Int. Journal of Math. Analysis, Vol. 5, 2011, no. 3, 129 - 133

Semigroups and Ideal Retraction Property

A. Zohri and Gh. Ghadimi

Payame Noor University, Ardebil, Iran
zohri_a@pnu.ac.ir
ghadirghadimi@gmail.com

Abstract. In this article we give some new definitions and prove some results about semigroups with ideal retraction property.Some examples and counterexamples is also presented.

1. Introduction

Semigroups with the ideal retraction property were studied by Wallace in 1957 [7]. Aucoin and others studied Semigroups with the ideal retraction property from other point of view say algebrically in [2]. Some examples were presented in that paper to show how this property compares with other properties of semigroups which have been studied in the recent literature.The structure of Abelian semigroups with ideal retraction property is studied in [3]. In this article we give some new definitions "inner ideals" and "enbloc semigroups " and "first type ideal" and then investigate about ideal retraction property in semigroups and then get some new results in ch.3. In next section we collect definitions and related results.
AMS Classification 20M10 20M99

2. Definitions and related results

Definition 2.1. *A semigroup S is said to have the ideal retraction property provided that S is not simple and for each ideal I of S, there exists a homomorphism $\phi : S \to I$ such that the restriction $\phi|_I$ of ϕ to I is the identity map on I. The map ϕ is called a homomorphic retraction of S onto I.*

proposition 2.1. *Let S be a semigroup with the ideal retraction property and let I be an ideal of S. Then S/I has the ideal retraction property.*

Definition 2.2. *Let S be a semigroup and $x \in S$ we define right wing of x and left wing of x as*

$$[x]_R = \{s \in S : xs = x\} \ [x]_L = \{s \in S : sx = x\}$$

Definition 2.3. *$I \subseteq S$ is said to be of first type if*

$$\forall\ x, y \in I\ if\ ([x]_L \cap [y]_L) \neq \phi \Rightarrow ([x]_L \cap [y]_L) \cap I \neq \phi$$

and

$$\forall\ x, y \in I\ if\ ([x]_R \cap [y]_R) \neq \phi \Rightarrow ([x]_R \cap [y]_R) \cap I \neq \phi$$

Definition 2.4. *Let S be a semigroup, the ideal $I \subseteq S$ is said to be inner if*

$$x \in S^2 \cap I \Rightarrow x \in I^2.$$

Let$S$ be a semigroup and I an ideal of S and $x \in I$ then we define the symbols R^x and L^x as follow:

$$R^x = \{y \in S : xy = xs, for s \in S \backslash I\}$$

and

$$L^x = \{y \in S : yx = sx, for s \in S \backslash I\}$$

Definition 2.5. *A semigroup S is said to be enbloc if every ideal $I \subseteq S$ be inner and for any $x \in I$ if $R^x \neq \phi$ then $R^x \cap I \neq \phi$ also if $L^x \neq \phi$ then $L^x \cap I \neq \phi$*

Definition 2.6. *A semigroup S is said to be a t-semigroup if whenever J is an ideal of S and I is an ideal of J, then I is an ideal of S ,i.e., the property of being an ideal is transitive.*

It is established in [5] that if S is either a regular semigroup, a semigroup with the ideal extension property, or has the congruence extension property, then S is a t-semigroup. That paper presents examples to demonstrate that the converse dose not generally hold for any of these properties. the same is true for the ideal retraction property.

proposition 2.2. *If S is a semigroup with the ideal retraction property,then S is a t-semigroup.*

3. The new results

Theorem 3.1. *Let S be a semigroup with the ideal retraction property, then all of its ideals are of first type.*

Proof: Let S be a semigroup with the ideal retraction property and I be an ideal of S which is not of first type, then

$$\exists\ x, y \in I\ s.t\ ([x]_L \cap [y]_L) \neq \phi\ \&([x]_L \cap [y]_L) \cap I = \phi$$

then there exists $s \in S \backslash I$ such that $s \in ([x]_L \cap [y]_L)$ then $sy = y, sx = x$.
Now let ϕ be the retraction mapping for I, since $x, y \in I$ then we have
$x = \phi(x)$ and $y = \phi(y)$ and therefore

$$y = sy = \phi(sy) = \phi(s)\phi(y) = \phi(s)y$$
$$x = sx = \phi(sx) = \phi(s)\phi(x) = \phi(s)x$$

then $\phi(s) \in ([x]_L \cap [y]_L)$ and so $\phi(s) \in ([x]_L \cap [y]_L) \cap I$ which is contradiction.

Example 3.2. *The semigroup $S = \{1, 2, 3, 4\}$ with following production table has no ideal retraction property:*

1 1 1 1

1 2 1 2

1 1 3 3

1 2 3 4

because $I = \{1, 2, 3\}$ is not from first type,since $[2] = \{2, 4\}$ and $[3] = \{3, 4\}$ then

$$([2] \cap [3]) \cap I = \phi$$

proposition 3.1. *Infinite cyclic semigroups has no ideal retraction property.*

proof: It is easy to see that every Infinite cyclic semigroup is isomorphic with $(\mathbb{N}, +)$. Also the ideal$I = \{2, 3, 4, 5, ...\}$ from $(\mathbb{N}, +)$is not inner, because $2 \in I$ but $2 = 1 + 1$ and $1 \notin I$.

In the next example the we consider a semigroup with no ideal retraction property.

Example 3.3. *The semigroup $S = \{1, 2, 3, \}$ with following production table*

1 1 1

1 1 1

1 1 2

is a t-semigroup but not enbloc then it doesn't have ideal retraction property.

Example 3.4. *The semigroup $S = \{1, 2, 3, 4\}$ with following production table*

1 1 1 1

1 1 1 1

1 1 3 3

1 1 4 4

is a t-semigroup and is an enbloc semigroup with the first type ideals. This semi group has ideal retraction property.

In the next theorem we characterize the ideals of a semigroup with ideal retraction property.

Theorem 3.5. *For every semigroup S with ideal retraction property,all of ideals are inner.*

Proof: Let $I \subseteq S$ be an ideal of S which is not inner. Then

$$\exists\ x, y \in S\ s.t\ xy \in I and x \notin I\ or\ y \notin I, \forall x, y \in I, st \neq xy$$

Since I has ideal retraction property then

$$\exists \phi : S \to I\ s.t\ \phi|_I = 1_I$$

then we can conclude that

$$xy = \phi(xy) = \phi(x)\phi(y) \Rightarrow \phi(x), \phi(y) \in I$$

which is contradiction.

Example 3.6. *The semigroup $S = \{1, 2, 3, 4, 5\}$ with following production table, is a semigroup with first type ideals which is not enbloc and then has no ideal retraction property.*

1	1	1	1	1
1	1	2	1	1
1	1	3	1	1
1	1	4	1	1
1	1	1	2	1

Example 3.7. *The semigroup $S = \{1, 2, 3, 4, 5\}$ with following production table, is a semigroup with prime ideals has no ideal retraction property because the ideal $\{1, 4, 5\}$ is not inner.*

1	1	1	1	1
1	1	5	1	1
1	4	1	1	1
1	1	1	1	1
1	1	1	1	1

References

[1] Au1 K.D. Aucoin, J.A. Dumesnil, and J.A. Hildebrant,Commutative periodic semigroups with the ideal retraction property,preprint,2004.
[2] K.D.Aucoin, J.A.Dumesnil, and J.A.Hildebrant, Semigroups with the ideal retraction property C Forum 66 (2003), 416432.
[3] K.D.Aucoin, J.A.Dumesnil, and J.A.Hildebrant, The structure of commutative semigroups with the ideal retraction property, Semigroup Forum 68 (2004), 202208.
[4] J.M. Howie, Introduction to semi group theory, Academic Press London (1976)
[5] Clifford, A. H. and G.B. Preston,"The Algebric Theory of semigroups",Math.Surveys,No.7 Amer.Math. Soc., Providence,1961.
[6] Dumesnil, J.A.,On t-semigroups,Semigroup Frum 50(1995),383-387.
[7] Wallace,A.D. Retractions in semigroups,Pacific J.Math. 7(1957),1513-1517.

Received: July, 2010