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Integral Representations for the Product of Krawtchouk, Meixner, Charlier and Gottlieb Polynomials

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Abstract

The present paper deals with an integral representations for the product of two polynomials. Some hypergeometric form for the product of two polynomials are also indicated in the given note.

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1 Introduction

In 1938, Watson[4; p.207] gave the integeral representation for $L_m^{(\alpha)}(x)L_n^{(\alpha)}(y)$, which was generalized by L. Carlitz,[12] in the form

$$L_m^{(\alpha)}(x)L_n^{(\alpha)}(y) = \frac{2^{\alpha+\beta+m+n}}{\pi^2}\frac{\Gamma(\alpha+m+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+m+n+1)}$$
$$\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m+n)\phi i+(\alpha-\beta)\theta i}\cos^{(m+n)}\phi\cos^{(\alpha+\beta)}\theta$$
$$\times L_{m+n}^{(\alpha+\beta)}\left(\frac{xe^{(\theta-\phi)i}+ye^{-(\theta-\phi)i}}{\cos\phi}\cos\theta\right)d\phi d\theta \tag{1.1}$$

where $L_n^{(\alpha)}(x)$ denotes the generalized Laguerre polynomials of order α and degree n in x, defined as

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!} \qquad , n = 0, 1, 2, ... \tag{1.2}$$

By following the method adopted by Carlitz[12], integral representations for the product of several other polynomials have appeared in the literature.

In 1963, Chatterjea [15] gave the integral representation for the product of two generalized Bessel polynomials $y_n(x, a, b)$ due to Krall and Frink[11; p.108, eq.(34)], and are defined as

$$y_n(x, a, b) = {}_2F_0\left[-n, a-1+n; .....; -\frac{x}{b}\right] \tag{1.3}$$

In 1964,Chatterjea[16] further gave the formula for the product of two Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, defined as

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}\ {}_2F_1\left[\begin{array}{ll} -n\ ,\alpha+\beta+n+1\ ; & \frac{1-x}{2} \\ \alpha+1\ \ ; & \end{array}\right] \tag{1.4}$$

In 1969, H.L. Manocha,[6] gave the integral representation for the product of two generalized Rice polynomials $H_n^{(\alpha,\beta)}(\xi, p; x)$, defined as

$$H_n^{(\alpha,\beta)}(\xi, p; x) = \binom{n+\alpha}{n}\ {}_3F_2\left[\begin{array}{ll} -n\ ,\alpha+\beta+n+1\ ,\xi\ ; & x \\ \alpha+1\ \ ,\ \ p\ \ ; & \end{array}\right] \tag{1.5}$$

A detailed account may be found in the paper of Srivastava and Joshi([7]; see also[5]), who gave a multiple integral representation for the polynomials product (cf.[7; p.923, eq.(1.3)]).

$$\phi_m\left[(a_p),\ \alpha;\ (b_q)+1;\ x\right]\phi_n\left[(a'_p),\ \alpha';\ (b'_q)+1;\ y\right].$$

where (a_p) denotes the sequence of p parameters $a_1, ...a_p$, with similar interpretation for (b_q), and

$$\phi_n\left[(a_p),\ \alpha;\ (b_q);\ x\right] = \frac{(\alpha)_n}{n!}\ {}_{p+2}F_q\left[\begin{matrix} -n\ , \alpha+n\ , a_1, ..., a_p; \\ b_1, ..., b_q \qquad ; \end{matrix}\ x\right] \tag{1.6}$$

Further in 1976, H.M. Srivastava, and R. Panda,[10] derived an integral representation for the product of two Jacobi polynomials and also gave some generalization involving Kampé de Fériet's double hypergeometric function [14; p.150].

The object of present note is to derive an integral representations for the product of several other polynomials which have not been investigated in the literature as yet.

2 New Integral Representations

The Krawtchouk polynomials defined by [see[9], p.75] is represented as

$$K_n(x;p,N) = {}_2F_1\left[\begin{matrix} -n\ ,\ \ -x\ ; \\ -N\ \ ; \end{matrix}\ p^{-1}\right], \qquad 0<p<1,\ \ x=0,1,....,N \tag{2.1}$$

First, we obtain the product of two Krawtchouk polynomials in the form

$$K_m(x;p,M)K_n(y;q,N) = \frac{m!\ n!\ x!\ y!}{N!\ M!}\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-1)^{r+s}\left(\frac{1}{p}\right)^r\left(\frac{1}{q}\right)^s}{r!\ s!}$$
$$\times\frac{\Gamma(M-r+1)\Gamma(N-s+1)}{\Gamma(M+N-r-s+2)}\frac{\Gamma(m+n-r-s+1)}{\Gamma(m-r+1)\Gamma(n-s+1)}$$
$$\times\frac{\Gamma(x+y-r-s+1)}{\Gamma(x-r+1)\Gamma(y-s+1)}\frac{(M+N-r-s+1)!}{(m+n-r-s)!\ (x+y-r-s)!} \tag{2.2}$$

Now by using results [3],

$$\frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)(\nu+1)} = \frac{2^{\mu+\nu}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\mu-\nu)\theta i}\ \cos^{(\mu+\nu)}\theta\, d\theta\ , \quad (\mu+\nu>-1) \tag{2.3}$$

$$\frac{\Gamma(\mu)\ \Gamma(\nu)}{\Gamma(\mu+\nu)} = \int_0^1 t^{\mu-1}\,(1-t)^{\nu-1}dt\ , \qquad (\mu>0,\ \nu>0) \tag{2.4}$$

It is readily observed that

$$K_m(x;p,M)K_n(y;q,N) = \frac{2^{m+n+x+y}}{\pi^2} \frac{m!\ n!\ x!\ y!\ (M+N+1)!}{(m+n)!\ (x+y)!\ M!\ N!}$$

$$\times \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (t)^{M-1}(1-t)^{N-1} e^{(m-n)\theta i+(x-y)\phi i} \cos^{(m+n)}\theta\ \cos^{(x+y)}\phi$$

$$\times \sum_{k=0}^{m+n} \frac{(-m-n)_k(-x-y)_k}{(-M-N-1)_k\ k!} \sum_{r+s=k} \binom{k}{r} \left(\frac{e^{-(\theta+\phi)i}}{4pt\ \cos\theta\ \cos\phi}\right)^r$$

$$\times \left(\frac{e^{(\theta+\phi)i}}{4q(1-t)\ \cos\theta\ \cos\phi}\right)^{k-r} dtd\theta d\phi \tag{2.5}$$

Finally, using (2.1) we obtain from (2.5) our desired result in the form

$$K_m(x;p,M)K_n(y;q,N) = \frac{2^{m+n+x+y}}{\pi^2} \frac{m!\ n!\ x!\ y!\ (M+N+1)!}{(m+n)!\ (x+y)!\ M!\ N!}$$

$$\times \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (t)^{M-1}(1-t)^{N-1} e^{(m-n)\theta i+(x-y)\phi i} \cos^{(m+n)}\theta\ \cos^{(x+y)}\phi$$

$$\times K_{m+n}\left[x+y; \frac{4t(1-t)\cos\theta\ \cos\phi}{p(1-t)e^{-(\theta+\phi)i}+qte^{(\theta+\phi)i}}, M+N+1\right] dtd\theta d\phi \tag{2.6}$$

where $M > -1,\ N > -1$ and $m, n, x, y \in \{0, 1, 2, ...\}$.

In similar manner we further derive the integral representations for similar products of several other polynomials.

The product of two Meixner polynomials can be written in integral form as

$$M_m(x;\alpha,c)M_n(y;\beta,d) = \frac{2^{m+n+\alpha+\beta+x+y}}{4\pi^3} \frac{m!\ n!\ x!\ y!\ \Gamma\alpha\ \Gamma\beta}{(m+n)!\ (x+y)!\ \Gamma(\alpha+\beta-1)}$$

$$\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-n)\theta i+(x-y)\phi i+(\alpha-\beta)\psi i} \cos^{(m+n)}\theta\ \cos^{(x+y)}\phi\ \cos^{(\alpha+\beta-2)}\psi$$

$$\times M_{m+n}\left[x+y; \alpha+\beta-1, \frac{2\cos\theta\cos\phi}{2\cos\theta\cos\phi-(ce^{-(\theta+\phi-\psi)i}+de^{(\theta+\phi-\psi)i})\cos\psi}\right]$$

$$\times\ d\theta d\phi d\psi \tag{2.7}$$

where $\alpha + \beta > 1$ and $m, n, x, y \in \{0, 1, 2, ...\}$ and $M_n(x; \beta, c)$ denotes the Meixner polynomials defined by (see[9], p.75),

$$M_n(x;\beta,c) = {}_2F_1\left[-n\ ,-x\ ;\ \beta\ ;\ 1-c^{-1}\right]$$
$$\beta > 0\ ,\ 0 < c < 1\ ,\ x = 0,1,2,............. \tag{2.8}$$

The Goetllieb polynomials $l_n(x; \lambda)$ defined by (see[9], p.185)

$$l_n(x;\lambda) = e^{-n\lambda} \sum_{k=o}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \begin{pmatrix} x \\ k \end{pmatrix} (1-e^{\lambda})^k \tag{2.9}$$

It follows from (2.9) that,

$$l_m(x;\lambda) l_n(y;\mu) = \frac{2^{m+n+x+y}}{\pi^3} \frac{\Gamma(m+1)\Gamma(n+1)\Gamma(x+1)\Gamma(y+1)}{\Gamma(m+n+1)\Gamma(x+y+1)}$$
$$\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-n)\theta i + (x-y)\phi i} \cos^{(m+n)}\theta\ \cos^{(x+y)}\phi$$
$$\times l_{m+n}\left[x+y; log(1 - \frac{\lambda e^{-(\theta+\phi-\psi)i} + \mu e^{(\theta+\phi-\psi)i}}{2\cos\theta\ \cos\phi} \cos\psi\right] d\theta d\phi d\psi \tag{2.10}$$

provided that m, n, x, y are non negative integers.

The integral representation which appears in paper [5; p.119, eq.(2.1)] is a special case of (1.6) at $p = q - 1 = 1$, may be written in equivalent form

$$\phi_m \left[\begin{matrix} a & , & \lambda & ; \\ b_1+1 & , & b_2+1 & ; \end{matrix}\ x\right] \phi_n \left[\begin{matrix} a' & , & \lambda' & ; \\ {b_1}'+1 & , & {b_2}'+1 & ; \end{matrix}\ y\right]$$
$$= \frac{2^{m+n+b_1+b_1'+b_2+b_2'}}{\pi^3} \frac{\Gamma(b_1+1)\Gamma(b_1'+1)\Gamma(b_2+1)\Gamma(b_2'+1)\Gamma(\lambda+m)\Gamma(\lambda'+n)}{\Gamma(b_1+b_1'+1)\Gamma(b_2+b_2'+1)\Gamma(\lambda)\Gamma(\lambda')\Gamma(m+n+1)}$$
$$\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-n)\theta i + (b_1-b_1')\phi i + (b_2+b_2')\psi i} \cos^{(m+n)}\theta\ \cos^{(b_1+b_1')}\phi\ \cos^{(b_2+b_2')}\psi$$
$$\times M_5\left[-m-n\ ,\ \lambda+m\ ,\ \lambda'+n\ ,\ a\ ,\ a'\ ;\ b_1+b_1'+1\ ,\ b_2+b_2'+1\ ;\right.$$
$$\left.\frac{2x\cos\phi\cos\psi}{\cos\theta} e^{-(\theta-\phi-\psi)i}\ ,\ \frac{2y\cos\phi\cos\psi}{\cos\theta} e^{(\theta-\phi-\psi)i}\right] d\theta d\phi d\psi \tag{2.11}$$

where, as before $\{\lambda > 0,\ \lambda' > 0,\ b_1 + b_1' > -1,\ b_2 + b_2' > -1,\ \ m, n \in (0, 1, 2, ...)\}$ and M_5 is the fifth kind generalization of Appell's functions of two variables obtained by using the product of two ${}_3F_2$ hypergeometric function defined by Khan, M.A. and Abukhammash, G.S.[13, p.65, eq.(1.9)] as follows:

$$M_5\,(a,\ b,\ b',\ c,\ c';\ d,\ e\ ;\ x,\ y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n(c)_m(c')_n}{(d)_{m+n}(e)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \qquad (2.12)$$

In view of the relation (1.5), the last formula (2.15) under the replacement of some aforementioned conditions,
(i) $a\ ,a'$ by $\xi\ ,\eta$
(ii) $\lambda\ \lambda'$ by $\alpha+\beta+1\ ,\gamma+\delta+1$
(iii) $b_1\ ,b_1'$ by $\alpha\ ,\gamma$
(iv) $b_2\ ,b_2'$ by $p-1\ ,q-1$

and multiply both sides by $\frac{(\alpha+1)_m(\gamma+1)_n}{(\alpha+\beta+1)_m(\gamma+\delta+1)_n}$, would evidently yields the hypergeometric form of the integral representation for the product of two generalized Rice polynomials defined by [6] in the equivalent form

$$H_m^{(\alpha,\beta)}(\xi,p;x)H_n^{(\nu,\delta)}(\eta,q;y) = \frac{2^{m+n+\alpha+\gamma+p+q}}{4\pi^3}\frac{\Gamma(1+\alpha+m)\Gamma(1+\gamma+n)\Gamma p\ \Gamma q}{\Gamma(1+\alpha+\gamma)\Gamma(p+q-1)(m+n)!}$$

$$\times\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-n)\theta i+(\alpha-\gamma)\phi i+(p-q)\psi i}\cos^{(m+n)}\theta\ \cos^{(\alpha+\gamma)}\phi\ \cos^{(p+q-2)}\psi$$

$$\times M_5\left[-m-n,\ 1+\alpha+\beta+m,\ 1+\gamma+\delta+n,\ \xi,\ \eta;\ 1+\alpha+\gamma,\ p+q-1;\ \frac{2x\cos\phi\cos\psi}{\cos\theta}e^{-(\theta-\phi-\psi)i},\ \frac{2y\cos\phi\cos\psi}{\cos\theta}e^{(\theta-\phi-\psi)i}\right]d\theta d\phi d\psi \qquad (2.13)$$

where $\alpha+\gamma>-1, p+q>1, \alpha+\beta>-1, \gamma+\delta>-1$ and $m,n\in\{0,1,2,........\}$.

The integral representation for the product of two Poisson-Charlier polynomials can be represented in elegant form

$$C_m(x;\alpha)C_n(y;\beta) = \frac{2^{m+n+x+y}}{\pi^2}\frac{(m)!\ (n)!\ (x)!\ (y)!}{(m+n)!\ (x+y)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-n)\theta i+(x-y)\phi i}$$

$$\times\cos^{(m+n)}\theta\cos^{(x+y)}\phi\, C_{m+n}\left(x+y;\frac{4e^{(\theta+\phi)i}}{\beta+\alpha e^{2(\theta+\phi)i}}\cos\theta\cos\phi\right)d\theta d\phi \qquad (2.14)$$

where $m,n,x,y\in\{0,1,2,........\}$ and $C_n(x;\alpha)$ denote the Poisson-Charlier polynomials defined by [cf. Erdélyi et al.(1953), vol.II, p.226, eq.(14)],

$$C_n(x;\alpha) = \sum_{k=0}^{n}(-1)^k\begin{pmatrix} n \\ k \end{pmatrix}\begin{pmatrix} x \\ k \end{pmatrix}k!\alpha^{-k}, \qquad \alpha>0,\ x=0,1,2..... \qquad (2.15)$$

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