) of all fuzzy sets of \( X \) is called a fuzzy closure operator on \( X \) and the pair \( (X, u) \) is called a fuzzy closure space, if the following conditions are satisfied

1) \( u\phi = \phi \)
2) \( A \leq u(A) \) for all \( A \in I^X \).
3) \( u(A \lor B) = u(A) \lor u(B) \) for all \( A, B \in I^X \).

**Definition 2.2** [4]: A fuzzy subset \( A \) of a fuzzy closure space \( (X, u) \) is said to be fuzzy closed, if \( uA = A \) and it is fuzzy open if its complement \( X - A \) is fuzzy closed.

The empty set and the whole set are both fuzzy open and fuzzy closed.

**Definition 2.3** [4]: A fuzzy closure space \( (Y, v) \) is said to be a fuzzy subspace of \( (X, u) \) if \( Y \leq X \) and \( vA = uA \cap Y \) for each fuzzy subset \( A \leq Y \). If \( Y \) is fuzzy closed in \( (X, u) \) then the fuzzy subspace \( (Y, v) \) of \( (X, u) \) is also said to be fuzzy closed.

**Definition 2.4**[6]: Let \( (X, u) \) and \( (Y, v) \) be fuzzy closure spaces. A map \( f : (X, u) \rightarrow (Y, v) \) is said to be fuzzy continuous if \( f(uA) \leq vf(A) \) for every fuzzy subset \( A \leq X \). In other words a map \( f : (X, u) \rightarrow (Y, v) \) is fuzzy continuous if and only if \( u(f^{-1}(B)) \leq f^{-1}(vB) \) for every fuzzy subset \( B \leq Y \).

Clearly, if map \( f : (X, u) \rightarrow (Y, v) \) is fuzzy continuous, then \( f^{-1}(F) \) is a fuzzy closed subset of \( (X, u) \) for every fuzzy closed subset \( F \) of \( (Y, v) \).

**Definition 2.5**[6]: Let \( (X, u) \) and \( (Y, v) \) be fuzzy closure spaces. A map \( f : (X, u) \rightarrow (Y, v) \) is said to be fuzzy closed (resp. fuzzy open) if \( f(F) \) is a fuzzy closed (resp. fuzzy open) subset of \( (Y, v) \) whenever \( F \) is a fuzzy closed (resp. fuzzy open) subset of \( (X, u) \).
**Definition 2.6** [5]: The product of a family \( \{(X_\alpha, u_\alpha) : \alpha \in J\} \) of fuzzy closure spaces denoted by \( \prod_{\alpha \in J} (X_\alpha, u_\alpha) \), is the fuzzy closure space \((\prod_{\alpha \in J} X_\alpha, u)\) where \( \prod_{\alpha \in J} X_\alpha \) denotes the cartesian product of fuzzy sets \( X_\alpha, \alpha \in J \) and \( u \) is the fuzzy closure operator generated by the projections \( \pi_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha, \alpha \in J \), i.e., is defined by \( uA = \prod_{\alpha \in J} u_\alpha \pi_\alpha(A) \) for each \( A \leq \prod_{\alpha \in J} X_\alpha \).

Clearly, if \( \{(X_\alpha, u_\alpha) : \alpha \in J\} \) is a family of fuzzy closure spaces, then the projection map \( \pi_\beta : \prod_{\alpha \in J} (X_\alpha, u_\alpha) \to (X_\beta, u_\beta) \) is fuzzy closed and fuzzy continuous for every \( \beta \in J \).

**Proposition 2.7.** Let \( \{(X_\alpha, u_\alpha) : \alpha \in J\} \) be a family of fuzzy closure spaces and let \( \beta \in J \). Then \( F \) is a fuzzy closed subset of \((X_\beta, u_\beta)\) if and only if \( F \times \prod_{\alpha \in J \setminus \{\beta\}} X_\alpha \) is a fuzzy closed subset of \( \prod_{\alpha \in J} (X_\alpha, u_\alpha) \).

**Proposition 2.8.** Let \( \{(X_\alpha, u_\alpha) : \alpha \in J\} \) be a family of fuzzy closure spaces and let \( \beta \in J \). Then \( G \) is a fuzzy open subset of \((X_\beta, u_\beta)\) if and only if \( G \times \prod_{\alpha \in J \setminus \{\beta\}} X_\alpha \) is a fuzzy open subset of \( \prod_{\alpha \in J} (X_\alpha, u_\alpha) \).

Most of the properties of ordinary sets are still true for fuzzy sets and fuzzy closure spaces.

### 3. Fuzzy Biclosure Spaces

In this section we introduce the concept of fuzzy biclosure spaces and investigate some of their characterizations.

**Definition 3.1.** A fuzzy biclosure space is a triple \((X, u_1, u_2)\) where \( X \) is a set and \( u_1, u_2 \) are two fuzzy closure operators on \( X \) which satisfies the following properties:

1. \( u_1 \phi = \phi \) and \( u_2 \phi = \phi \)
(ii) \( A \leq u_1A \) and \( A \leq u_2A \) for all \( A \leq I^X \)

(iii) \( u_1(A \lor B) = u_1A \lor u_1B \) and \( u_2(A \lor B) = u_2A \lor u_2B \) for all \( A, B \leq I^X \).

**Definition 3.2.** A subset \( A \) of a fuzzy biclosure space \((X, u_1, u_2)\) is called fuzzy closed if \( u_1u_2A = A \). The complement of fuzzy closed set is called fuzzy open.

Clearly, \( A \) is a fuzzy closed subset of fuzzy biclosure space \((X, u_1, u_2)\) if and only if \( A \) is both a fuzzy closed subset of \((X, u_1)\) and \((X, u_2)\).

Let \( A \) be a fuzzy closed subset of a fuzzy biclosure space \((X, u_1, u_2)\). The following conditions are equivalent

(i) \( u_2u_1A = A \)

(ii) \( u_1A = A \), \( u_2A = A \).

The following statement is obvious.

**Proposition 3.3.** Let \((X, u_1, u_2)\) be a fuzzy biclosure space and let \( A \leq X \). Then

(i) \( A \) is open if and only if \( A = X - u_1u_2(X - A) \).

(ii) If \( G \) is open and \( G \leq A \), then \( G \leq X - u_1u_2(X - A) \).

**Definition 3.4.** Let \((X, u_1, u_2)\) be a fuzzy biclosure space. A fuzzy biclosure space \((Y, v_1, v_2)\) is called a fuzzy subspace of \((X, u_1, u_2)\) if \( Y \leq X \) and \( v_iA = u_iA \land Y \) for each \( i \in \{1, 2\} \) and each subset \( A \leq Y \).

**Proposition 3.5.** Let \((X, u_1, u_2)\) be a fuzzy biclosure space and let \((Y, v_1, v_2)\) be a fuzzy closed subspace of \((X, u_1, u_2)\). If \( F \) is a fuzzy closed subset of \((Y, v_1, v_2)\), then \( F \) is a fuzzy closed subset of \((X, u_1, u_2)\).

**Proof.** Let \( F \) be a fuzzy closed subset of \((Y, v_1, v_2)\). Then \( v_1F = F \) and \( v_2F = F \).

Since \( Y \) is fuzzy closed subset of both \((X, u_1)\) and \((X, u_2)\), \( u_1F = F \) and \( u_2F = F \). Consequently, \( F \) is both a fuzzy closed subset of \((X, u_1)\) and \((X, u_2)\). Therefore, \( F \) is a fuzzy closed subset of \((X, u_1, u_2)\).

**Proposition 3.6.** Let \( \{(X_{\alpha}, u_{\alpha}^1, u_{\alpha}^2) : \alpha \in J\} \) be a family of fuzzy biclosure spaces and let \( \beta \in J \). Then \( F \) is a fuzzy closed subset of \((X_{\beta}, u_{\beta}^1, u_{\beta}^2)\) if and only if \( F \times \prod_{\alpha \in \beta} X_{\alpha} \) is a fuzzy closed subset of \( \prod_{\alpha \in J} (X_{\alpha}, u_{\alpha}^1, u_{\alpha}^2) \).
Proof. Let $\beta \in J$ and let $F$ be a fuzzy closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$. Then $F$ is a fuzzy closed subset of $(X_\beta, u_\beta^1)$ and $(X_\beta, u_\beta^2)$, respectively. Since

$$\pi_\beta : \prod_{a \in J} (X_a, u_a^1) \rightarrow (X_\beta, u_\beta^1)$$

is fuzzy continuous, $\pi_\beta^{-1}(F) = F \times \prod_{a \in J} X_a$ is a fuzzy closed subset of $\prod_{a \in J} (X_a, u_a^1)$. Similarly, since $\pi_\beta : \prod_{a \in J} (X_a, u_a^2) \rightarrow (X_\beta, u_\beta^2)$ is fuzzy continuous, $\pi_\beta^{-1}(F) = F \times \prod_{a \in J} X_a$ is a fuzzy closed subset of $\prod_{a \in J} (X_a, u_a^2)$. Consequently, $F \times \prod_{a \in J} X_a$ is a fuzzy closed subset of $\prod_{a \in J} (X_a, u_a^1, u_a^2)$.

Conversely, let $F \times \prod_{a \in J} X_a$ is a fuzzy closed subset of $\prod_{a \in J} (X_a, u_a^1, u_a^2)$. Then $F \times \prod_{a \in J} X_a$ is a fuzzy closed subset of $\prod_{a \in J} (X_a, u_a^1)$ and $\prod_{a \in J} (X_a, u_a^2)$ respectively.

Since $\pi_\beta : \prod_{a \in J} (X_a, u_a^1) \rightarrow (X_\beta, u_\beta^1)$ is fuzzy closed, $\pi_\beta(F \times \prod_{a \in J} X_a) = F$ is a fuzzy closed subset of $(X_\beta, u_\beta^1)$. Similarly, since $\pi_\beta : \prod_{a \in J} (X_a, u_a^2) \rightarrow (X_\beta, u_\beta^2)$ is fuzzy closed, $\pi_\beta(F \times \prod_{a \in J} X_a) = F$ is a fuzzy closed subset of $(X_\beta, u_\beta^2)$. Consequently, $F$ is a closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$.

Proposition 3.7. Let $\{(X_a, u_a^1, u_a^2) : \alpha \in J\}$ be a family of fuzzy biclosure spaces and let $\beta \in J$. Then $G$ is a fuzzy open subset of $(X_\beta, u_\beta^1, u_\beta^2)$ if and only if $G \times \prod_{a \in J} X_a$ is a fuzzy open subset of $\prod_{a \in J} (X_a, u_a^1, u_a^2)$.

Definition 3.8. Let $(X, u_1, u_2)$ and $(Y, v_1, v_2)$ be fuzzy biclosure spaces and let $i \in \{1, 2\}$. A map $f : (X, u_i, u_j) \rightarrow (Y, v_1, v_2)$ is called fuzzy $i-$closed (resp. fuzzy $i-$open) if the map $f : (X, u_i) \rightarrow (Y, v_i)$ is fuzzy $i-$closed (resp. fuzzy $i-$open). A map $f$ is called fuzzy closed (resp. fuzzy open) if $f$ is fuzzy $i-$closed (resp. fuzzy $i-$open) for each $i \in \{1, 2\}$. 
Definition 3.9. Let \((X,u_i,u_2)\) and \((Y,v_1,v_2)\) be fuzzy biclosure spaces and let \(i \in \{1,2\}\). A map \(f:(X,u_i,u_2) \rightarrow (Y,v_1,v_2)\) is called fuzzy \(i-continuous\) if the map \(f:(X,u_i) \rightarrow (Y,v_i)\) is fuzzy continuous. A map \(f\) is called fuzzy continuous if \(f\) is fuzzy \(i-continuous\) for each \(i \in \{1,2\}\).

Definition 3.10. A subset \(A\) of a fuzzy biclosure space \((X,u_1,u_2)\) is called generalized fuzzy closed briefly , \(g\)-fuzzy closed , if \(u_i(A) \leq G\) whenever \(G\) is a fuzzy open subset of \((X,u_2)\) with \(A \leq G\). The complement of a fuzzy \(g\)-closed set is called fuzzy \(g\)-open.

Proposition 3.11. Let \((X,u_1,u_2)\) be a fuzzy biclosure space. Then \(A\) is a fuzzy \(g\)-open subset of \((X,u_1,u_2)\) if and only if \(F \leq X-u_i(X-A)\) for every \(F\) which is fuzzy closed subset of \((X,u_2)\) with \(F \leq A\).

Proof. Assume that \(A\) is a fuzzy \(g\)-open and let \(F\) be a fuzzy closed subset of \((X,u_2)\) such that \(F \leq A\). Then \((X-A) \leq (X-F)\). Since \((X-A)\) is fuzzy \(g\)-closed and \((X-F)\) is fuzzy open subset of \((X,u_2)\) , \(u_i(X-A) \leq X-F\). Therefore , \(F \leq X-u_i(X-A)\). Conversely, let \(U\) be an fuzzy open subset of \((X,u_2)\) such that \((X-A) \leq U\). Then \((X-A) \leq U\). Since \((X-U)\) is a fuzzy closed subset of \((X,u_2)\), \((X-U) \leq X-u_i(X-A)\). Consequently, \(u_i(X-A) \leq U\). Hence, \((X-A)\) is fuzzy \(g\)-closed and so \(A\) is fuzzy \(g\)-open.

References


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