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Strong Convergence of Iterative Sequence to a Common Fixed Point of a Finite Family Nonexpansive Mappings in Banach Spaces

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Abstract

The purpose of this paper is to give a regularization method to obtain the strong convergence of iterative sequence to a solution of the problem of finding a common fixed point of a finite family nonexpansive mappings in an uniformly convex and uniformly smooth Banach space.

Keywords: Accretive operators, uniformly smooth and uniformly convex Banach space, sunny nonexpansive retraction, weak sequential continuous mapping, and regularization

1 Introduction and Preliminaries

Let E be a Banach space with its dual space E^*. For the sake of simplicity, the norm of E and E^* are denoted by the symbol $\|.\|$. We write $\langle x, x^* \rangle$ instead of $x^*(x)$ for $x^* \in E^*$ and $x \in E$. We use the symbols $\rightharpoonup$ and $\longrightarrow$ to denote the weak convergence and strong convergence, respectively.

Definition 1.1. A Banach space E is said to be uniformly convex if for any $\varepsilon \in (0, 2]$ the inequalities $\|x\| \leq 1,\ \|y\| \leq 1,\ \|x - y\| \geq \varepsilon$ imply there exists a $\delta = \delta(\varepsilon) \geq 0$ such that

$$\frac{\|x + y\|}{2} \leq 1 - \delta.$$

The function

$$\delta_E(\varepsilon) = \inf\{1 - 2^{-1}\|x + y\| :\ \|x\| = \|y\| = 1,\ \|x - y\| = \varepsilon\} \tag{1.1}$$

is called the modulus of convexity of the space E and it defined on the interval $[0, 2]$ is continuous, increasing and $\delta_E(0) = 0$. The space E is uniformly convex

if and only if $\delta_E(\varepsilon) > 0,\ \forall\varepsilon > 0$.
The function

$$\rho_E(\tau) = \sup\{2^{-1}\big(\|x+y\| + \|x-y\|\big) - 1 :\ \|x\| = 1,\ \|y\| = \tau\}, \tag{1.2}$$

is called the modulus of smoothness of the space E and it defined on the interval $[0, +\infty)$ is convex, continuous, increasing and $\rho_E(0) = 0$.

Definition 1.2. A Banach space E is said to be uniformly smooth, if

$$\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau} = 0. \tag{1.3}$$

Definition 1.3. A mapping j from E onto E^* satisfying the condition

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 \text{ and } \|f\| = \|x\|\} \tag{1.4}$$

is called the normalized duality mapping of E.

Definition 1.4. An operator $A :\ E \longrightarrow 2^{E^*}$ is said to be monotone on $D(A) \subseteq E$ if

$$\langle u - v, x - y\rangle \geq 0,\ \forall x, y \in D(A),\ \forall u \in A(x),\ \forall v \in A(y). \tag{1.5}$$

Definition 1.5. An operator $A :\ D(A) \subseteq E \longrightarrow 2^E$ is called accretive if for all $x, y \in D(A)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y)\rangle \geq 0,\ \forall u \in A(x),\ v \in A(y). \tag{1.6}$$

Definition 1.6. An operator $A :\ E \longrightarrow 2^E$ is called $m-$accretive if it is an accretive operator and the range $R(\lambda A + I) = E$ for all $\lambda > 0$, where I denote the identity of E.

Definition 1.7. A mapping $T :\ C \longrightarrow E$ is said to be nonexpansive on the closed convex subset C of Banach space E if

$$\|Tx - Ty\| \leq \|x - y\|,\ \forall x, y \in C. \tag{1.7}$$

If $T :\ C \longrightarrow E$ is a nonexpansive then $I - T$ is accretive operator. In case of the subset C coincides E then $I - T$ is $m-$accretive operator.

Definition 1.8. Let G be a nonempty closed convex subset of E. A mapping $Q_G :\ E \longrightarrow G$ is said to be

i) a retraction onto G if $Q_G^2 = Q_G$;

ii) a nonexpansive retraction if it also satisfies the inequality

$$\|Q_Gx - Q_Gy\| \leq \|x - y\|,\ \forall x, y \in E; \tag{1.8}$$

iii) a sunny retraction if for all $x \in E$ and for all $t \in [0, +\infty)$,

$$Q_G(Q_G x + t(x - Q_G x)) = Q_G x. \tag{1.9}$$

Proposition 1.9. *[9] Let G be a nonempty closed convex subset of E. A mapping $Q_G : E \longrightarrow G$ is a sunny nonexpansive retraction if and only if*

$$\langle x - Q_G x, J(\xi - Q_G x)\rangle \leq 0, \ \forall x \in E, \ \forall \xi \in G. \tag{1.10}$$

We consider the problem

$$\text{Finding an element } x^* \in S = \cap_{i=1}^{N} F(T_i), \tag{1.11}$$

where $N \geq 1$ is an integer and each $F(T_i)$ is assumed to be the set of fixed points of the nonexpansive mapping $T_i : C \longrightarrow E$ from a closed convex subset C of an uniformly convex and uniformly smooth Banach space E into E.

The problem of finding a fixed point of a nonexpansive mapping is equivalent to the problem of finding a zero of the following operator equation

$$0 \in A(x), \tag{1.12}$$

involving the accretive mapping A.

One popular method of solving equation $0 \in A(x)$ is the proximal point algorithm of Rockafellar [7] which is recognized as a powerful and successful algorithm in finding a zero of monotone operators. Starting from any initial guess $x_0 \in H$, this proximal point algorithm generates a sequence $\{x_n\}$ given by

$$x_{n+1} = J_{c_n}^{A}(x_n + e_n), \tag{1.13}$$

where $J_r^A = (I + rA)^{-1}$, $\forall r > 0$ is the resolvent of A on the space H. Rockafellar [7] proved the weak convergence of his algorithm (1.13) provided that the regularization sequence $\{c_n\}$ remains bounded away from zero and the error sequence $\{e_n\}$ satisfies the condition $\sum_{n=0}^{\infty} \|e_n\| < \infty$. Güler's example [6] however shows that in infinite dimensional Hilbert space, proximal point algorithm (1.13) has only weak convergence. Recently several authors proposed modifications of Rockafellar's proximal point algorithm (1.13) to have strong convergence. For example, Solodov and Svaiter [4] and Kamimura and Takahashi [8] studied a modified proximal point algorithm by an additional projection at each step of iteration. Lehdili and Moudafi [5] obtained the convergence of the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = J_{c_n}^{A_n}(x_n), \tag{1.14}$$

where $A_n = \mu_n I + A$ is viewed as a Tikhonov regularization of A. When A is maximal monotone in Hilbert space H, Xu [1]; Song and Yang [11] used the technique of nonexpansive mappings to get convergence theorems for $\{x_n\}$ defined by the perturbed version of the algorithm (1.14)

$$x_{n+1} = J_{r_n}^A(t_n u + (1 - t_n)x_n). \tag{1.15}$$

The equation (1.15) can be written in the form equivalent following

$$r_n A(x_{n+1}) + x_{n+1} \ni t_n u + (1 - t_n)x_n. \tag{1.16}$$

Alber, Reich and Yao [10] modified proximal point algorithm (1.13) in the form

$$\lambda_n A(Q_C y) + y = x_n, \ x_{n+1} = Q_C y, \ x_n \in C \tag{1.17}$$

for the problem of finding a fixed point of nonexpasive $T: \ C \longrightarrow E$, where $A = I - T$ and Q_C is a sunny nonexpansive retraction from E onto C. The main result of this algorithm is the weak convergence of iterative sequence $\{x_n\}$ to an element in $F(T)$.

In this paper we combine algorithms (1.16) and (1.17) to obtain the strong convergence of iterative sequence $\{x_n\}$ to a solution of the problem (1.11) in the form

$$r_n \sum_{i=1}^{N} A_i(Q_C y) + y = t_n u + (1 - t_n)x_n, \ x_{n+1} = Q_C y, \ n \geq 0, \tag{1.18}$$

where $x_0, \ u \in C, \ A_i = I - T_i, \ i = 1, 2, ..., N$.

2 Main results

First, we need the following lemmas in the proof of our results.

Lemma 2.1. *[9] If $A = I - T$ with a nonexpansive mapping T then for all $x, y \in D(T)$, the domain of T,*

$$\langle Ax - Ay, J(x - y)\rangle \geq L^{-1}R^2\delta_E\left(\frac{\|Ax - Ay\|}{4R}\right), \tag{2.1}$$

where $\|x\| \leq R, \ \|y\| \leq R$ and $1 < L < 1.7$ is Figiel constant.

Lemma 2.2. *[2] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property:*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n\beta_n + \sigma_n, \ \forall n \geq 0$$

where $\{\lambda_n\}, \ \{\beta_n\}$ and $\{\sigma_n\}$ satisfy the conditions

i) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

ii) $\limsup_{n\longrightarrow\infty} \beta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\lambda_n \beta_n| < \infty$;

iii) $\sigma_n \geq 0,\ \forall n \geq 0$ *and* $\sum_{n=0}^{\infty} \sigma_n < \infty$.

Then $\{a_n\}$ *converges to zero as* $n \longrightarrow \infty$.

Lemma 2.3. *[9] In a uniformly smooth Banach space, for all* $x, y \in E$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + c\rho_E(\|y\|), \tag{2.2}$$

where $c = 48 \max(L, \|x\|, \|y\|)$.

Lemma 2.4 (demiclosedness principle). *Let* E *be a reflexive Banach space having a weakly sequentially continuous duality mapping,* C *a nonempty closed convex subset of* E*, and* $T :\ C \longrightarrow E$ *a nonexpansive mapping. Then the mapping* $I - T$ *is demiclosed on* C*, where* I *is the identity mapping; that is,* $x_n \rightharpoonup x$ *in* E *and* $(I - T)x_n \longrightarrow y$ *imply that* $x \in C$ *and* $(I - T)x = y$.

Theorem 2.5. *Suppose that* E *is a uniformly convex and uniformly smooth Banach space which admits a weakly sequentically continuous normalized duality mapping* j *from* E *to* E^*. *Let* C *be a nonempty closed convex subset of* E *and let* $T_i :\ C \longrightarrow E,\ i = 1, 2, ..., N$ *be nonexpansive mappings with* $S = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\{x_n\}$ *be any sequence of iterates generated by* (1.18). *If the sequences* $\{r_n\} \subset (0, +\infty)$ *and* $\{t_n\} \subset (0, 1)$ *satisfy*

i) $\lim_{n\to\infty} t_n = 0$, $\sum_{n=0}^{\infty} t_n = +\infty$;

ii) $\lim_{n\to\infty} r_n = +\infty$,

then the sequence $\{x_n\}$ *converges strongly to* $Q_S u$*, where* $Q_S :\ E \longrightarrow S$ *is a sunny nonexpansive retraction from* E *onto* S.

Proof. For each $x^* \in S$, we have

$$\langle r_n \sum_{i=1}^{N} A_i(x_{n+1}), j(x_{n+1} - x^*)\rangle \geq 0,\ \forall n \geq 0. \tag{2.3}$$

Therefore,

$$\langle t_n u + (1 - t_n)x_n - y, j(x_{n+1} - x^*)\rangle \geq 0,\ \forall n \geq 0. \tag{2.4}$$

So, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq [t_n\|u - x^*\| + (1 - t_n)\|x_n - x^*\|].\|x_{n+1} - x^*\| \\
&+ \langle x_{n+1} - y, j(x_{n+1} - y)\rangle \\
&= [t_n\|u - x^*\| + (1 - t_n)\|x_n - x^*\|]\|x_{n+1} - x^*\| \\
&+ \langle Q_C y - y, j(Q_C y - x^*)\rangle \\
&\leq [t_n\|u - x^*\| + (1 - t_n)\|x_n - x^*\|].\|x_{n+1} - x^*\|.
\end{aligned}$$

Consequently,

$$\begin{aligned}\|x_{n+1}-x^*\| &\le t_n\|u-x^*\|+(1-t_n)\|x_n-x^*\|\\ &\le \max(\|u-x^*\|,\|x_n-x^*\|)\\ &\vdots\\ &\le \max(\|u-x^*\|,\|x_0-x^*\|),\ \forall n\ge 0.\end{aligned}$$

Therefore, the sequence $\{x_n\}$ is bounded. Every bounded set in a reflexive Banach space is relatively weakly compact. This means that there exists some subsequence $\{x_{n_k}\}\subseteq\{x_n\}$ which converges to a limit point $\overline{x}$. Since C is closed and convex, it is also weakly closed. Therefore $\overline{x}\in C$.

Suppose $\|x_n\|\le R$ and $\|x^*\|\le R$ with $R>0$. By Lemma 2.1, we have

$$\begin{aligned}\delta_E(\frac{\|A_i(x_{n+1})\|}{4R}) &\le \frac{L}{R^2r_n}\langle r_nA_i(x_{n+1}),j(x_{n+1}-x^*)\rangle\\ &\le \frac{L}{R^2r_n}\langle r_n\sum_{k=1}^{N}A_k(x_{n+1}),j(x_{n+1}-x^*)\rangle\\ &\le \frac{L}{R^2r_n}\|t_nu+(1-t_n)x_n-x_{n+1}\|.\|x_{n+1}-x^*\|\\ &+\frac{L}{R^2r_n}\langle Q_Cy-y,j(Q_Cy-x^*)\rangle\\ &\le \frac{L}{R^2r_n}\|t_nu+(1-t_n)x_n-x_{n+1}\|.\|x_{n+1}-x^*\|\longrightarrow 0,\ n\longrightarrow\infty,\end{aligned}$$

for every $i=1,2,...,N$.
Since modulus of convexity δ_E is continuous and E is a uniformly convex, $A_i(x_{n+1})\longrightarrow 0,\ i=1,2,...,N$. It is clear that $\overline{x}\in S$ because the operators A_i are demiclosed. Hence, noting inequality (1.10), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle u-Q_Su,j(x_n-Q_Su)\rangle &= \lim_{k\to\infty}\langle u-Q_Su,j(x_{n_k}-Q_Su)\rangle\\ &=\langle u-Q_Su,j(\overline{x}-Q_Su)\rangle\le 0.\end{aligned}\tag{2.5}$$

Next, we have

$$\begin{aligned}\|x_{n+1} - Q_S u\|^2 &= \langle x_{n+1} - y + y - Q_S u, j(x_{n+1} - Q_S u)\rangle \\ &= \langle Q_C y - y, j(Q_C y - Q_S u)\rangle + \langle y - Q_S u, j(x_{n+1} - Q_S u)\rangle \\ &\le \langle y - Q_S u, j(x_{n+1} - Q_S u)\rangle \\ &= -r_n\langle \sum_{i=1}^{N} A_i(x_{n+1}), j(x_{n+1} - Q_S u)\rangle \\ &+ \langle t_n u + (1-t_n)x_n - Q_S u, j(x_{n+1} - Q_S u)\rangle \\ &\le \frac{1}{2}[\|t_n(u - Q_S u) + (1-t_n)(x_n - Q_S u)\|^2 + \|x_{n+1} - Q_S u\|^2].\end{aligned}$$

Therefore,

$$\begin{aligned}\|x_{n+1} - Q_S u\|^2 &\le \|t_n(u - Q_S u) + (1-t_n)(x_n - Q_S u)\|^2 \\ &\le (1-t_n)^2\|x_n - Q_S u\|^2 + 2t_n(1-t_n)\langle u - Q_S u, j(x_n - Q_S u)\rangle \\ &+ c\rho_E(t_n\|u - Q_S u\|).\end{aligned}$$

Consequently,

$$\|x_{n+1} - Q_S u\|^2 \le (1-t_n)\|x_n - Q_S u\|^2 + t_n\beta_n, \tag{2.6}$$

where

$$\beta_n = 2(1-t_n)\langle u - Q_S u, j(x_n - Q_S u)\rangle + c\frac{\rho_E(t_n\|u - Q_S u\|)}{t_n}.$$

Since E is uniformly smooth, hence $\dfrac{\rho_E(t_n\|u - Q_S u\|)}{t_n} \longrightarrow 0,\ n \longrightarrow \infty$. Combine with (2.5), we obtain $\limsup_{n\to\infty}\beta_n \le 0$. So, an application of Lemma 2.2 onto (2.6), we have $x_n \longrightarrow Q_S u$.

□

Theorem 2.6. *Suppose that E is a uniformly convex and uniformly smooth Banach space which admits a weakly sequentically continuous normalized duality mapping j from E to E^*. Let C be a nonempty closed convex subset of E and let $T_i :\ C \longrightarrow E,\ i = 1, 2, ..., N$ be nonexpansive mappings with $S = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{u_n\}$ be any sequence of iterates generated by*

$$\begin{cases} r_n\sum_{i=1}^{N} A_i(Q_C z) + z = Q_C(t_n u + (1-t_n)u_n + e_n),\ u_0,\ u \in C, \\ u_{n+1} = Q_C z,\ n \ge 0. \end{cases} \tag{2.7}$$

If the sequences $\{r_n\} \subset (0, +\infty)$, $\{t_n\} \subset (0, 1)$ and the error sequence $\{e_n\}$ satisfy *i)* $\lim_{n\to\infty} t_n = 0$, $\sum_{n=0}^{\infty} t_n = +\infty$, $\lim_{n\to\infty} r_n = +\infty$;

ii) $\sum_{n=0}^{\infty} \|e_n\| < +\infty$,

then the sequence $\{u_n\}$ *converges strongly to* $Q_S u$*, where* $Q_S : \ E \longrightarrow S$ *is a sunny nonexpansive retraction from* E *onto* S*.*

Proof. From (1.18) and (2.7) we have

$$r_n \sum_{i=1}^{N} \big(A_i(Q_C z) - A_i(Q_C y)\big) + z - y = Q_C(t_n u + (1-t_n)u_n + e_n) \\ - t_n u - (1-t_n)x_n. \tag{2.8}$$

By the accretivenees of operators A_i, we obtain

$$r_n \langle \sum_{i=1}^{N} \big(A_i(Q_C z) - A_i(Q_C y)\big), j(u_{n+1} - x_{n+1})\rangle \geq 0, \ \forall n \geq 0.$$

Therefore,

$$\langle z - y, j(u_{n+1} - x_{n+1})\rangle \leq \langle Q_C(t_n u + (1-t_n)u_n + e_n) \\ - Q_C(t_n u + (1-t_n)x_n), j(u_{n+1} - x_{n+1})\rangle. \tag{2.9}$$

The inequality (2.9) equivalent to

$$\langle z - Q_C z, j(Q_C z - x_{n+1})\rangle + \langle Q_C z - Q_C y, j(u_{n+1} - x_{n+1})\rangle \\ + \langle Q_C y - y, j(u_{n+1} - Q_C y)\rangle \leq \big[(1-t_n)\|u_n - x_n\| + \|e_n\|\big]\|u_{n+1} - x_{n+1}\|.$$

From Proposition 1.9, we have $\langle z - Q_C z, j(Q_C z - x_{n+1})\rangle \geq 0$ and $\langle Q_C y - y, j(u_{n+1} - Q_C y)\rangle \geq 0$. Hence, we obtain

$$\|u_{n+1} - x_{n+1}\| \leq (1-t_n)\|u_n - x_n\| + \|e_n\|, \ \forall n \geq 0. \tag{2.10}$$

Apply Lemma 2.2 for $a_n = \|u_n - x_n\|$, $\beta_n = 0$ and $\sigma_n = \|e_n\|$, we obtain $\|u_n - x_n\| \longrightarrow 0, \ n \longrightarrow \infty$. Since the sequence $\{x_n\}$ converges strongly to $Q_S u$, hence the sequence $\{u_n\}$ also converges strongly to $Q_S u$.

□

Corollary 2.7. *Let* C *be a nonempty closed convex subset of an uniformly convex and uniformly smoothness Banach space* E *which admits a weakly sequentically continuous normalized duality mapping* j *from* E *to* E^* *and let* $T_i : C \longrightarrow C, \ i = 1, 2, ..., N$ *be nonexpansive mappings such that* $S = \cap_{i=1}^{N} F(T_i) \neq \emptyset$*. Let* $\{r_n\}, \ \{t_n\}$ *be two sequences such that* $r_n > 0, \ 0 < t_n < 1$ *for all* $n \geq 0$*. Then* *i) for each* $u \in C,\ x_0 \in C$ *the equation*

$$r_n \sum_{i=1}^{N} A_i(x_{n+1}) + x_{n+1} = Q_C(t_n u + (1-t_n)x_n + e_n),\ \{e_n\} \subset E,\ n \geq 0 \tag{2.11}$$

has unique solution x_n.

ii) if, in addition, the sequences $\{r_n\}$, $\{t_n\}$ *and the error sequence* $\{e_n\}$ *satisfy* $\lim_{n\to\infty} r_n = +\infty,\ \lim_{n\to\infty} t_n = 0,\ \sum_{n=0}^{\infty} t_n = +\infty,\ \sum_{n=1}^{\infty} \|e_n\| < +\infty$, *then the sequence* $\{x_n\}$ *converges strongly to* $Q_S u$, *where* $Q_S : E \longrightarrow S$ *is a sunny nonexpansive retraction from* E *onto* S.

Proof. i) Let $z = Q_C(t_n u + (1-t_n)x_n + e_n) \in C$, equation (2.11) is equivalent to the equation

$$x_{n+1} = \frac{r_n}{Nr_n + 1} \sum_{i=1}^{N} T_i(x_{n+1}) + \frac{1}{Nr_n + 1} z. \tag{2.12}$$

Since C is a convex set, we can define a function $f : C \longrightarrow C$ is given by

$$f(x) = \frac{r_n}{Nr_n + 1} \sum_{i=1}^{N} T_i(x) + \frac{1}{Nr_n + 1} z.$$

We have

$$\|f(x) - f(x')\| \leq \frac{r_n N}{r_n N + 1} \|x - x'\|,\ \forall x, x' \in C.$$

Hence, the function f is a contraction mapping from C to C. Thus equation (2.12) has unique solution $x_{n+1} \in C$.
ii) Apply Theorem 2.6 for $y = Q_C y = x_{n+1}$, we obtain the sequence $\{x_n\}$ converges strongly to $Q_S u$. □

Corollary 2.8. *Suppose that* E *is a uniformly convex and uniformly smooth Banach space which admits a weakly sequentically continuous normalized duality mapping* j *from* E *to* E^*. *Let* $T_i : E \longrightarrow E,\ i = 1, 2, ..., N$ *be nonexpansive mappings with* $S = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. *If the sequences* $\{r_n\} \subset (0, +\infty)$, $\{t_n\} \subset (0, 1)$ *and the error sequence* $\{e_n\}$ *satisfy*

i) $\lim_{n\to\infty} t_n = 0$, $\sum_{n=0}^{\infty} t_n = +\infty$, $\lim_{n\to\infty} r_n = +\infty$;

ii) $\sum_{n=0}^{\infty} \|e_n\| < +\infty$,

then the sequence $\{v_n\}$ *is defined by* $v_0, u \in E$,

$$r_n \sum_{i=1}^{N} A_i(v_{n+1}) + v_{n+1} = t_n u + (1-t_n)v_n + e_n,\ n \geq 0. \tag{2.13}$$

converges strongly to $Q_S u$, *where* $Q_S : E \longrightarrow S$ *is a sunny nonexpansive retraction from* E *onto* S.

Proof. By the operator $\sum_{i=1}^{N} A_i$ is $m-$accretive, so the equation (2.13) has unique solution v_{n+1}. Apply Theorem 2.6 we obtain the proof of this Corollary. □

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