Int. Journal of Math. Analysis, Vol. 5, 2011, no. 3, 99 - 104

Monotonicity-Preserving Piecewise Rational Cubic Interpolation

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Abstract

An explicit representation of a C^1 piecewise rational cubic spline has been developed, which can produce a monotonic interpolant to given monotonic data. The explicit representation is easily constructed, and numerical experiments indicate that the method produces visually pleasing curves. Furthermore, an error analysis of the interpolant is given.

Keywords: Interpolation, Monotonicity-preserving, Approximation.

1 Introduction

The problem of shape preserving interpolation has been considered by a number of authors in recent years. Brodlie and Butt [1] and Delbourgo [2] have discussed the piecewise cubic interpolation of convex data. Gregory and Delbourgo [4] consider the piecewise interpolation to monotonic data. In this paper, a shape preserving interpolant has been discussed for monotonic data, using rational cubic spline with quadratic denominator. The theory of method, in this paper, has number of advantageous features. It is C^1. No additional restriction needed to exert on the data. The interpolant is not concerned with an arbitrary degree, and when the shape parameters are selected, the curve representation is unique in its solution. The Monotonicity-preserving curve based on function value only can be obtained automatically and the interpolant is stable.

2 The Rational Cubic Interpolant

Let $\{(x_i, f_i), i = 1, 2, \cdots, n\}$ be a given set of data points, where $x_1 < x_2 < \cdots < x_n$, and f_i are the function values at the knots. Let $h_i = x_{i+1} - x_i, \Delta_i = (f_{i+1} - f_i)/h_i, (i = 1, 2, \cdots, n-1)$. For $x \in [x_i, x_{i+1}]$, let $\theta = (x - x_i)/h_i, 0 \le \theta \le 1$. A piecewise rational cubic function $S(x) \in C^1[x_1, x_n]$ and is defined for $x \in [x_i, x_{i+1}]$ as :

$$S_i(x) = \frac{P_i(\theta)}{Q_i(\theta)} = \frac{\begin{array}{c}(1-\theta)^3 v_i f_i + \theta(1-\theta)^2[(2u_iv_i + v_i)f_i + v_ih_id_i] \\ +\theta^2(1-\theta)[(2u_iv_i + u_i)f_{i+1} - u_ih_id_{i+1}] + \theta^3u_if_{i+1}\end{array}}{(1-\theta)^2v_i + 2u_iv_i\theta(1-\theta) + \theta^2u_i}. \quad (1)$$

where d_i are the derivative values at the knots, u_i, v_i are named shape parameters, and $u_i > 0, v_i > 0$. The rational cubic interpolant has the following interpolatory properties:

$$S(x_i) = f_i, S(x_{i+1}) = f_{i+1}, S^{'}(x_i) = d_i, S^{'}(x_{i+1}) = d_{i+1}.$$

In most application, derivative parameters d_i are not given and hence must be determined from the data (x_i, f_i). An obvious choice is mentioned here [6]:

$$\begin{array}{lcl} d_i &=& (h_i\Delta_{i-1} + h_{i-1}\Delta_i)/(h_i + h_{i-1}), \quad i = 2, \cdots, n-1, \\ d_1 &=& \Delta_1 + (\Delta_1 - \Delta_2)h_1/(h_1 + h_2), \\ d_n &=& \Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2})h_{n-1}/(h_{n-2} + h_{n-1}). \end{array} \quad (2)$$

This method is based on three-point difference approximation for the d_i. A Newton series expansion analysis shows that $f_i^{'} - d_i = O(h^2), h = max(h_i)$.

For given bounded data, the derivative approximation Eq. (2) are bounded. Hence, for bounded values of the appropriate shape parameters u_i, v_i, the interpolant Eq. (1)is bounded and unique.

3 Monotonicity-preserving Spline Interpolation

We assume a monotonic increasing data, so that $f_1 \le f_2 \le \cdots \le f_n$, or equivalently $\Delta_i \ge 0, (i = 1, 2, \cdots, n-1)$.(In a similar fashion one can deal with a monotonic decreasing data.) To have a monotonic interpolant $S(x)$, it is necessary that the derivative parameters d_i should satisfy:

$$d_i \ge 0 \quad (d_i \le 0, \ for \ monotonic \ decreasing \ data) i = 1, 2, \cdots, n. \quad (3)$$

Now $S(x)$ is monotonic if and only if $S^{(1)}(x) \ge 0, x \in [x_1, x_n]$. After some simplification, it can be shown that for $x \in [x_i, x_{i+1}]$:

$$S^{(1)}(x) = \frac{\sum_{j=1}^{5} C_{j,i}\theta^{j-1}(1-\theta)^{5-j}}{[Q_i(\theta)]^2}, \quad (4)$$

where

$$C_{1,i} = v_i^2 d_i,$$

$$\begin{aligned}
C_{2,i} &= 2u_iv_i[(2v_i+1)\Delta_i - d_{i+1}],\\
C_{3,i} &= u_iv_i[(2v_i+1)\Delta_i - d_{i+1} + (2u_i+1)\Delta_i - d_i + 4u_iv_i\Delta_i + 2\Delta_i - 2v_id_i - 2u_id_{i+1}\\
C_{4,i} &= 2u_iv_i[(2u_i+1)\Delta_i - d_i],\\
C_{5,i} &= u_i^2 d_{i+1}.
\end{aligned}$$

For the selection of $u_i, v_i > 0$, the denominator in Eq. (4) is positive, therefore the sufficient conditions for Monotonicity on $[x_i, x_{i+1}]$ are $C_{j,i} \geq 0, j = 1, 2, \cdots, 5$. Thus, from Eq. (4) we have the sufficient conditions for monotonicity preservation:

$$\begin{cases} (2v_i+1)\Delta_i - d_{i+1} \geq 0, \\ 4u_iv_i\Delta_i - 2v_id_i - 2u_id_{i+1} \geq 0, \\ (2u_i+1)\Delta_i - d_i \geq 0. \end{cases} \tag{5}$$

If $\Delta_i = 0$, it is necessary to set $d_i = d_{i+1} = 0$,and thus $S_i(x) = f_i = f_{i+1}$. If $\Delta_i > 0$, for the case where the data is strictly monotonic, the sufficient conditions (5) lead to the following constraints:

$$u_i \geq \frac{d_i}{\Delta_i}, \quad v_i \geq \frac{d_{i+1}}{\Delta_i}, \tag{6}$$

It should be noted that if we let $u_i = \frac{d_i}{\Delta_i}, v_i = \frac{d_{i+1}}{\Delta_i}$, the denominator $Q_i(\theta)$ may be equal to 0 when $\theta = 0$ or $\theta = 1$. Thus, we let

$$u_i = \frac{d_i}{\Delta_i} + \alpha, \quad v_i = \frac{d_{i+1}}{\Delta_i} + \alpha, where \begin{cases} \alpha \geq 0, & if d_i = 0 \; or d_{i+1} = 0 \\ \alpha = 0, & other \end{cases} \tag{7}$$

This choice satisfies (5) and it provides visually very pleasant results. When $u_i, v_i \to \infty$, the curve is getting tightened and it may not be desired. So we let $\alpha = 0.1$

Theorem 3.1 *Given a strictly monotonic data , the constraint Eq. (7) is the sufficient condition for the interpolant Eq. (1) to be monotonic .*

4 Error Estimation

For the error estimation of the piecewise rational cubic interpolant Eq. (1), where d_i were replaced with Eq. (2), u_i, v_i were replaced with Eq. (8), we deal with the case when the knots are equally spaced. Let $h = x_{i+1} - x_i, (i = 2, \cdots, n-2)$. Without loss of generality, it is necessary to consider just the subinterval $[x_i, x_{i+1}]$. When $f(x) \in C^2[x_1, x_n]$ and $S(x)$ is the rational cubic interpolatory function of $f(x)$, using the Peano-Kernel Theorem [3] gives

$$R[f] = f(x) - S(x) = \int_{x_{i-1}}^{x_{i+2}} f^{(2)}(\tau) R_x[(x-\tau)_+] d\tau$$

where

$$R_x[(x-\tau)_+] = \begin{cases} (x-\tau)-(x_i-\tau)(1-\theta)-(x_{i+1}-\tau)\theta \\ -\dfrac{\theta(1-\theta)^2 v_i(x_i-\tau-h)}{2\omega(u_i,v_i,\theta)}, & x_{i-1}<\tau<x_i; \\ (x-\tau)-(x_{i+1}-\tau)\theta \\ -\dfrac{\theta^2(1-\theta)u_i(x_{i+1}-\tau-h)-\theta(1-\theta)^2 v_i(x_{i+1}-\tau)}{2\omega(u_i,v_i,\theta)}, & x_i<\tau<x; \\ -(x_{i+1}-\tau)\theta \\ -\dfrac{\theta^2(1-\theta)u_i(x_{i+1}-\tau-h)-\theta(1-\theta)^2 v_i(x_{i+1}-\tau)}{2\omega(u_i,v_i,\theta)}, & x<\tau<x_{i+1}; \\ \dfrac{\theta^2(1-\theta)u_i(x_{i+2}-\tau)}{2\omega(u_i,v_i,\theta)}, & x_{i+1}<\tau<x_{i+2}; \end{cases}$$

$$= \begin{cases} p(\tau), & x_{i-1}<\tau<x_i; \\ q(\tau), & x_i<\tau<x; \\ r(\tau), & x<\tau<x_{i+1}; \\ k(\tau), & x_{i+1}<\tau<x_{i+2}; \end{cases}$$

and $\omega(u_i,v_i,\theta)=(1-\theta)^2v_i+2u_iv_i\theta(1-\theta)+\theta^2u_i, h=x_{i+1}-x_i$.

Study the properties of the function $p(\tau)$ in $[x_{i-1},x_i]$. we can get

$$p(x_{i-1})=0, p(x_i)=\frac{\theta(1-\theta)^2v_ih}{2\omega(u_i,v_i,\theta)} \quad \geq \quad 0.$$

Thus

$$\int_{x_{i-1}}^{x_i}|p(\tau)|d\tau=\int_{x_{i-1}}^{x_i}p(\tau)d\tau=\frac{\theta(1-\theta)^2v_ih^2}{4\omega(u_i,v_i,\theta)}.$$

Similarly, consider the properties of$q(\tau)$, $r(\tau)$ and $k(\tau)$, we can get

$$\int_{x_i}^{x}|q(\tau)|d\tau=\int_{x_i}^{\tau^*}q(\tau)d\tau+\int_{\tau^*}^{x}-q(\tau)d\tau=\frac{\theta^2(1-\theta)h^2[(1-\theta)^2v_i^2+l^2(u_i,v_i,\theta)]}{4\omega(u_i,v_i,\theta)m(u_i,v_i,\theta)}.$$

$$\int_{x}^{x_{i+1}}|r(\tau)|d\tau=\int_{x}^{\tau_*}-r(\tau)d\tau+\int_{\tau_*}^{x_{i+1}}r(\tau)d\tau=\frac{\theta(1-\theta)^2h^2[\theta^2u_i^2+l^2(u_i,v_i,\theta)]}{4\omega(u_i,v_i,\theta)n(u_i,v_i,\theta)}.$$

$$\int_{x_{i+1}}^{x_{i+2}}|k(\tau)|d\tau=\int_{x_{i+1}}^{x_{i+2}}k(\tau)d\tau=\frac{\theta^2(1-\theta)u_ih^2}{4\omega(u_i,v_i,\theta)}.$$

where

$$l(u_i,v_i,\theta)=(1-\theta)^2v_i+4u_iv_i\theta(1-\theta)+u_i\theta^2.$$

$$m(u_i,v_i,\theta)=(1-\theta)(2-\theta)v_i+4u_iv_i\theta(1-\theta)+u_i\theta^2.$$

$$n(u_i,v_i,\theta)=(1-\theta)^2v_i+4u_iv_i\theta(1-\theta)+u_i\theta(1+\theta).$$

Finally,

$$\begin{aligned}\|R(f)\| &\leq \|f^{(2)}(x)\|[\int_{x_{i-1}}^{x_i}|p(\tau)|d\tau+\int_{x_i}^{x}|q(\tau)|d\tau \\ &\quad +\int_{x}^{x_{i+1}}|r(\tau)|d\tau+\int_{x_{i+1}}^{x_{i+2}}|k(\tau)|d\tau] \\ &\leq \|f^{(2)}(x)\|h^2W(u_i,v_i,\theta).\end{aligned}$$

$$W(u_i,v_i,\theta) = \frac{\begin{array}{c}\theta(1-\theta)\{(1-\theta)v_i+\theta u_i \\ +\theta(1-\theta)[(1-\theta)v_i^2/m(u_i,v_i,\theta)+\theta u_i^2/n(u_i,v_i,\theta)] \\ +l^2(u_i,v_i,\theta)[\theta/m(u_i,v_i,\theta)+(1-\theta)/n(u_i,v_i,\theta)]\}\end{array}}{4\omega(u_i,v_i,\theta)}$$

Theorem 4.1 *If $f(x)\in C^2[x_1,x_n], x_1<x_2<\cdots<x_n$ is an equal-knot spacing, for the given u_i,v_i , $S(x)$ is the corresponding rational cubic interpolation function given in Eq. (1), where d_i were replaced with Eq. (2), and for $x\in[x_i,x_{i+1}],(i=2,\cdots,n-2)$ gives*

$$\|R[f]\|\leq h^2\|f^{(2)}(x)\|c_i, \tag{8}$$

where $c_i=\max\limits_{0\leq\theta\leq 1}W(u_i,v_i,\theta)$, and c_i was named the optimal error constant.

5 Demonstration

We consider a set of monotonic increasing data shown in Table (1)(a data taken randomly) and a set of monotonic decreasing data shown in Table (2)(a data taken randomly). For the demonstration of C^1 monotonic rational cubic curve scheme, the derivatives $d_i's$ will be computed with Eq. (2). The result of applying the piecewise rational cubic scheme to the given data, with the choice of the shape parameters $u_i's, v_i's$, described in Eq. (7),is shown in Fig.1, Fig.2, respectively.Table 3 and Table 4 show the values of u_i,v_i,d_i for Fig.1 and Fig.2 respectively.

Table 1: Data set taken at random

i	1	2	3	4	5	6	7	8
x_i	0.0	2.0	5.0	6.0	10.5	17.0	25.0	26.0
f_i	10.0	10.0	10.5	15.0	18.0	50.0	55.0	70.0

Table 2: Data set taken at random

i	1	2	3	4	5	6	7
x_i	0.0	2.0	8.0	13.0	13.5	20.0	21.0
f_i	75.0	55.0	55.0	53.0	40.0	32.0	25.0

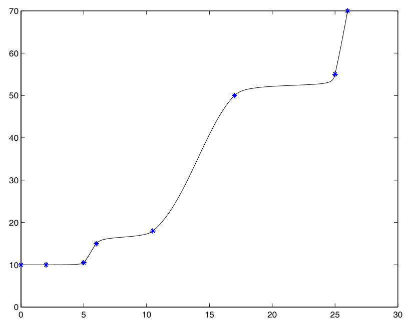

Figure 1: The shape preserving spline for the monotonic increasing data in Table 1.

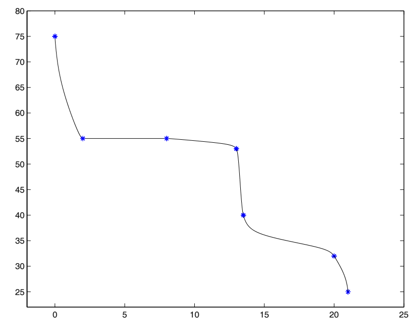

Figure 2: The shape preserving spline for the monotonic decreasing data in Table 2.

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Received: August, 2010