Int. Journal of Math. Analysis, Vol. 5, 2011, no. 3, 145 - 155

Canonical Reduction of the Self-Dual Yang Mills Equations to Nonlinear Modified Korteweg-de Vries Equations with Exact Solutions and its Conservation Laws

A. R. Shehata[1,2] and J. F. Alzaidy[2]

[1] Mathematics Department, Faculty of Science
Minia University, El-Minia,Egypt
j-f-h-z@hotmail.com

[2] Mathematics Department, Faculty of Science
Taif University, Kingdom of Saudi Arabia

Abstract

The (constrained) canonical reduction of four-dimensional self-dual Yang-Mills (SDYM) theory to two-dimensional modified Korteweg-de Vries (mKdV) equations are considered. On the other hand, other methods and transformations are developed to obtain exact solutions for the original two dimensional modified Korteweg-de Vries (mKdV) equations. The corresponding gauge potential $A_\mu$and the gauge field strengths $F_{\mu\nu}$ are also obtained.

For these nonlinear evolution equations (NLEEs) which describe pseudo-spherical surfaces (pss) two new exact solution classes are generated from known solutions by using the B$\ddot{a}$cklund transformations with the aid of Mathematica , either the seed solution is constant or a traveling wave. Also a procedure for generating an infinite number of conservation laws is given.

Mathematics Subject Classification: 35; 53C; 58J; 58Z05;14Q10; 37K40; 58A10; 65M32

Keywords: SDYM; modified Korteweg-de Vries (mKdV) equations; B$\ddot{a}$cklund transformations;Gauge filed theories; Inverse Problems; Pseudo-spherical Surfaces; Nonlinear Evolution Equations

[1,2] Present address: Mathematics Department, Faculty of Science, El-Taif, P.O. Box 888, Kingdom of Saudi Arabia.E-mail address: shehata1423@yahoo.com

1. Introduction

The self-dual Yang–Mills (SDYM) equations (a system of equations for Lie algebra-valued functions of C^4) play a central role in the field of integrable systems and also play a fundamental role in several other areas of mathematics and physics [10,15]. It arises in relativity [14,25] and in field theory [7]. The SDYM equations describe a connection for a bundle over the Grassmannian of two-dimensional subspaces of the twistor space. Integrability for a SDYM connection means that its curvature vanishes on certain two-planes in the tangent space of the Grassmannian. As shown in [20,23]. This allows one to characterize SDYM connections in terms of the splitting problem for a transition function in a holomorphic bundle over the Riemann sphere, i.e. the trivialization of the bundle [18,19].

The theory of integrable systems has been an active area of mathematics for the past thirty years. Different aspects of the subject have fundamental relations with mechanics and dynamics, applied mathematics, algebraic structures, theoretical physics, analysis including spectral theory and geometry. In recent decades, a class of transformations having their origin in the work by B$\ddot{a}$cklund in the late nineteenth century has provided a basis for remarkable advances in the study of nonlinear partial differential equations (NLPDEs) [10,11],[13-15] .The importance of B$\ddot{a}$cklund transformations (BTs) and their generalizations is basically twofold. Thus, on one hand, invariance under a BT may be used to generate an infinite sequence of solutions for certain mKdV equation by purely algebraic superposition principles. On the other hand, BTs may also be used to link certain NLPDEs [9,12,27,28] (particularly (NLEEs) modelling nonlinear waves) to canonical forms whose properties are well known [1-5],[8,22].

Non-Abelian gauge theories first appeared in the seminal work of Yang and Mills [34] as a non-Abelian generalization of Maxwell's equations. Let *G* be a Lie group (referred to as the gauge group) with Lie algebra (LG) and let $\{x_\mu\}_{\mu=1,2,3,4}$ be coordinates on a four- dimensional manifold *M* which can be $R^4, R^{1,3}$ or $R^{2,2}$. Given the gauge potential $A_\mu(x) \in LG$, we introduce the covariant derivatives

$$D_\mu = \partial_\mu - A_\mu , \tag{1}$$

and their commutators

$$F_{\mu\nu} = -[D_\mu , D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu , A_\nu] , \tag{2}$$

where $F_{\mu\nu}$ are the gauge field strengths.

The Yang- Mills equations are a set of coupled, second-order NLPDEs in four dimensions for the LG-valued gauge potential functions A_μ's, and are extremely difficult to solve in general. It is however possible to obtain a special class of first-order reductions of the full Yang-Mills equations by noting that any $F_{\mu\nu}$ that satisfies

$$\lambda F_{\mu\nu} = {}^*F_{\mu\nu} , \quad \lambda = \begin{cases} \pm 1 & on \quad R^4,\ R^{2,2}; \\ \pm i & on \quad R^{3,1} . \end{cases} \tag{3}$$

All real solutions of the equations $^*F_{\mu\nu} = \pm\, iF_{\mu\nu}$ are trivial. On R^4 and $R^{2,2}$, the equations $^*F_{\mu\nu} = (-)F_{\mu\nu}$ are called the (anti) SDYM equations. Now consider four complex variables $\mathcal{Y}, \bar{\mathcal{Y}}, \mathcal{Z}$ and $\bar{\mathcal{Z}}$ defined in [30]

$$\sqrt{2}\mathcal{Y} = x_1 + ix_2 \ , \ \sqrt{2}\bar{\mathcal{Y}} = x_1 - ix_2 \ ,$$
$$\sqrt{2}\mathcal{Z} = x_3 - ix_4 \ , \ \sqrt{2}\bar{\mathcal{Z}} = x_3 + ix_4 \ , \tag{4}$$

it is simple to check that the self-duality equations $F_{\mu\nu} = {}^*F_{\mu\nu}$ reduces to

$$F_{\mathcal{Y}\mathcal{Z}} = 0, \qquad F_{\bar{\mathcal{Y}}\bar{\mathcal{Z}}} = 0\,, \qquad F_{\mathcal{Y}\bar{\mathcal{Y}}} + F_{\mathcal{Z}\bar{\mathcal{Z}}} = 0. \tag{5}$$

Equations (5) are the compatibility condition of the linear problem [34]

$$\left(\psi_{\mathcal{Y}} + i\zeta\psi_{\bar{\mathcal{Z}}}\right) = \left(\mathrm{A}_{\mathcal{Y}} + i\zeta\mathrm{A}_{\bar{\mathcal{Z}}}\right)\psi\,, \tag{6}$$
$$\left(\psi_{\mathcal{Z}} - i\zeta\psi_{\bar{\mathcal{Y}}}\right) = \left(\mathrm{A}_{\mathcal{Z}} - i\zeta\mathrm{A}_{\bar{\mathcal{Y}}}\right)\psi\,, \tag{7}$$

where ζ is a parameter, independent of $\mathcal{Y}, \bar{\mathcal{Y}}, \mathcal{Z}$ and , $\bar{\mathcal{Z}}$.
The compatibility condition is simply

$$(\partial_{\mathcal{Z}} - i\zeta\partial_{\bar{\mathcal{Y}}})(\partial_{\mathcal{Y}} + i\zeta\partial_{\bar{\mathcal{Z}}})\,\psi = (\partial_{\mathcal{Y}} + i\zeta\partial_{\bar{\mathcal{Z}}})(\partial_{\mathcal{Z}} - i\zeta\partial_{\bar{\mathcal{Y}}})\psi. \tag{8}$$

On using equations (6) and (7), this gives

$$[F_{\mathcal{Y}\mathcal{Z}} - i\zeta\,(\,F_{\mathcal{Y}\bar{\mathcal{Y}}} + F_{\bar{\mathcal{Z}}\bar{\mathcal{Z}}}) - \zeta^2\,F_{\bar{\mathcal{Y}}\bar{\mathcal{Z}}}]\,\psi = 0 \qquad . \tag{9}$$

Equations (5) can be immediately integrated, since they are pure gauge, to give

$$A_{\mathcal{Y}} = \mathcal{D}^{-1}\mathcal{D}_{\mathcal{Y}}\,, \quad A_{\mathcal{Z}} = \mathcal{D}^{-1}\mathcal{D}_{\mathcal{Z}}, \quad A_{\bar{\mathcal{Y}}} = \bar{\mathcal{D}}^{-1}\bar{\mathcal{D}}_{\bar{\mathcal{Y}}}\,, \quad A_{\bar{\mathcal{Z}}} = \bar{\mathcal{D}}^{-1}\,\bar{\mathcal{D}}_{\bar{\mathcal{Z}}}\,, \tag{10}$$

where $\mathcal{D}$ and $\bar{\mathcal{D}}$are arbitrary 2 × 2 complex matrix functions of $\mathcal{Y}, \bar{\mathcal{Y}}, \mathcal{Z}$ and , $\bar{\mathcal{Z}}$ with determinant = 1 (for SU(2) gauge group) and $\mathcal{D}_{\mathcal{Y}} = \partial_{\mathcal{Y}}\mathcal{D}$, etc. For real gauge fields $A_\mu \doteq -A_\mu^+$ (the symbol$\doteq$ is used for equations valid only for real values of x_1, x_2, $x_3$and x_4), we require

$$\bar{\mathcal{D}} \doteq (\mathcal{D}^+)^{-1}. \tag{11}$$

Gauge transformations are the transformations

$$\mathcal{D} \to \mathcal{D}U\,, \qquad \bar{\mathcal{D}} \to \bar{\mathcal{D}}U\,, \qquad U^+U \doteq I \tag{12}$$

where U is a 2 × 2 matrix function of $\mathcal{Y}, \bar{\mathcal{Y}}, \mathcal{Z}, \bar{\mathcal{Z}}$ with determined = 1. Under transformation (12), equation (11) remains unchanged. We now define the hermitian matrix $\mathcal{J}$ as

$$\mathcal{J} \equiv \mathcal{D}\,\bar{\mathcal{D}}^{-1} \doteq \mathcal{D}\,\mathcal{D}^+. \tag{13}$$

$\mathcal{J}$ has the very important property of being invariant under the gauge transformation (12). The only non vanishing field strengths in terms of $\mathcal{J}$ becomes

$$F_{u\bar{v}} = -\bar{\mathcal{D}}^{-1}\,(\,\mathcal{J}^{-1}\mathcal{J}_u)_{\bar{v}}\,\bar{\mathcal{D}}\,, \tag{14}$$

$(u, v = \mathcal{Y}, \mathcal{Z})$ and the remaining self-duality equation (5) takes the form

$$(\,\mathcal{J}^{-1}\mathcal{J}_{\mathcal{Y}})_{\bar{\mathcal{Y}}} + (\mathcal{J}^{-1}\mathcal{J}_{\mathcal{Z}})_{\bar{\mathcal{Z}}} = 0. \tag{15}$$

The action density in terms of $\mathcal{J}$ is

$$\emptyset(\,\mathcal{J}) = -\tfrac{1}{2}\,Tr\,F_{\mu\nu}F_{\mu\nu}$$
$$= -2\mathrm{Tr}(\,F_{\mathcal{Y}\bar{\mathcal{Y}}}F_{\mathcal{Z}\bar{\mathcal{Z}}} + F_{\mathcal{Y}\bar{\mathcal{Z}}}F_{\bar{\mathcal{Y}}\mathcal{Z}}) \tag{16}$$
$$= -2\mathrm{Tr}\left\{(\mathcal{J}^{-1}\mathcal{J}_{\mathcal{Y}})_{\bar{\mathcal{Y}}}(\,\mathcal{J}^{-1}\mathcal{J}_{\mathcal{Z}})_{\bar{\mathcal{Z}}} - (\,\mathcal{J}^{-1}\mathcal{J}_{\mathcal{Y}})_{\bar{\mathcal{Z}}}(\,\mathcal{J}^{-1}\mathcal{J}_{\mathcal{Z}})_{\bar{\mathcal{Y}}}\right\}.$$

In this paper, the canonical reduction of four dimensional self-dual Yang-Mills theory to two dimensional mKdV equation [24] are considered. We give a new of exact solution for the mKdV equation by applying the BTs method with the aid of Mathematica [1-6],[30,32]. Consequently we find exact solutions for self-dual Yang Mills equations. In addition the corresponding gauge potentialA_μ and the gauge field strengths $F_{\mu\nu}$ are also obtained. We also obtain an infinite number of conserved charges by solving a set of coupled Riccati equations and apply the geometrical method to mKdV equation which describe pss.

The paper is organized as follows: On one hand the reduction of Yang-Mills theory to mKdV equation and exact solutions are presented in sections 2 and 3 respectively. Moreover the gauge potential A_μ and the gauge field strengths $F_{\mu\nu}$ are also obtained. Section 4 contains the derivation of an infinite number of conserved charges from the Riccati equations. Finally, we give some conclusions in section 5.

2. The canonical reduction of four-dimensional SDYM theory to two dimensional mKdV equation

Suppose that A_μ's depend on $x = \bar{y}$ and $t = \bar{z}$ only. If we use a gauge in which $A_y = 0$, in terms of the matrix-valued functions $P := A_z$, $Q := A_{\bar{y}}$, $R := A_{\bar{z}}$, the SDYM equations (5) are

$$P_t + [P, R] = 0 \ , \tag{17}$$

$$R_x - Q_t - [Q, R] = 0 \ . \tag{18}$$

Let P take the canonical form

$$P = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} , \tag{19}$$

for some constant k. We then find that

$$R = \begin{pmatrix} 0 & -u_{xx} - 2u^3 \\ u_{xx} + 2u^3 & 0 \end{pmatrix}, \tag{20}$$

$$Q = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} , \tag{21}$$

from Eq. (18), we obtain the mKdV equation

$$u_t + 6u_x u^2 + u_{xxx} = 0 \ . \tag{22}$$

3. The AKNS system for some NLEEs which describe pss and its BTs

We recall the definition [21,22] of a differential equation (DE) that describes a pss. Let M^2 be a two dimensional differentiable manifold with coordinates (x, t). A DE for a real function $u(x, t)$ describes a pss if it is a necessary and sufficient condition for the existence of differentiable functions

$$f_{ij}, \qquad 1 \le i \le 3, \quad 1 \le j \le 2, \tag{23}$$

depending on u and its derivatives such that the one-forms

$$\omega_1 = f_{11}dx + f_{12}dt, \quad \omega_2 = f_{21}dx + f_{22}dt, \quad \omega_3 = f_{31}dx + f_{32}dt, \tag{24}$$

satisfy the structure equations of a pss, i.e.,

$$d\omega_1 = \omega_3 \wedge \omega_2, \qquad d\omega_2 = \omega_1 \wedge \omega_3, \qquad d\omega_3 = \omega_1 \wedge \omega_2. \tag{25}$$

As a consequence, each solution of the DE provides a local metric on M^2, whose Gaussian curvature is constant, equal to −1. Moreover, the above definition is equivalent to saying that DE for u is the integrability condition for the problem [1,28]:

$$d\phi = \Omega\phi, \;\; \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{26}$$

where d denotes exterior differentiation, ϕ is a column vector and the 2 × 2 trix Ω (Ω_{ij} $i,j = 1,2$) is traceless

$$\Omega = \frac{1}{2}\begin{pmatrix} \omega_2 & \omega_1 - \omega_3 \\ \omega_1 + \omega_3 & -\omega_2 \end{pmatrix} .$$

Take

$$\Omega = \begin{pmatrix} \frac{\eta}{2}dx + Adt & qdx + Bdt \\ rdx + Cdt & -\frac{\eta}{2}dx - Adt \end{pmatrix} = Sdx + Tdt, \tag{27}$$

from Eqs. (26) and (27), we obtain

$$\phi_x = S\phi, \quad \phi_t = T\phi, \tag{28}$$

where S and T are two 2 × 2 null-trace matrices

$$S = \begin{pmatrix} \frac{\eta}{2} & q \\ r & -\frac{\eta}{2} \end{pmatrix}, \tag{29}$$

$$T = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{30}$$

here η is a parameter, independent of x and t, while q and r are functions of x and t. Now

$$0 = d^2\phi = d\Omega\phi - \Omega \wedge d\phi = (d\Omega - \Omega \wedge \Omega)\phi ,$$

which requires the vanishing of the two form

$$\Theta \equiv d\Omega - \Omega \wedge \Omega = 0 , \tag{31}$$

or in component form

$$\begin{aligned} -A_x + qC - rB &= 0 , \\ q_t - 2Aq - B_x + \eta B &= 0 , \\ r_t - C_x + 2Ar - \eta C &= 0 , \end{aligned} \tag{32}$$

Chern and Tenenblat [13] obtained Eq. (32) directly from the structure equations (25). By suitably choosing r, A, B and C in (32), we shall obtain various NLEEs which q must satisfy. Konno and Wadati introduced the function [26]

$$\Gamma = \frac{\phi_1}{\phi_2}, \tag{33}$$

this function first appeared used and explained in the geometric context of pss equations in [11,27], and see also the classical papers by Sasaki [29] and Chern–Tenenblat [13]. Then Eq. (28) is reduced to the Riccati equations:

$$\frac{\partial\Gamma}{\partial x} = \eta\Gamma - r\Gamma^2 + q , \tag{34}$$

$$\frac{\partial\Gamma}{\partial t} = 2A\Gamma - C\Gamma^2 + B . \tag{35}$$

Our procedure in the following is that we construct a transformation Γ' satisfying the same equation as (34) and (35) with a potential u' where

$$u' = u + f(\Gamma, \eta) \,, \tag{36}$$

Chern and Tenenblat [13] introduced several examples of (36) for pss equations. For use in the sequel, we list the mKdV equation and their corresponding BT in the following.

The mKdV equation

We have $\omega_1 = 2\eta u_x dt \,, \qquad \omega_2 = \eta dx - (2\eta u^2 + \eta^3)dt,$

$$\omega_3 = 2udx - (2u_{xx} + 4u^3 + 2\eta^2 u)dt, \tag{37}$$

For any solution $u(x,t)$ of the mKdV equation (22), the matrices S and T are

$$S = \begin{pmatrix} \frac{\eta}{2} & -u \\ u & -\frac{\eta}{2} \end{pmatrix} \,, \tag{38}$$

$$T = \begin{pmatrix} -(\eta u^2 + \frac{\eta^3}{2}) & \eta u_x + u_{xx} + 2u^3 + \eta^2 u \\ -(-\eta u_x + u_{xx} + 2u^3 + \eta^2 u) & (\eta u^2 + \frac{\eta^3}{2}) \end{pmatrix}, \tag{39}$$

the above matrices S , T satisfy Eqs. (32). Then Eq. (34) becomes

$$\frac{\partial \Gamma}{\partial x} = \eta\Gamma - u - u\Gamma^2. \tag{40}$$

If we choose Γ' and u' as [22]

$$\Gamma' = \frac{1}{\Gamma} \,, \tag{41}$$

$$u' = u + 2\frac{\partial}{\partial x}\tan^{-1}\Gamma \,. \tag{42}$$

Now we shall choose some known solution of the mKdV equation and substitute this solution into the corresponding matrices S and T. Next, we solve Eqs. (28) for ϕ_1 and ϕ_2. Then, by (33) and the corresponding BT we shall obtain the new solution for the mKdV equation.

Substitute $u = 0$ into the matrices S and T in (38) and (39), then by (28) we have

$$d\phi = \phi_x dx + \phi_t dt = S\phi d\rho \,, \tag{43}$$

where

$$S = \begin{pmatrix} \frac{\eta}{2} & 0 \\ 0 & -\frac{\eta}{2} \end{pmatrix}, \tag{44}$$

$$\rho = x - \eta^2 t, \qquad . \tag{45}$$

The solution of Eq. (43) is

$$\phi = e^{S\rho}\phi_0 = \left(I + \rho S + \frac{\rho^2 S^2}{2!} + \frac{\rho^3 S^3}{3!} + \cdots\right)\phi_0 \,, \tag{56}$$

where ϕ_0 is a constant column vector. The solution of Eq. (46) is

$$\phi = \begin{pmatrix} \cosh\frac{\eta}{2}\rho + \sinh\frac{\eta}{2}\rho & 0 \\ 0 & \cosh\frac{\eta}{2}\rho - \sinh\frac{\eta}{2}\rho \end{pmatrix}\phi_0 \,. \tag{57}$$

Now, we choose $\phi_0 = (1,1)^T$ in (47), then we have

$$\phi = \begin{pmatrix} e^{\frac{\eta\rho}{2}} \\ e^{-\frac{\eta\rho}{2}} \end{pmatrix} \,. \tag{58}$$

Substitute (48) into (33), then by (42), we obtain the new solutions of the mKdV equation (22)

$$u' = \eta \text{ sech } (\eta\rho) \quad . \tag{59}$$

We can calculate the gauge potential A_μ and the gauge field strengths $F_{\mu\nu}$ from equations (6)-(10) and (19)-(21),then

$$A_y = 0, \qquad A_{\bar{y}} = \begin{pmatrix} 0 & \eta \text{ sech } (\eta\rho) \\ -\eta \text{ sech } (\eta\rho) & 0 \end{pmatrix},$$

$$A_z = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}, \qquad A_{\bar{z}} = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}, \tag{60}$$

where $\quad a = \eta^3 \, sech(\eta\rho).$

Consequently, we obtain the gauge field strengths $F_{\mu\nu}$ as follows:

$$F_{yz} = -[A_y, A_z], \qquad F_{\bar{y}\bar{z}} = \partial_x A_{\bar{z}} - \partial_t A_{\bar{y}} - [A_{\bar{y}}, A_{\bar{z}}],$$

$$F_{y\bar{y}} = -\partial_x A_y - [A_y, A_{\bar{y}}], \qquad F_{z\bar{z}} = -\partial_t A_z - [A_z, A_{\bar{z}}]. \tag{61}$$

4. Conservation laws for the mKdV equation which describe pss

In this section, from equations (32),(33),(34) and (35) the Riccati system of equations can be put in explicit conservation form [16,17,31,33] . In doing this, by using equations (34) and (35) imply that

$$C\Gamma_x - r\Gamma_t = (qC - rB) + (\eta C - 2Ar)\,, \tag{62}$$

adding $-r_t\Gamma$ to both sides, we obtain

$$C\Gamma_x - (r\Gamma)_t = (qC - rB) + (\eta C - 2Ar - r_t)\Gamma\,, \tag{63}$$

we using eq. (32), eq. (63) takes the form

$$(r\Gamma)_t = (A + C\,\Gamma)_x\,. \tag{64}$$

The Riccati equations for Γ in the $x -$ variable can be rearranged to take the form

$$\eta(r\Gamma) = -rq + (r\Gamma)^2 + r\left[\frac{\partial}{\partial x}\left(\frac{r\Gamma}{r}\right)\right]. \tag{65}$$

A similar pair of equations can be obtained for the t derivatives. Expand $(r\,\Gamma)$ into a power series in inverse powers of η so that

$$\text{r}\Gamma(x,t,\eta) = \sum_{n=1}^{\infty} \phi_n(x,t)\eta^{-n}. \tag{66}$$

The $\phi_n(x,t)$ are unknown at this point, however a recursion relation can be obtained for the $\phi_n(x,t)$ by using (65). Substituting (66) into Γ equation in (65), we find that

$$\sum_{n=1}^{\infty} \phi_n(x,t)\eta^{-n+1}$$
$$= -rq + \left(\sum_{n=1}^{\infty} \phi_n(x,t)\eta^{-n}\right)^2 + r\,\sum_{n=1}^{\infty}\left(\frac{\phi_n(x,t)}{r}\right)_x \eta^{-n}\,. \tag{67}$$

Applying the Cauchy product formula to the square in (67), it then takes the form

$$\phi_1 + \phi_2\eta^{-1} + \sum_{n=2}^{\infty} \phi_{n+1}(x,t)\eta^{-n}$$
$$= -rq + \sum_{n=2}^{\infty}\left(\sum_{j=1}^{n-1} \phi_j\phi_{n-j}\right)\eta^{-n} + r\sum_{n=2}^{\infty}\left(\frac{\phi_n(x,t)}{r}\right)_x \eta^{-n} + r\left(\frac{\phi_1(x,t)}{r}\right)_{..} \tag{68}$$

Now equate powers of η on both sides of this expression to produce the set of recursions

$$\phi_1 = -rq, \quad \phi_2 = -rq_x\,,$$
$$\phi_{n+1} = \sum_{k=1}^{n-1} \phi_k(x,t)\phi_{n-k}(x,t) + r\left(\frac{\phi_n(x,t)}{r}\right)_x, \quad n \geq 2\,. \tag{69}$$

Substituting (66) into (64), the following system of conservation laws appears

$$\sum_{n=1}^{\infty}\frac{\partial\phi_n(x,t)}{\partial t}\eta^{-n}=\frac{\partial}{\partial x}\left(A+\mathrm{C}\sum_{n=1}^{\infty}\frac{\phi_n(x,t)}{r}\eta^{-n}\right). \quad (70)$$

In general, A and C will depend on parameter η , the function r and higher derivatives of r. Substituting A and C into (70) a particular case, eq. (70) will simplify under relations (69), and then like powers of η can be equated on both sides of (70). This procedure generates an infinite number of conservation laws for the equation under examination.

To obtain conservation laws using (70) in a particular example using this procedure, let us consider the mKdV equation which describe pss.

For the mKdV equation

$$q=-u\,,\qquad r=u\,,$$

$$A=-\left(\eta u^2+\frac{\eta^3}{2}\right),\qquad B=\eta u_x+u_{xx}+2u^3+\eta^2u,$$

$$C=-(-\eta u_x+u_{xx}+2u^3+\eta^2u).$$

Substituting (71) into (32), the first equation (32) reduces to an identity, and the remaining two hold modulo mKdV equation (22). Putting (71) into (70), it is found that

$$\sum_{n=1}^{\infty}\frac{\partial\phi_n(x,t)}{\partial t}\eta^{-n}$$

$$=-2\eta uu_x-\frac{\partial}{\partial x}\left[(-\eta u_x+u_{xx}+2u^3+\eta^2u)\sum_{n=1}^{\infty}\frac{\phi_n(x,t)}{r}\eta^{-n}\right]. \quad (72)$$

However, from the recursions in (69), it follows that $\phi_1=u^2$ and $\phi_2=uu_x$. Using these to simplify this, the remaining coefficients of η^{-n} can be equated on both sides, and the following set of conservation laws are obtained for $n\geq 1$,

$$\frac{\partial\phi_n(x,t)}{\partial t}=-\frac{\partial}{\partial x}\left\{(u_{xx}+2u^3)\frac{\phi_n(x,t)}{r}-u_x\frac{\phi_{n+1}(x,t)}{r}+u\frac{\phi_{n+2}(x,t)}{r}\right\}. \quad (73)$$

5. Conclusions

In this paper, we considered the construction of exact solutions to mKdV equation. We obtain travelling wave solutions for the above equations by using BTs method with the aid of Mathematica.

The soliton phenomena and integrable NLEEs represent an important and well established field of modern physics, mathematical physics and applied mathematics. Solitons are found in various areas of physics from hydrodynamics and plasma physics, nonlinear optics and solid state physics, to field theory and gravitation. NLEEs which describe soliton phenomena have an universal character.

A travelling wave of permanent form has already been met; this is the solitary wave solution of the NLEE itself. Such a wave is a special solution of the governing equation which does not change its shape and which propagates at constant speed. The SDYM equations play a central role in the field of integrable systems and also play a fundamental role in several other areas of mathematics and physics.

In addition the SDYM equations are a rich source of integrable systems suggested by the fact that they are the compatibility condition of an associated linear problem which admits enormous freedom if one allows the associated gauge algebra to be arbitrary. The classical soliton equations in 1+1, 2+1 and 3+1 dimensions are reductions of the SDYM equations with finite-dimensional gauge algebra. In this paper we have demonstrated the reductions of the SDYM equations to mKdV equation and also obtained travelling wave solution. Also an infinite number of conserved charges for the mKdV equation are derived.

References

1. M.J. Ablowitz, S. Chakravarty and R. Halburd, Integrable systems and reductions of the self-dual Yang-Mills equations, J. Math. Phys. 44, 3147(2003).
2. M.J. Ablowitz, S. Chakravarty and R. Halburd, On Painlevé and Darboux-Halphen type equations, in the Painlevé property, One Century Later, edited By R. Conte, CRM Series in Mathematical Physics (Springer, Berlin, 1998).
3. Ablowitz, M.J., Chakravarty, S. and Halburd, R., The generalized Chazyequation and Schwarzian triangle functions, Asian J. Math. 2, 619(1998).
4. M.J. Ablowitz, S. Chakravarty and R. Halburd, The generalized Chazy equation from the self-duality equations, Stud. Appl. Math. 103, 75(1999).
5. M.J. Ablowitz, S. Chakravarty and , L. A. Takhtajan, A self-dual Yang- Mills hierarchy and its reductions to integrable systems in 1+1 and 2+1 dimensions, Commun. Math. Phys. 158, 289(1993).
6. M .J. Ablowitz, D. J. Kaup, A.C. Newell and H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems, Stud. Appl. Math. 53, 249(1974).
7. M. J. Abolwitz, and P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering (Cambridge University Press, Cambridge,1991).
8. M.F. Atiyah and R.S. Ward, Instants and algebraic geometry, Commun. Math. Phys. 55, 117(1977).
9. R. Beals, M. Rabelo and K. Tenenblat, Bäcklund transformations and inverse scattering solutions for some pseudo-spherical surfaces, Stud. Appl. Math. 81, 125(1989).
10. J. P. Bourguignon, H.B. Lawson and J. Simons, Stability and gap Phenomena for Yang- Mills fields. Differential Geometry of Yang-Mills fields, Proc.Nat.Acad Sci.76,1550 (1979).
11. J. A. Cavalcante and K. Tenenblat, Conservation laws for nonlinear evolution equations, J. Math. Phys. 29, 1044(1988).
12. K. Chadan and P. C. Sabatier, Inverse problem in quantum scattering theory (springer, New York, 1977).
13. S.S. Chern and K. Tenenblat, Pseudospherical surfaces and evolution equations, Stud. Appl. Math. 74, 1 (1986).
14. T. Chuu-Lian, and K. Uhlenbeck, Introduction to Surveys in Differential Geometry: Integrable Systems, J. Differential Geometry 4, 5 (1999).
15. R. Donagi, and E. Witten, Supersymmetric Yang-Mills theory and integrable systems, Nucl. Phys. B460, 299(1996).

16. S. Ghosh and S. Nandy , Inverse Scattering Method and Vector Higher Order Nonlinear Schrödinger Equation Nucl. Phys. B561 451-66 (1999).
17. S. Ghosh, A. Kundu and S. Nandy , Soliton Solutions, Liouville Integrability and Gauge Equivalence of Sasa Satsume Equation J. Math. Phys. 40 1993-2000 (1999).
18. R. Hermann, Vector Bundles in Mathematical Physics, Vol. I and II(W.A. Benjamin, Inc.) (1970).
19. N. Hitchin, Stable bundles and integrable systems, Duke Math. Journal54, 91(1987).
20. A. H. Khater, D.K. Callebaut, A. A. Abdalla, A.R. Shehata and S.M. Sayed, Bäcklund transformations and exact solutions for self-dual *SU*(3) Yang-Mills equations, IL Nuovo Cimento B114, 1(1999).
21. A. H. Khater, D.K. Callebaut and S.M. Sayed, Conservation laws for some nonlinear evolution equations which describe pseudo-spherical surfaces, J. Geometry and Phys. 51, 332(2004).
22. A. H. Khater, D.K. Callebaut and S.M. Sayed, Exact solutions for some nonlinear evolution equations which describe pseudo-spherical surfaces, J. of Comput. and Appl. Math. 189, 387(2006).
23. A. H. Khater, A.R. Shehata, D.K. Callebaut and S.M. Sayed, Self-Dual Solutions for SU(2) and SU(3) Gauge Fields on Euclidean Space, International J. of Theoretical Physics 43, 151(2004).
24. A. H. Khater, R.S. Ibrahim, O.H. El-LKalaawy and D.K. Callebaut, Bäcklund transformations and Exact Soluations for Some Nonlinear Evolution Equations of the ZS/AKNS System (1998)
25. W. Klingenberg, Riemannian Geometry,Walter de Gruyter, Berlin, New York, 1982.
26. K. Konno and M. Wadati, Simple derivation of Bäcklund transformation from Riccati Form of inverse method, Progr. Theor. Phys. 53, 1652(1975).
27. E. G. Reyes, Conservation laws and Calapso-Guichard deformations of equations describing pseudo-spherical surfaces, J. Math. Phys. 41, 2968(2000).
28. E.G. Reyes, On Geometrically integrable equations and Hierarchies of pseudo-spherical type, Contemporary Mathematics, 285, 145(2001).
29. R. Sasaki, Soliton equations and pseudo-spherical surfaces, Nucl. Phys. B154, 343 (1979).
30. S.M. Sayed and G.M. Gharib, Canonical reduction of Self-Dual Yang- Mills equations to Fitzhugh-Nagumo equation and exact solutions, Chaos, Solitons & Fractals. 39,492(2009).
31. M. Wadati, H. Sanuki and K. Konno, Relationships among inverse method, Bäcklund transformation and an infinite number of conservation laws Progr. Theor. Phys. **53** 419-36 (1975).
32. E. Witten, Two dimensional gravity and intersection theory on moduli space, Survey in Differential Geometry 1, 243(1991).
33. T .Xia, F. You and D. Chen , A generalized AKNS hierarchy and its bi Hamiltonian structures Chaos, Solitons & Fractals **23** 1911-19 (2005).

34. C. N. Yang and R.L. Mills, Conservation of isotopic spin and isotopic gauge invariance, Phys. Rev. 96, 191(1954).

Received: August, 2010