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Sandwich Theorems for Mcshane Integration

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Abstract

In this article, we characterize the S*M integrable functions by the S*HK integrable functions in Banach space with method of sandwich theorems and generalize similar classic propositions in the real line case. We also included some ideas and similar formulations for the case of complete locally convex space.

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1. Introduction

In this paper, we extend some relations between Mcshane integrals and Kurzweil-Henstock integrals for the vector-valued functions, well-known for the real case. In the second section, we deal with properties on Banach space and take a theorem for approximately of Mchane functions. This is the main preposition. In the third section we discuss some new relations in locally convex spaces and we consider the relations between Bochner, Mcshane, and Pettis integrals. In the all material we adopted terminology of known concepts of integration in these spaces from [1] and [2]. All the material consisting of the propositions on mean value ones or so called the theorems of ” squeeze functions” that often are characterized as the sandwich propositions.

2. Definitions and notations

In the section 3 we consider space X as a Banach space while in section 4 we consider it as the locally convex space. Because many definitions on Banach space are well-known, by using [7], we emphasize here some relatively new definitions on locally convex spaces, which are simple generalization of the Banach space case. We follow definitions from [3] and [1].
In the section 4., we consider that the locally convex space X is equipped with the topology τ. Its dual space is tagged X^*. $P(X)$ denotes a family of τ-continuous semi norms on X so that topology of X is generated by $P(X)$.

For the set E of real numbers, μ(E) and $1_E(x)$ denote respectively the Lebesgue outer measure and characteristic function of E. ℵ denotes the family of all Lebesgue measurable subsets of [0,1]. An interval I is a compact subinterval of **R**. A collection of intervals is called no overlapping if their interiors are disjoint. A partition C in [0,1] is a collection $\{(I_i,t_i): i=1,2,...,r\}$, where $I_1,\cdots I_r$ are no overlapping subintervals of [0,1] and $t_1,\cdots,t_r \in [0,1]$. Given a set E⊂R, we say that C is

(i) a partition in E if $\bigcup_{i=1}^{r} I_i \subset E$

(ii) a partition of E if $\bigcup_{i=1}^{r} I_i = E$

(iii) a Perron partition (or K-partition) if $t_i \in I_i$, $i=1,...,r$.

Given $f: S \to X$ and partition $C=\{(I_i,t_i): i=1,...,r\}$ in [0,1], we set

$$\sigma(f,C) = \sum_{i=1}^{r} f(t_i)\mu(I_i) .$$

A gauge δ on E⊂S=[a,b] is a positive function on E. For a given gauge δ on E a partition $C=\{(I_i,t_i): i=1,...,r\}$ in S is called $\delta-$ *fine* if $I_i \subset]t_i-\delta(t_i), t_i+\delta(t_i)[$, $i=1,2,...,r$.

A function $f: S \to X$ is called

(a) Strongly measurable if a sequence $(f_n)_n$ of simple functions exists, such that by norms (in Banach space) almost everywhere $f_n(t) \overset{n}{\to} f(t)$ $a.e.$

(b) measurable by semi norm if for any $p \in P(X)$ there exists a sequence $(f_n^p)_n$ of simple functions such that $\lim_{n\to\infty} p(f_n^p(t) - f(t)) = 0$,a.e.

(c) weakly measurable if the function $x^* f$ is measurable for every $x^* \in X^*$.

Definition 1. A function $f: S \to X$ is called strongly measurable or Bochner integrable if there exists a sequence of simple functions $(f_n)_n$ such that

(i) $f_n(t) \to f(t)$, *a.e.*

(ii) $p(f(t) - f_n(t)) \in L^1(S)$ for each $n \in \mathbf{N}$ and $p \in P(X)$ and $\lim_{n\to\infty} \int_S p(f(t) - f_n(t))dt = 0$ for every $p \in P(X)$;

(iii) $\int_A f_n$ converge in X for each measurable subset A of S. In this case, we put

$$(B)\int_A f = \lim_{n\to\infty} \int_A f_n \ .$$

Definition 2. A function $f : S \to X$ is called integrable by semi norms if for any $p \in P(X)$ there exists a sequence of simple functions $(f_n)_n$ such that

(i) $\lim_{n\to\infty} p(f_n^p(t) - f(t)) = 0$, *a.e*.

(ii) $p(f(t) - f_n(t)) \in L^1(S)$ for each $n \in \mathbf{N}$ and $p \in P(X)$ and $\lim_{n\to\infty} \int_S p(f(t) - f_n(t))dt = 0$ for every $p \in P(X)$;

(iii) for each measurable subset A in S there exists an element $y_A \in X$ such that $\lim_{n\to\infty} p(\int_A f_n^p(t) - y_A) = 0$ for every $p \in P(X)$. We put $y_A = \int_A f$.

Definition 3. A function $f : S \to X$ is said to be Pettis integrable if $x^* f$ is Lesbegue integrable on S: for every $x^* \in X^*$ and for every $E \in \aleph$ there is $\nu_f(E) \in X$ such that $x^*(\nu_f(E)) = \int_E x^* f(t)dt$ for all $x^* \in X^*$.

Following [2] and [8], recall the definitions of Mcshane and Kurzweil-Henstock integrals on locally convex spaces

Definition 4. A function $f : S \to X$ is said to be Mcshane integrable, respectively Kurzweil-Henstock integrable (briefly McS-integrable, respectively KH-integrable) on S if there exists a vector $\omega \in X$ satisfying following property: given $\varepsilon > 0$ and $p \in P(X)$, then exists a gauge δ_p on S such that for every δ_p-fine, respectively Perron partition $C = \{(I_i, t_i) : i = 1,..., r\}$ of S we have

$$p(\sigma(f, C) - \omega) < \varepsilon .$$

3. Extension of the mean theorems on Banach space

Let recall some basic results of the integration on Banach spaces that used for our main proposition. We mainly refer to [7]

Definition 5. A function $f: I \to X$ is said to be strongly Mcshane integrable on I if there is a additive function $F: Z \to X$ (Z –collection of intervals of I) such that for every $\varepsilon > 0$ there exists a gauge δ on I such that

$$\sum_{i=1}^{r} \| f(t_i)\mu(I_i) - F(I_i) \| < \varepsilon$$

for every δ-fine partition $C = \{(I_i, t_i) : i = 1,...,r\}$ of I.

Definition 6. A function $f: I \to X$, where X Banach space, has the property $S*M$ $(S*HK)$ if for every $\varepsilon > 0$ there is a gauge δ on I such that for every M-partions (K-partions) $\delta - fine$ $\{(I_i, t_i) : i = 1,...,r\}$ and $\{(E_j, s_j) : j = 1, 2, \cdots, m\}$ holds

$$\sum_{i=1}^{r} \sum_{j=1}^{m} \| f(t_i) - f(s_j) \| \mu(I_i \cap L_j) < \varepsilon .$$

Theorem 7. ([7] p.150). A function $f: I \to X$ is Bochner integrable if and only if function f has the property $S*M$ or equivalently, if and only if the function f is Mcshane strongly integrable.

Lemma 8. L et X be normed vectorial space and K a konvex set of X. The set K is open if and only if for every two inner points x,y of K there is another inner point z∈K such that the equality $x = \theta z + (1-\theta)y$, for $0 \le \theta \le 1$, holds .

Proof. Since the K is open set for the point x there is a ball R(x,δ) ⊂K. Inner this ball there is an point z of K such that $x = \theta z + (1-\theta)y$, $0 \le \theta \le 1$. It follows that the point x is an inner point of the segment [z,y]. In contrary , from the definition of extremal point ([9], p.390),that cannt' happened if and only if the point x is an extremal point but we suppose that x is a inner point.

Main result

Theorem 9. Let f be a measurable function, $f: S \to X$, the following statements are equivalent

a) function f is S*M integrable

there exists $S*HK$ integrable functions $g(x)$ and $h(x)$ such that for every $\eta > 0$

$$\| f(x) - [g(x) + \theta(h(x) - g(x))] \|_X < \eta \quad (1)$$

and for every $\varepsilon > 0$

$$(HK)\int_S \| h(x) - g(x) \| < \varepsilon \quad (2)$$

Proof. Let $\alpha > 0$ be real number. Considering ([4] p.9), function f is measurable if and only if the function f has the separable range a.e. on S, so we can find a set V on S such that $\mu(V) < \alpha$ and the range $f(S \setminus V)$ is separable. Construct the set

$\{x_n \in f(S\setminus V): n \in N\}$ which is dense everywhere in $f(S\setminus V)$. Fix any $k \in \boldsymbol{N}$ and denote

$$E_n^k = \{s \in S\setminus V : \| f(s) - x_n \| < \frac{1}{k}\} = (S\setminus V) \cap f^{-1}[R_{\frac{1}{k}}(x_n)] .$$

These sets are measurable. Since for every $s \in S\setminus V$ and for every k it follows that exists the natural number n such that

$$\| f(s) - x_n \| < 1/k ,$$

then we get

$$\bigcup_{n=1}^{\infty} E_n^k = S\setminus V .$$

Construct the sequence of sets

$$B_n^k = E_n^k \setminus (E_1^k \cup E_2^k \cup ... \cup E_{n-1}^k) .$$

We see that these sets are disjoint and

$$\bigcup_{n=1}^{\infty} B_n^k = S\setminus V .$$

Construct now the function with countable range

$$d^k(x) = \sum_{n=1}^{\infty} x_n 1_{B_n^k}(x) + 0.1_{S\setminus V}(x) \quad (k \in N).$$

We get that for every $s \in S\setminus V$ and every k

$$\| f(s) - d^k(x) \| < 1/k .$$

it follows that

$$\lim_{n\to\infty} \| f(s) - d^k(x) \| = 0$$

uniformly on $S\setminus V$. Since the set $S\setminus V$ is measurable and its measure is not greater than b-a, it follows that exists a number $n_k \in \mathbf{N}$ for $k > n_k$ we obtain

$$\sum_{n=n_k+1}^{\infty} m(B_n^k) < \frac{1}{k} .$$

Considering the neighborhood $R_{\frac{1}{k}}(x_n)$ above mentioned, we construct $R_{\frac{1}{2k}}(x_n) \subset R_{\frac{1}{k}}(x_n)$. Let y_n and z_n be elements of range $f(S\setminus V)$ such that $\| x_n - y_n \| < 1/2k$ and $|\,|x_n - z_n|\,| < 1/2k$ and by virtue of Lemma 8 fulfill the equality $x_n = y_n + \theta(z_n - y_n)$, where θ is real number $0 \le \theta \le 1$. It easy to see that y_n and z_n are inner the $R_{\frac{1}{k}}(x_n)$

$$\| z_n - y_n \| < \| z_n - x_n \| + \| x_n - y_n \| < 1/k .$$

Construct two measurable functions

$$g(x) = \sum_{n=1}^{\infty} y_n 1_{B_n^k}(x) + 0.1_V(x) \;\; \textit{and} \;\; h(x) = \sum_{n=1}^{\infty} z_n 1_{B_n^k}(x) + 0.1_V(x) .$$

First, we prove the inequality (1).We obtain:

$$\| f(s) - [g(s) + \theta(h(s) - g(s)) \| \le \| f(s) - x_n \| + \| x_n - [g(s) + \theta(h(s) - g(s)) \| <$$

$$< 1/k + 1/k = 2/k$$

for every B_n^k, $n = 1, \ldots, n_k$ but this inequality may not hold for $x \notin \bigcup_{n=1}^{n_k} B_n^k$. Since the function *f* is S*M integrable, then it is Bochner integrable by ([7], p.146). In order to show that *g* and *h* are S*M integrable we can prove that the function $d^k(x)$ is S*M integrable.

Let *p* be a natural number $p \in \boldsymbol{N}$. By virtue of (3); for every $x \in V$ and $q>p$ we get $\left\| f(x) - f_q(x) \right\| < 1/k$. Since this inequality is satisfied for every B_n^k, it follows that for q>*p*

$$\left\| d^k(x) - f_q(s) \right\| \le \left\| d^k(x) - f(s) \right\| + \left\| f(s) - f_q(s) \right\| < 2/k \,.$$

Observing the inequalities

$$\sum_{n=1}^{\infty} \|x_n\| m(B_n^k) \le \sum_{n=1}^{\infty} \| x_n - f_q(x) \| \, m(B_n^k) + \sum_{n=1}^{\infty} \left\| f_q(x) \right\| m(B_n^k) <$$

$$2/\mathrm{k} \sum_{n=1}^{\infty} m(B_n^k) + \sum_{n=1}^{\infty} \| f_q(x) \| m(B_n^k) < (\mathrm{B}) \int_S \| f \| < \infty$$

we obtain, by Lemma 1.4.1. ([7] p.23), that the function $d^k(x)$ is Bochner absolute integrable and satisfies the condition S*M, therefore also the S*HK condition.

To prove the inequality (2), we can prove the inequality for the Bochner integral and $x \in S \setminus V$. We observe that

$$(B) \int_{S \setminus V} \| g(x) - h(x) \| dm = \sum_{n=1}^{\infty} \| z_n - y_n \| \, m(B_n^k) < \frac{1}{k} m(S) \,.$$

The right side goes to zero if for every ε>0 we choose the number k such that

$$\frac{1}{k} < \frac{\varepsilon}{2(b-a)} \,.$$

For the second part of theorem, we suppose that (b) holds.

Set

$$f_1(x) = g(x) + \theta\big(h(x) - g(x)\big).$$

By the condition, for every $\varepsilon>0$

$$\left\| f(x) - f_1(x) \right\| < \varepsilon \,,$$

it follows that

$$\left\| f(x) \right\| \le \left\| f_1(x) \right\| + \left\| f(x) - f_1(x) \right\| < \varepsilon + \left\| g(x) \right\| + \left\| h(x) - g(x) \right\| \,.$$

By the ([7], p.2), if the function *f* is measurable then ||*f*|| is also measurable. By the ([4] consequence 1.1.4., p.29), if the function *f(x)* and *g(x)* are KH absolute integrable then its are Mcshane integrable. This implies that ||*f(x)*|| is Bochner integrable. According to ([7], proposition 5.1.2.,p.135) we obtain that *f(x)* is S*M integrable.

Corollary 10. (**Theorem Vitaly- Carathedory**) [5]

Let $f : S \to \mathbf{R}$ be a function, the following statements are equivalent

(a) *f* is M-integrable on S.
(b) *f* is absolutely KH-integrable on S
(c) for every $\varepsilon>0$ there are absolutely KH-integrable functions *g* and *h* such that

$$g(x) \le f(x) \le h(x) \text{ on S}$$

and

$$(KH)\int_S (h(x) - g(x) < \varepsilon \;.$$

Proof. In the case for X=**R**, it is obvious that equality

$$f(x) = g(x) + \theta\big(h(x) - g(x)\big), \quad (0 \le \theta \le 1)$$

implies

$$g(x) \le f(x) \le h(x).$$

For example, if $\theta = 1/2$ we have

$$\mathrm{g}(\mathrm{x}) \le \mathrm{f}(\mathrm{x}) = \frac{g(x)+h(x)}{2} \le h(x).$$

Lemma 11. ([7], p.133). Assume that $f: S \to X$ is Bochner integrable and let be *ε>0*. Then there is a gauge $\delta : S \to]0,\infty[$ and $\eta \in]0,\varepsilon[$ such that the following statement holds.

If there is an $\{(H_{m,}t_m)\}$ K-system(M-system) δ-fine for which holds

$$\sum_{i=1}^{r} m(H_m) < \eta$$

Then

$$\sum_{m=1}^{r} \| f(t_m) \| \, m(H_m) < \varepsilon \;.$$

Theorem 12. Let *f* be a function $f: S \to X$, the following statements are equivalent:

a) function *f* is S*M -integrable
b) for every *ε*>0 there are absolutely S*KH-integrable functions *g* and *h* such that

$$f(x) = g(x) + \phi(x)h(x) \quad \text{where} \quad \phi(x) : S \to \{0,1\} \quad (1)$$

and

$$(KH)\int_S \| h(x) - g(x) \| < \varepsilon \;. \qquad (2)$$

Proof. Let us choose a gauge $\delta : S \to]0,\infty[$ as in Lemma 11 and $\eta \in]0,\frac{\varepsilon}{2}[$. Since function *f* is S*M integrable it follows that it is Bochner integrable and according to definition there exists a consequence of simple functions (f_q) convergent everywhere on $S \setminus Z_\alpha$ with $\mu(\mathrm{Z}_\alpha) = 0$. By the Egorov theorem, there exists a subsequence of this sequence which is uniformly convergent for every $x \in S \setminus V$, when $\mathrm{S} \supset \mathrm{Z}_\alpha$ and $\alpha < \eta / 4$. This implies, that there exist the measurable disjoint sets $S_i \subset S$, such that $\bigcup_{i=1}^{\infty} S_i = V$ and

$$f(x)=\sum_{i=1}^{\infty} C_i.1_{S_i}(x).$$

Since function f is Bochner integrable, it follows that below series is absolute convergent

$$\sum_{i=1}^{\infty} \| C_i \| \mu(S_i) = (B)\int_V \| f(x) \| < +\infty .$$

We obtain

$$\sum_{i=N+1}^{\infty} \| C_i \| \mu(S_i) < \frac{\eta}{3}.$$

By the Lesbegue theorem there exists a closed set F_i and open set G_i such that

$$F_i \subset S_i \subset G_i$$

and $\mu\left(G_i \setminus F_i\right) < \varepsilon / 2^{i+1}$.

We observe that for every i the equality holds

$$1_{S_i}(x) = 1_{F_i}(x) + \phi(x).1_{G_i}(x).$$

where $\phi(x): I \to \{0,1\}$. According this equality, we construct the functions

$$g(x)=\sum_{i=1}^{\infty} C_i.1_{F_i}(x) + 0.1_{S\setminus U}(x)$$

with $U=\bigcup_{i=1}^{\infty} F_i \subset V$ and

$$h(x)=\sum_{i=1}^{\infty} C_i.1_{G_i}(x) + 0.1_{S\setminus T}(x)$$

with $T=\bigcup_{i=1}^{\infty} G_i \supset V$. Reviewing the proof arguments of theorem 9, we conclude that these functions are Bochner integrable and it follows that they are absolute KH-integrable.

In order to prove (2), we consider inclusion $S\setminus U \subset S\setminus T \cup T\setminus U$.

Since

$$T\setminus U = \bigcup_{i=1}^{\infty} G_i \setminus \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty}(G_i \setminus F_i)$$

we get

$$m(T\setminus U) = m(\bigcup_{i=1}^{\infty}(G_i \setminus F_i)) \le \sum_{i=1}^{\infty} m(G_i \setminus F_i) < \eta/2 .$$

If above K-system of the set, S\U has been taken δ-fine and satisfy

$$\sum_{m=1}^{r} m(H_m) \le \sum_{i=1}^{\infty} m(G_i \setminus F_i) < \eta$$

then we have

$$\sum_{m=1}^{r} \| h(t_m) - g(t_m) \| m(H_m) < \varepsilon .$$

This proves (2).

Second part of proof is the same as in Theorem 11.

Corollary 13.

Let function *f* to be S*M (-Bochner) integrable then

$$f(x) = g(x) + k(x),$$

with $g(x) = \sum_{i=1}^{\infty} x_n 1_{G_n}(x)$ and series $\sum_{i=1}^{\infty} x_n m(G_n)$ are unconditionally convergent, whereas *k(x)* is a measurable and bounded function

4. An extension on locally convex spaces

Proposition 14([3] Prop. 1**)** Let $(E_n)_n$ be a sequence of disjoint measurable sets in S and let $(x_n)_n$ be a sequence in X and let $f : S \to X$ defined by

$$f(t) = \sum_n x_n 1_{E_n}(x).$$

If series $\sum_n x_n \mu(E_n)$ is unconditionally convergent, then the function f is McS-integrable on S and

$$(McS)\int_S f = \sum_n x_n \mu(E_n)$$

Further on, we will use the fact that every convex space is projective limit of normed spaces (see [6] p.86) . For each continuous semi norm *p* on the convex space X, $p^{-1}(0)$ is a vector subspace and *p* defines a norm on $X \backslash p^{-1}(0)$. X_p is associated Banach space, namely the completion of normed linear space $X \backslash p^{-1}(0)$ and π_p is a canonical mapping of X to X_p. Then X is projective limit of the spaces X_p by the canonical mapping π_p of X onto X_p.
For the function $f : S \to X$ and for each $p \in P(X)$, we define the function $\mathrm{f_p} : \mathrm{S} \to \mathrm{X_p}$ by the formula

$$f_p(t) = (\pi_{p\circ} f)(t) = \pi_p(f(t))$$

for $t \in S$.

Proposition 15. Let $f : S \to X$ be a measurable by semi norm Pettis integrable function. Then for each $p \in P(X)$ there are two functions *g(x)* and *h(x)* Mcshane-integrable such that for every $\eta > 0$

$$p(f(x) - [g(x) + \theta(h(x) - g(x)]) < \eta \qquad (1)$$

and for each $\varepsilon > 0$ and every *p*

$$(McS)\int_E p(h(x) - g(x)) < \varepsilon . \qquad (2)$$

Proof. Construction of functions *g* and *h* repeats construction in proposition 8 for every *p*. So, let $p \in P(X)$ be a semi norm, $\varepsilon > 0$ and $\eta \in]0, \varepsilon[$. Since the function *f* is measurable by the semi norms it follows that $\mathrm{f}(\mathrm{S})$ is almost separable on S.

Therefore, exists a set V such that $\mu(S\setminus V)<\eta/4$ and $f(S\setminus V)$ is separable. Let the set $\{x_n:\ n\in \mathbf{N}\}$ be everywhere dense on $f(S\setminus V)$ and we construct the sets

$$U^p_{n,k}=\{s\in S\setminus V : p(f(s)-x_n)<\frac{1}{k}\}=(S\setminus V)\cap R^p_{1/k}(x_n)\ .$$

$$E_1=U^p_{1,k}$$

and

$$E_n=U^p_{n,k}\setminus\bigcup_{i=1}^{n-1}U^p_{i,k}\ .$$

These sets are disjoint and $S\setminus V=\bigcup_{i=1}^{\infty}E_i$.

Since $R^p_{1/2k}\subset R^p_{1/k}$, then we take two points y_n and z_n such that $p(y_n-x_n)<1/2k$ and $p(z_n-x_n)<1/2k$, and also by virtue of lemma 8., its satisfy the equality $x_n=y_n+\theta(z_n-y_n)$, where $0\le\theta\le 1$. Construct the functions

$$g(x)=\sum_{n=1}^{\infty}y_n 1_{E_n}(x)+0.1_{S\setminus V}(x) \text{ and } h(x)=\sum_{n=1}^{\infty}z_n 1_{B^k_n}(x)+0.1_{S\setminus V}(x)$$

which realize the equality (1). The functions *f* and *g* are strongly measurable and with measurable ranges. We can prove that *g* is Mcshane integrable, the same holds for *h*. The function $k(x)=h(x)-g(x)$ is strongly measurable and $\mathrm{p}(k(x))=p(h(x)-g(x))<\varepsilon/2$. Construct the function

$$k_p=p_p\cdot \mathrm{k}\ :\mathrm{S}\to \mathrm{X_p}$$

which is the strongly measurable and has essentially bounded range. Therefore it is Bochner integrable and in particular Pettis integrable. The function $g=f(x)-\theta(h(x)-g(x))$ as the difference of two Pettis integrable functions is scalar integrable. Then for each $x^*\in X^*,\ x^*g\in L^1(S)$ and every measurable set E

$$\int_E x^*g=\sum_{n=1}^{\infty}x^*(y_n)\mu(E\cap E_n)$$

and

$$\int_E |x^*g|=\sum_{n=1}^{\infty}|x^*(y_n)|\,\mu(E\cap E_n)<\infty \qquad (3)$$

To prove that the series

$$\sum_{n=1}^{\infty}y_n\mu(E\cap E_n) \qquad (4)$$

converges unconditionally in Banach space X_p for every E, by the Pettis-Orlitz theorem it is sufficient to show that every sub series of it converges weakly to an element in X_p. If $(n_k)_k$ is a subconseguence of natural numbers and $A=\bigcup_k E_{n_k}$, then

$$x^*(\upsilon_g(E\cap A)=\int_{E\cap A}x^*g=\sum_{n=1}^{\infty}x^*(y_n)\mu(E\cap E_n\cap A)$$

$$= \sum_{n=1}^{\infty} x^*(y_{n_k})\mu(E \cap E_{n_k}) .$$

From (3), the last series converges for all $x^*\in X^*$, then we obtain that (4) converges unconditionally in X_p. By the proposition 14, the function g is Mcshane integrable. Inequality (2) is proved from the fact that

$$(McS)\int_{S\setminus V} g = \sum_{n=1}^{\infty} y_n \mu(E_n)$$

and the integral

$$\int_{S\setminus V} p(h(x) - g(x))$$

goes to zero. To prove the same for the integral on V we use the theorem 9 choosing the suitable gauge δ for every p.

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