Int. Journal of Math. Analysis, Vol. 5, 2011, no. 4, 165 - 173

Some Remarks on an Equivalence Theorem for a Singularly Perturbed Semilinear Neumann Problem with Non-Normally Hyperbolic Critical Manifold

Robert Vrábel'

Institute of Applied Informatics, Automation and Mathematics
Faculty of Materials Science and Technology
Hajdoczyho 1, 917 01 Trnava, Slovakia
robert.vrabel@stuba.sk

Abstract

We investigate the Neumann boundary value problem for singularly perturbed second-order differential equation

$$\epsilon y'' + ky = f(t, y), \quad t \in \langle a, b\rangle, \quad k > 0, \quad 0 < \epsilon << 1$$

with non normally-hyperbolic critical manifold. We give the equivalence theorem for existence of the solutions of problem above.

Mathematics Subject Classification: 34K10, 34K26

Keywords: singular perturbation, boundary value problem, method of lower and upper solutions

1 Preliminaries

In the work [3], we have established the sufficient condition for existence and gave some remarks on asymptotic behavior (for $\epsilon \to 0^+$) of the solutions of Neumann problem

$$\epsilon y'' + ky = f(t, y), \quad t \in \langle a, b\rangle, \quad k > 0, \quad 0 < \epsilon << 1 \tag{1}$$

$$y'(a) = 0, \quad y'(b) = 0, \tag{2}$$

for

$$f \in C^1\left(\langle a, b\rangle \times [u(t) - z_{\epsilon,\alpha_\epsilon}(t), u(t) + z_{\epsilon,\beta_\epsilon}(t)]\right)$$

such that

$$\left|\frac{\partial f(t,y)}{\partial y}\right| \le w < k \quad (t,y) \in \langle a,b\rangle \times \left[u(t) - z_{\epsilon,\alpha_\epsilon}(t), u(t) + z_{\epsilon,\beta_\epsilon}(t)\right], \tag{3}$$

where u is a solution of reduced problem (in (1) letting $\epsilon \to 0^+$)

$$ky = f(t,y). \tag{4}$$

We remark, that the assumption (3) is the condition of non-hyperbolicity for a solution u of a reduced problem (4). Further, we will assume that $u \in C^2(\langle a,b\rangle)$. For the functions $z_{\epsilon,i}(t)$, $i = \alpha_\epsilon, \beta_\epsilon$, see (9) below and α_ϵ, β_ϵ are the lower and upper solutions to (1), (2), for details see e.g. [1].

The purpose of this paper is to exhibit the conditions which are equivalent to the existence of the solutions for (1), (2). The problem is very difficult unless several special cases, but those ones may be instructive for future considerations.

Hereafter we follow the notation and definition of [3]. Let

$$J_n(\lambda) = \left\langle m\left(\frac{b-a}{(n+1)\pi - \lambda}\right)^2, m\left(\frac{b-a}{n\pi + \lambda}\right)^2\right\rangle, \quad n = 0,1,2,\dots,$$

where $m = k + w$, $\lambda > 0$ is an arbitrarily small, but fixed constant and let

$$M = \left\{\bigcup J_n, n = 0,1,2,\dots\right\}.$$

Let

$$\begin{aligned} v_{1,\epsilon}(t) &= \quad |u'(a)| \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(b-t)\right]}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]} \\ v_{2,\epsilon}(t) &= -|u'(b)| \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right]}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]}. \end{aligned}$$

Denote

$$\omega_{0,\epsilon}(t) = v_{2,\epsilon}(t) - v_{1,\epsilon}(t) \left(= O(\sqrt{\epsilon})\right), \epsilon \in M.$$

Let

$$\begin{aligned} \omega_{1,\epsilon,i}(t) = & \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right] \int\limits_a^b \cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right](\pm u''(s))\,\mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]} \\ & + \int\limits_a^t \frac{\sin\left[\sqrt{\frac{m}{\epsilon}}(t-s)\right](\pm u''(s))\,\mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}}\mathrm{d}s \left(= O(\epsilon)\right), \epsilon \in M, \end{aligned}$$

i.e. $\omega_{1,\epsilon,i}(t)$ is a solution of Neumann problem (2) for linear differential equation

$$\epsilon y'' + my = \pm\epsilon u''(t),$$

where the sign $+$ and $-$ is considered for $i = \alpha_\epsilon$ and $i = \beta_\epsilon$, respectively.

Hence

$$\omega_{1,\epsilon,\alpha_\epsilon}(t) + \omega_{1,\epsilon,\beta_\epsilon}(t) \equiv 0 \quad \text{on} \quad \langle a, b\rangle. \tag{5}$$

Let $r_{\epsilon,i}(t)$ is a solution of Fredholm integral equation of second kind

$$\Gamma(\epsilon)\int_a^b K_\epsilon(t,s)y(s)\mathrm{d}s + \tilde{\Omega}_{\epsilon,i}(t) = y(t), \quad i = \alpha_\epsilon, \beta_\epsilon, \tag{6}$$

where $\tilde{\Omega}_{\epsilon,i}(t) = \omega_{0,\epsilon}(t) + \omega_{1,\epsilon,i}(t) - \tilde{z}_{\epsilon,i}(t)$, $\tilde{z}_{\epsilon,i}(t)$ is an appropriate chosen continuous function (see the assumption $(A1')$ below),

$$\Gamma(\epsilon) = \frac{1}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]}\cdot\frac{1}{\epsilon}, \quad \epsilon \in M$$

and the kernel

$$K_\epsilon(t,s) = \begin{cases} K_{1,\epsilon}(t,s), & a \le s \le t \le b \\ K_{2,\epsilon}(t,s), & a \le t \le s \le b, \end{cases}$$

$$\begin{aligned} K_{1,\epsilon}(t,s) = {} & \cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right]\cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right] \\ & + \sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]\sin\left[\sqrt{\frac{m}{\epsilon}}(t-s)\right] \\ K_{2,\epsilon}(t,s) = {} & \cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right]\cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right] \end{aligned}$$

for $\epsilon \in M$. Obviously, $r_{\epsilon,i}(t) = r_{\epsilon,i}\left(\tilde{z}_{\epsilon,i}(t)\right)$. Let

$$\begin{aligned} v_{c,\epsilon,i}(t) = {} & \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right]\int_a^b \cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right]\frac{r_{\epsilon,i}(s)}{\epsilon}\mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}\sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]} \\ & + \int_a^t \frac{\sin\left[\sqrt{\frac{m}{\epsilon}}(t-s)\right]\frac{r_{\epsilon,i}(s)}{\epsilon}\mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}}\mathrm{d}s \,(= O(r_{\epsilon,i}(t)))\,, \epsilon \in M, i = \alpha_\epsilon, \beta_\epsilon. \end{aligned}$$

i.e. $v_{c,\epsilon,i}(t)$ is a solution of Neumann problem (2) for linear differential equation

$$\epsilon y'' + my = r_{\epsilon,i}(t).$$

We will assume that

$(A1')$ There exist the continuous functions $\tilde{z}_{\epsilon,i}(t)$, $i=\alpha_\epsilon,\beta_\epsilon$ such that for every $\epsilon\in(0,\epsilon_0]\cap M$

$$\big[(7)\big]\vee\big[(8)\wedge(v_{c,\epsilon,\alpha_\epsilon}(t)+v_{c,\epsilon,\beta_\epsilon}(t)\geq-2\omega_{0,\epsilon}(t))\big]_{i=\alpha_\epsilon}$$

and

$$\big[(7)\big]\vee\big[(8)\wedge(v_{c,\epsilon,\alpha_\epsilon}(t)+v_{c,\epsilon,\beta_\epsilon}(t)\geq-2\omega_{0,\epsilon}(t))\big]_{i=\beta_\epsilon}$$

on $\langle a,b\rangle$

where

$$-\tilde{z}_{\epsilon,i}(t)\leq r_{\epsilon,i}(t)\leq 0,\quad i=\alpha_\epsilon,\beta_\epsilon \tag{7}$$

$$(1-2w)r_{\epsilon,i}(t)\leq 2w\tilde{z}_{\epsilon,i}(t)\leq-2wr_{\epsilon,i}(t)\quad i=\alpha_\epsilon,\beta_\epsilon. \tag{8}$$

2 Main results

Theorem 2.1 *Let $w=0$ and $u'(a)=u'(b)=0$. Then the following three statements are equivalent*

(i) $(A1')$

(ii) For $\epsilon\in(0,\epsilon_0]\cap M$ there exist $\alpha_\epsilon,\beta_\epsilon\in C^2(\langle a,b\rangle)$ satisfying $\epsilon\alpha''_\epsilon(t)+k\alpha_\epsilon(t)\geq f(t,\alpha_\epsilon(t))$ and $\alpha'_\epsilon(a)\geq 0$, $\alpha'_\epsilon(b)\leq 0$, $\epsilon\beta''_\epsilon(t)+k\beta_\epsilon(t)\leq f(t,\beta_\epsilon(t))$ and $\beta'_\epsilon(a)\leq 0$, $\beta'_\epsilon(b)\geq 0$ and $\alpha_\epsilon(t)\leq\beta_\epsilon(t)$ for all $t\in\langle a,b\rangle$

(iii) For $\epsilon\in(0,\epsilon_0]\cap M$ there exists a solution $y=y_\epsilon(t)$ of (1), (2).

Proof.

(i)⇒(ii) This result is proven in [3], of a more general form. We recall, that

$$\alpha_\epsilon(t)=u(t)-z_{\epsilon,\alpha_\epsilon}(t)\quad\text{and}\quad\beta_\epsilon(t)=u(t)+z_{\epsilon,\beta_\epsilon}(t),$$

where

$$z_{\epsilon,i}(t)=r_{\epsilon,i}(t)+\tilde{z}_{\epsilon,i}(t)=\omega_{0,\epsilon}(t)+\omega_{1,\epsilon,i}(t)+v_{c,\epsilon,i}(t),\quad i=\alpha_\epsilon,\beta_\epsilon. \tag{9}$$

The condition $\alpha_\epsilon(t)\leq\beta_\epsilon(t)$ on $\langle a,b\rangle$ requires

$$z_{\epsilon,\alpha_\epsilon}(t)+z_{\epsilon,\beta_\epsilon}(t)\geq 0\quad\text{for all}\quad t\in\langle a,b\rangle. \tag{10}$$

(ii)⇒(iii) follows from the method of lower and upper solutions [1, 2].

(iii)⇒(i):

Firstly, the assumption $u'(a)=u'(b)=0$ implies that $\omega_{0,\epsilon}(t)\equiv 0$ on $\langle a,b\rangle$. Denote

$$I_{+,\epsilon,i}=\{t\in\langle a,b\rangle: z_{\epsilon,i}(t)\geq 0\},\quad i=\alpha_\epsilon,\beta_\epsilon$$

and

$$I_{-,\epsilon,i} = \{t \in \langle a, b\rangle : z_{\epsilon,i}(t) \le 0\}, \quad i = \alpha_\epsilon, \beta_\epsilon.$$

We set

$$z_{\epsilon,\alpha_\epsilon}(t) = u(t) - y_\epsilon(t) \,(= r_{\epsilon,\alpha_\epsilon}(t) + \tilde{z}_{\epsilon,\alpha_\epsilon}(t))$$

and

$$z_{\epsilon,\beta_\epsilon}(t) = y_\epsilon(t) - u(t) \,(= r_{\epsilon,\beta_\epsilon}(t) + \tilde{z}_{\epsilon,\beta_\epsilon}(t))$$

Obviously $I_{-,\epsilon,\alpha_\epsilon} \cup I_{-,\epsilon,\beta_\epsilon} = \langle a, b\rangle$. From the required conditions (7) and (8) we obtain for $\tilde{z}_{\epsilon,i}(t)$, $i = \alpha_\epsilon, \beta_\epsilon$ the equivalent inequalities with (7) and (8)

$$z_{\epsilon,\alpha_\epsilon}(t) \le \tilde{z}_{\epsilon,\alpha_\epsilon}(t), \quad z_{\epsilon,\beta_\epsilon}(t) \le \tilde{z}_{\epsilon,\beta_\epsilon}(t) \quad \text{for} \quad z_{\epsilon,i}(t) \ge 0 \tag{11}$$

$$(1-2w) z_{\epsilon,\alpha_\epsilon}(t) \le \tilde{z}_{\epsilon,\alpha_\epsilon}(t), \quad (1-2w) z_{\epsilon,\beta_\epsilon}(t) \le \tilde{z}_{\epsilon,\beta_\epsilon}(t) \quad \text{for} \quad z_{\epsilon,i}(t) \le 0. \tag{12}$$

i.e.

$$\tilde{z}_{\epsilon,\alpha_\epsilon}(t) \ge \begin{cases} u(t) - y_\epsilon(t) & \text{for } u(t) - y_\epsilon(t) \ge 0 \\ (1-2w)\,(u(t) - y_\epsilon(t)) & \text{for } u(t) - y_\epsilon(t) \le 0 \end{cases}$$

$$\tilde{z}_{\epsilon,\beta_\epsilon}(t) \ge \begin{cases} y_\epsilon(t) - u(t) & \text{for } y_\epsilon(t) - u(t) \ge 0 \\ (1-2w)\,(y_\epsilon(t) - u(t)) & \text{for } y_\epsilon(t) - u(t) \le 0 \end{cases}$$

on $\langle a, b\rangle$.

In the case $w = 0$ only we may choose $\tilde{z}_{\epsilon,\alpha_\epsilon}(t)$, $\tilde{z}_{\epsilon,\beta_\epsilon}(t)$ such that

$$\tilde{z}_{\epsilon,\alpha_\epsilon}(t) + \tilde{z}_{\epsilon,\beta_\epsilon}(t) \equiv 0 \quad \text{on} \quad \langle a, b\rangle$$

and consequently

$$v_{c,\epsilon,\alpha_\epsilon}(t) + v_{c,\epsilon,\beta_\epsilon}(t) \equiv 0 \quad \text{on} \quad \langle a, b\rangle,$$

as illustrated in Figure 1. More precisely,

$$v_{c,\epsilon,\alpha_\epsilon}(t) = v_{c,\epsilon,\beta_\epsilon}(t) \equiv 0 \quad \text{on} \quad \langle a, b\rangle.$$

It follows from the fact that $v_{c,\epsilon,i}(t)$ are the solutions of Neumann problem for differential equation

$$\epsilon y'' + my = r_{\epsilon,i}(t) \,(= z_{\epsilon,i}(t) - \tilde{z}_{\epsilon,i}(t)) \quad i = \alpha_\epsilon, \beta_\epsilon.$$

Thus, $(A1')$ holds.

Remark 2.2 We remark, that the function $r_{\epsilon,\alpha_\epsilon}(t) \equiv 0 \wedge r_{\epsilon,\beta_\epsilon}(t) \equiv 0$ on $\langle a, b\rangle$ in following two cases only

1. $w = 0$, $u'(a) = u'(b) = 0$
2. $w \ne 0$, $u'(a) = u'(b) = 0$, $u''(t) \equiv 0$ on $\langle a, b\rangle$.

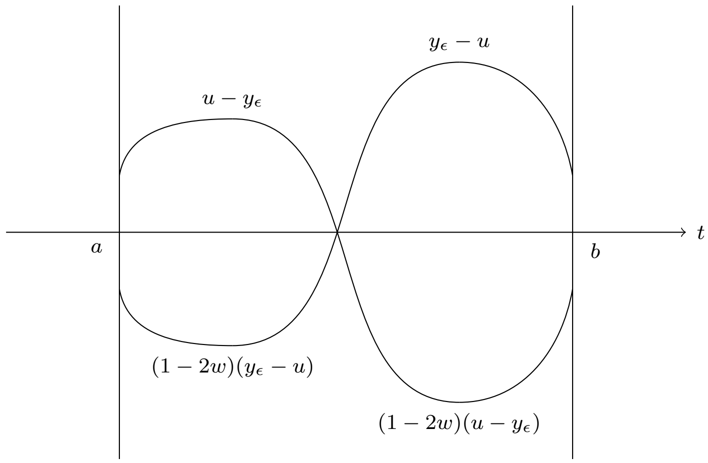

Figure 1: The regions for $\tilde{z}_{\epsilon,\alpha_\epsilon}(t)$ and $\tilde{z}_{\epsilon,\beta_\epsilon}(t)$, $w = 0$

In both cases $z_{\epsilon,i}(t) \equiv \tilde{z}_{\epsilon,i}(t)$, $i = \alpha_\epsilon, \beta_\epsilon$ and

1. $z_{\epsilon,i}(t) = \omega_{1,\epsilon,i}(t)$, $i = \alpha_\epsilon, \beta_\epsilon$
2. $z_{\epsilon,i}(t) = 0$, $i = \alpha_\epsilon, \beta_\epsilon$,

respectively. The second one is trivial, $y_\epsilon(t) = u(t) = const$ on $\langle a, b\rangle$.

In other cases (in all ones is $|u'(a)|+|u'(b)| \neq 0$), if $r_{\epsilon,\alpha_\epsilon}(t) \equiv 0 \wedge r_{\epsilon,\beta_\epsilon}(t) \equiv 0$ for $w = 0$ or $r_{\epsilon,\alpha_\epsilon}(t) \equiv 0 \vee r_{\epsilon,\beta_\epsilon}(t) \equiv 0$ for $w \neq 0$ then, as follows from the integral equation (6),

$$z_{\epsilon,\alpha_\epsilon}(t) = \tilde{z}_{\epsilon,\alpha_\epsilon}(t) = \omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\alpha_\epsilon}(t)$$

or

$$z_{\epsilon,\beta_\epsilon}(t) = \tilde{z}_{\epsilon,\beta_\epsilon}(t) = \omega_{0,\epsilon}(t) + \omega_{1,\epsilon,\beta_\epsilon}(t)$$

are not identically equal to zero on $\langle a, b\rangle$ and it leads to contradiction with (12) and consequently with $(A1')$ (for $w \neq 0$) or (10) (for $w = 0$).

Example 2.3 Consider Neumann problem (2) for linear Diff. Eq.

$$\epsilon y'' + ky = f(t), \quad t \in \langle a, b\rangle, \quad k > 0, \quad 0 < \epsilon << 1 \tag{13}$$

where $f \in C^1(\langle a, b\rangle)$.

Assume that $f'(a) = f'(b) = 0$. This is case 1 from Remark 2.2. Therefore,

$$z_{\epsilon,i}(t) = \tilde{z}_{\epsilon,i}(t) = \omega_{1,\epsilon,i}(t), \quad i = \alpha_\epsilon, \beta_\epsilon$$

and we may define the lower and upper solutions by

$$\alpha_\epsilon(t) = u(t) - z_{\epsilon,\alpha_\epsilon}(t) = u(t) - \omega_{1,\epsilon,\alpha_\epsilon}(t) = \frac{f(t)}{k} - \omega_{1,\epsilon,\alpha_\epsilon}(t)$$

and

$$\beta_\epsilon(t) = u(t) + z_{\epsilon,\beta_\epsilon}(t) = u(t) + \omega_{1,\epsilon,\beta_\epsilon}(t) = \frac{f(t)}{k} + \omega_{1,\epsilon,\beta_\epsilon}(t)$$

On the basis of Theorem 2.1 and the theory of lower and upper solutions [1] there exists solution y_ϵ, $\epsilon \in M$ of problem (13), (2) such that

$$\frac{f(t)}{k} - \omega_{1,\epsilon,\alpha_\epsilon}(t) \le y_\epsilon(t) \le \frac{f(t)}{k} + \omega_{1,\epsilon,\beta_\epsilon}(t), \quad \epsilon \in M$$

i.e.

$$y_\epsilon(t) = \frac{f(t)}{k} + O(\epsilon), \quad \epsilon \in M.$$

In detail, we obtain that

$$\beta_\epsilon(t) - \alpha_\epsilon(t) = z_{\epsilon,\beta_\epsilon}(t) + z_{\epsilon,\alpha_\epsilon}(t) = \omega_{1,\epsilon,\beta_\epsilon}(t) + \omega_{1,\epsilon,\alpha_\epsilon}(t) \equiv 0 \quad \text{on} \quad \langle a, b\rangle,$$

therefore

$$y_\epsilon(t) = \alpha_\epsilon(t) = \beta_\epsilon(t), \quad \epsilon \in M.$$

On the other side, using the fact that a solution of (13), (2) we may write in the form

$$y_\epsilon(t) = \frac{\cos\left[\sqrt{\frac{m}{\epsilon}}(t-a)\right] \int\limits_a^b \cos\left[\sqrt{\frac{m}{\epsilon}}(b-s)\right] \frac{f(s)}{\epsilon} \mathrm{d}s}{\sqrt{\frac{m}{\epsilon}} \sin\left[\sqrt{\frac{m}{\epsilon}}(b-a)\right]} + \int\limits_a^t \frac{\sin\left[\sqrt{\frac{m}{\epsilon}}(t-s)\right] \frac{f(s)}{\epsilon} \mathrm{d}s}{\sqrt{\frac{m}{\epsilon}}} \mathrm{d}s$$

it is easy to verify, by integrating twice per-partes in $\omega_{1,\epsilon,i}(t)$, $i = \alpha_\epsilon, \beta_\epsilon$ that

$$y_\epsilon(t) = u(t) - \omega_{1,\epsilon,\alpha_\epsilon}(t) = u(t) + \omega_{1,\epsilon,\beta_\epsilon}(t),$$

i.e. the choice of $\alpha_\epsilon, \beta_\epsilon$ in your example is optimal.

Now we introduce the notion of optimality of $v_{c,\epsilon,i}(t)$.

3 Optimality of $v_{c,\epsilon,i}(t)$

We say that $r^*_{\epsilon,i}(t) \le 0$ on $\langle a, b\rangle$ is $v_{c,\epsilon,i}$ − optimal, if $v^*_{c,\epsilon,i}(t) \ge e_{c,\epsilon,i}(t)$ on $\langle a, b\rangle$ where

$$e_{c,\epsilon,\alpha}(t) = u(t) - y_\epsilon(t) - \omega_{0,\epsilon}(t) - \omega_{1,\epsilon,\alpha_\epsilon}(t)$$
$$e_{c,\epsilon,\beta}(t) = y_\epsilon(t) - u(t) - \omega_{0,\epsilon}(t) - \omega_{1,\epsilon,\beta_\epsilon}(t), \quad i = \alpha_\epsilon, \beta_\epsilon,$$

y_ϵ is a solution of (1), (2) and

1. The functions

$$\tilde{z}^*_{\epsilon,i}(t) = z^*_{\epsilon,i}(t) - r^*_{\epsilon,i}(t) = \omega_{0,\epsilon}(t) + \omega_{1,\epsilon,i}(t) + v^*_{c,\epsilon,i}(t) - r^*_{\epsilon,i}(t), \quad i = \alpha_\epsilon, \beta_\epsilon$$

satisfy $(A1')$.

2. For every $v^{\triangle}_{c,\epsilon,i}(t) \geq e_{c,\epsilon,i}(t)$ on $\langle a, b\rangle$ such that

$$\tilde{z}^{\triangle}_{\epsilon,i}(t) = z^{\triangle}_{\epsilon,i}(t) - r^{\triangle}_{\epsilon,i}(t) = \omega_{0,\epsilon}(t) + \omega_{1,\epsilon,i}(t) + v^{\triangle}_{c,\epsilon,i}(t) - r^{\triangle}_{\epsilon,i}(t), \quad i = \alpha_\epsilon, \beta_\epsilon$$

satisfy $(A1')$ is

$$\sup_{t\in\langle a,b\rangle} \left(v^{\triangle}_{c,\epsilon,i}(t) - e_{c,\epsilon,i}(t)\right) \geq \sup_{t\in\langle a,b\rangle} \left(v^*_{c,\epsilon,i}(t) - e_{c,\epsilon,i}(t)\right), \quad i = \alpha_\epsilon, \beta_\epsilon.$$

Remark 3.1 For problem (13), (2), the functions $e_{c,\epsilon,\alpha}(t)$, $e_{c,\epsilon,\beta}(t)$ may be computed exactly

$$e_{c,\epsilon,\alpha}(t) = \left(\frac{-f'(b) + |f'(b)|}{k}\right) \frac{\cos\left[\sqrt{\frac{k}{\epsilon}}(t-a)\right]}{\sqrt{\frac{k}{\epsilon}}\sin\left[\sqrt{\frac{k}{\epsilon}}(b-a)\right]} + \left(\frac{f'(a) + |f'(a)|}{k}\right) \frac{\cos\left[\sqrt{\frac{k}{\epsilon}}(b-t)\right]}{\sqrt{\frac{k}{\epsilon}}\sin\left[\sqrt{\frac{k}{\epsilon}}(b-a)\right]},$$

$$e_{c,\epsilon,\beta}(t) = \left(\frac{f'(b) + |f'(b)|}{k}\right) \frac{\cos\left[\sqrt{\frac{k}{\epsilon}}(t-a)\right]}{\sqrt{\frac{k}{\epsilon}}\sin\left[\sqrt{\frac{k}{\epsilon}}(b-a)\right]} + \left(\frac{-f'(a) + |f'(a)|}{k}\right) \frac{\cos\left[\sqrt{\frac{k}{\epsilon}}(b-t)\right]}{\sqrt{\frac{k}{\epsilon}}\sin\left[\sqrt{\frac{k}{\epsilon}}(b-a)\right]}.$$

These functions do not satisfy the boundary conditions (2) required for $v_{c,\epsilon,i}(t)$ if $|f'(a)| + |f'(b)| \neq 0$. Consequently, $v^*_{c,\epsilon,i}(t)$, are not identically equal to $e_{c,\epsilon,i}(t)$ on $\langle a, b\rangle$, therefore α_ϵ and β_ϵ are not identically equal to y_ϵ on $\langle a, b\rangle$.

For $|f'(a)| + |f'(b)| = 0$ is $v^*_{c,\epsilon,i}(t) = e_{c,\epsilon,i}(t) \equiv 0$ (i.e. $r^*_{\epsilon,i}(t) \equiv 0$), $i = \alpha_\epsilon, \beta_\epsilon$ which corresponds with Remark 2.2.

For $e_{c,\epsilon,\alpha}(t)$, $e_{c,\epsilon,\beta}(t)$ we obtain the following estimates

$$|e_{c,\epsilon,\alpha}(t)| \leq \left(\frac{-f'(b) + |f'(b)|}{k} + \frac{f'(a) + |f'(a)|}{k}\right) \frac{\sqrt{\epsilon}}{\sqrt{k}\sin\lambda},$$
$$|e_{c,\epsilon,\beta}(t)| \leq \left(\frac{f'(b) + |f'(b)|}{k} + \frac{-f'(a) + |f'(a)|}{k}\right) \frac{\sqrt{\epsilon}}{\sqrt{k}\sin\lambda}.$$

Thus, the functions $e_{c,\epsilon_n,i}$, $i = \alpha_\epsilon, \beta_\epsilon$ converge uniformly on $\langle a, b\rangle$ to zero for every sequence $\{\epsilon_n\}_{n=0}^{\infty}$, such that $\epsilon_n \in J_n$, i.e. $\epsilon_n \to 0^+$.

ACKNOWLEDGEMENTS. This research was supported by Slovak Grant Agency, Ministry of Education of Slovak Republic under grant number 1/0068/08.

References

[1] C. De Coster, P. Habets, *Two-Point Boundary Value Problems: Lower and Upper Solutions*, Volume 205 (Mathematics in Science and Engineering), Elsevier Science; 1 edition, 2006.

[2] J. Mawhin, *Points fixes, points critiques et problemes aux limites*, Semin. Math. Sup. no. 92, Presses Univ. Montreal, 1985.

[3] R. Vrábeľ, *Singularly perturbed semilinear Neumann problem with non-normally hyperbolic critical manifold*, E. J. Qualitative Theory of Diff. Equ., No. **9** (2010), 1-11.

Received: August, 2010