A Common Fixed Point Theorem in Menger Probabilistic Metric Spaces Using Compatibility

K. P. R. Sastry\textsuperscript{1}, G. A. Naidu\textsuperscript{2}, P. V. S. Prasad\textsuperscript{3} and S. S. A. Sastri\textsuperscript{4}

\textsuperscript{1} 8-28-8/1, Tamil Street, Chinna Waltair Visakhapatnam-530 017, India
kprsastry@hotmail.com

\textsuperscript{2} \textsuperscript{3} Department of Mathematics, Andhra University Visakhapatnam-530 003, India
drgolivean@yahoo.com, pvsprasad10@yahoo.in

\textsuperscript{4} Department of Basic Science and Humanities Coastal Institute of Technology and Management Narapatam, Vizianagaram-535 183, India
sambharasas@yahoo.co.in

Abstract

In this paper, we show that a result of Servet Kutukcu and Sushil Sharma of \cite{6} is not valid and provide also modifications Theorem 2.6 and corollary 2.7. For this we introduce the notion of a strict Menger space. We also give supporting examples. Two open problems also are given.

Mathematics Subject Classification: 47H10, 54H25

Keywords: common fixed point, compatible maps, Menger space, strict Menger space, triangular norm

1 Introduction

In this paper we show that Corollary 1 and hence Theorem 1 of \cite{6} is not valid, through an example and obtains necessary modifications to the above theorem.
We start with

Definition 1.1 \cite{4} A function $F: \mathbb{R} \to [0, 1]$ is called a distribution function if
(i) \( F \) is non-decreasing,
(ii) \( F \) is left continuous,
(iii) \( \inf_{x \in \mathbb{R}} F(x) = 0 \) and \( \sup_{x \in \mathbb{R}} F(x) = 1 \)

**Definition 1.2:** [2] A triangular norm \( * : [0, 1] \times [0, 1] \to [0, 1] \) is a function satisfying the following conditions
   (i) \( a * 1 = a \quad \forall a \in [0, 1] \)
   (ii) \( a * b = b * a \quad \forall a, b \in [0, 1] \)
   (iii) \( c * d \geq a * b \quad \forall a, b, c, d \in [0, 1] \) with \( c \geq a \) and \( d \geq b \)
   (iv) \( (a * b) * c = a * (b * c) \quad \forall a, b, c \in [0, 1] \)

A triangular norm is also denoted by t-norm.

For any \( a, b \in [0, 1] \), if we define \( a * b = \min\{a, b\} \), then * is a t-norm and is denoted by ‘min’.

We observe that if \( a * a \geq a \quad \forall a \in [0, 1] \), then * is min t-norm.

**Definition 1.3:** [4] Let \( X \) be a non-empty set and let \( F : X \times X \to \mathcal{D} \) (The set of distribution functions). For \( p, q \in X \), we denote the image of the pair \( (p, q) \) by \( F_{p,q} \) which is a distribution function so that \( F_{p,q}(x) \in [0, 1] \), for every real \( x \).

Suppose \( F \) satisfies:
   (i) \( F_{p,q}(x) = 1 \) for all \( x > 0 \) if and only if \( p = q \)
   (ii) \( F_{p,q}(0) = 0 \)
   (iii) \( F_{p,q}(x) = F_{q,p}(x) \)
   (iv) If \( F_{p,q}(x) = 1 \) and \( F_{q,r}(y) = 1 \) then \( F_{p,r}(x + y) = 1 \) where \( p, q, r \in X \).

Then \( (X, F) \) is called a probabilistic metric space.

**Definition 1.4:** [2] Let \( X \) be a non-empty set, * be a t-norm and \( F : X \times X \to \mathcal{D} \) be a function satisfying
   (i) \( F_{p,q}(0) = 0 \)
   (ii) \( F_{p,q}(x) = 1 \) for all \( x > 0 \) if and only if \( p = q \)
   (iii) \( F_{p,q}(x) = F_{q,p}(x) \)
   (iv) \( F_{p,q}(x + y) \geq F_{p,q}(x) \ast F_{q,r}(y) \) for all \( x, y \geq 0 \) and \( p, q, r \in X \).

Then the triplet \( (X, F, *) \) is called a Menger space.

**Definition 1.5:** [5]
   (i) Let \( (X, F, *) \) be a Menger space and \( p \in X \). For \( \epsilon > 0, 0 < \lambda < 1 \), the \((\epsilon, \lambda)\)-neighborhood of \( p \) is defined as \( U_{p} (\epsilon, \lambda) = \{ q \in X: F_{p,q}(\epsilon) > 1 - \lambda \} \).

It may be observed that, if * is continuous, then the topology induced by the family \( \{U_{p} (\epsilon, \lambda): p \in X, \epsilon > 0, 0 < \lambda < 1 \} \) is a Hausdorff topology on \( X \) and is known as the \((\epsilon, \lambda)\)-topology.

(ii) A sequence \( \{x_{n}\} \) in \( X \) is said to converge to \( p \in X \) in the \((\epsilon, \lambda)\)-topology, if for any \( \epsilon > 0 \) and \( 0 < \lambda < 1 \) there exists a positive integer \( N = N (\epsilon, \lambda) \) such that \( F_{x_{n},p}(\epsilon) > 1 - \lambda \) where \( n > N \).

(iii) A sequence \( \{x_{n}\} \) in \( X \) is said to be a Cauchy sequence in the \((\epsilon, \lambda)\)-topology, if for \( \epsilon > 0 \) and \( 0 < \lambda < 1 \) there exists a positive integer \( N = N (\epsilon, \lambda) \) such that \( F_{x_{m},x_{n}}(\epsilon) > 1 - \lambda \) for all \( m, n > N \).
(iv) A Menger space \((X, F, \ast)\) where \(\ast\) is continuous, is said to be complete if every Cauchy sequence in \(X\) is convergent in \((\epsilon, \lambda)\)-topology.

**Definition 1.6:**[6] Two self mappings \(A\) and \(B\) of a Menger space \((X, F, \ast)\) are said to be

(i) compatible of type \((P)\) if

\[ F_{ABx_n, BBx_n}(t) \to 1 \text{ and } F_{BAx_n, AAx_n}(t) \to 1 \text{ for all } t > 0 \]

where \(\{x_n\}\) is a sequence in \(X\) such that \(Ax_n, Bx_n \to z\) for some \(z\) in \(X\) as \(n \to \infty\).

(ii) compatible of type \((P_1)\) if \(F_{ABx_n, BBx_n}(t) \to 1\) for all \(t > 0\)

where \(\{x_n\}\) is a sequence in \(X\) such that \(Ax_n, Bx_n \to z\) for some \(z\) in \(X\) as \(n \to \infty\).

(iii) compatible of type \((P_2)\) if \(F_{BAx_n, AAx_n}(t) \to 1\) for all \(t > 0\)

where \(\{x_n\}\) is a sequence in \(X\) such that \(Ax_n, Bx_n \to z\) for some \(z\) in \(X\) as \(n \to \infty\).

**Lemma 1.7:**[3] Let \(\{x_n\}_{n=0}^{\infty}\) be a sequence in a Menger space \((X, F, \ast)\) with Hadzic-type \(t\)-norm \(\ast\). If there exists \(k \in (0, 1)\) such that

\[ F_{x_n,x_{n+1}}(kt) \geq F_{x_{n-1},x_n}(t) \text{ for all } t > 0, \]

then \(\{x_n\}\) is a Cauchy sequence.

We observe that ‘min’ \(t\)-norm is of Hadzic type.

**Lemma 1.8:**[7] Let \((X, F, \ast)\) be a Menger space. If there exists \(k \in (0, 1)\) such that

\[ F_{x,y}(kt) \geq F_{x,y}(t) \text{ for all } x, y \in X \text{ and } t > 0, \]

then \(x = y\).

## 2 Main Results

Now we state below Theorem 1 of [6], which we show to be not valid (Example 2.3) and obtain modifications (Theorem 2.6 and Corollary 2.7).

**Theorem 2.1 ([6], Theorem 1):** Let \(A, B, P, Q, S, T\) be self mappings of a complete Menger space \((X, F, \ast)\) with continuous \(t\)-norm \(\ast\) such that \(t \ast t \geq t \forall t \in [0, 1]\) (i.e. \(\ast\) is the ‘min’ \(t\)-norm), satisfying:

(i) \(P(X) \subseteq ST(X), Q(X) \subseteq AB(X)\)

(ii) there exists a constant \(k \in (0, 1)\) such that

\[ F_{Pz,Qy}(kt) \geq F_{ABz, STy}(t) \ast F_{Pz, ABx}(t) \ast F_{Qy, STy}(t) \ast F_{Pz, STy}(\alpha t) \ast F_{Qy, ABx}((2-\alpha)t) \]

for all \(x, y \in X, t > 0\) and \(\alpha \in (0, 2)\)

(iii) either \(P\) or \(C\) is continuous

(iv) the pairs \((P, C)\) and \((Q, R)\) are both compatible of type \((P_1)\) or type \((P_2)\)

(v) \(AB = BA, ST = TS, PB = BP, QT = TQ\)

Then \(A, B, P, Q, S, T\) have a unique common fixed point. The following Corollary of Theorem 2.1 is given in [6].
Corollary 2.2 ([6], Corollary 1): Let $P$, $Q$ be self maps on Menger space $(X, F, \ast)$ with continuous $t$-norm and $t \ast t \geq t \forall t \in [0, 1]$. If there exists a constant $k \in (0, 1)$ such that

$$F_{Px,Qy}(kt) \geq F_{x,y}(t) \ast F_{P,x}(t) \ast F_{Q,y}(t) \ast F_{P,x}((2 - \alpha)t) \ast F_{Q,y}((2 - \alpha)t)$$

for all $x, y \in X, t > 0$ and $\alpha \in (0, 2)$,

then $P$ and $Q$ have a unique common fixed point.

Corollary 2.2 is not valid in view of the following example.

Example 2.3: Let $X = Z^+$ and the function $F$ by for any $m, n \in Z^+$

$$F_{m,n}(t) = \begin{cases} 0 & \text{if } t \leq \max\{m, n\} \\ 1 & \text{if } t > \max\{m, n\} \end{cases}$$

Define the functions $P$ and $Q$ on $Z^+$ by $P_n = n + 1$ and $Q_n = 1$ for all $n \in Z^+$.

Let $k \in (0, 1)$ and take $\alpha = k$

Then for every $x, y \in X, t > 0$, we have

$$F_{x+1,1}(kt) \geq F_{x,y}(t) \ast F_{x+1,x}(t) \ast F_{1,y}(t) \ast F_{x+1,y}(at) \ast F_{1,x}((2 - \alpha)t) ... (1)$$

This is so, because $F_{x+1,1}(kt) = 1$ if $kt > x + 1$ and (1) is satisfied and if $kt < x + 1$, then $F_{x+1,1}(kt) = 0$ and $F_{x+1,1}(at) = 0$ ($\because \alpha = k$)

So that (1) is again satisfied. In either case the inequality (1) holds, but $P$ and $Q$ fail to have a common fixed point.

Hence Theorem 2.1 needs modification which we give below. In this connection it may be observed that $F_{m,n}(t)$ is not strictly increasing in $(0, \infty)$. We make use of this observation in obtaining the necessary modifications to Theorem 2.1.

Definition 2.4: Let $(X, F, \ast)$ be a Menger space such that $F_{x,y}(t)$ is strictly increasing in $t$ when ever $x \neq y$. Then $(X, F, \ast)$ is called a strict Menger space.

Example 2.5: Let $(X, d)$ be a metric space. Define $F_{x,y}(t) = \frac{t}{t + d(x,y)} \forall t > 0$ and $x, y \in X$. If $t$-norm $\ast$ is $a \ast b = \min\{a, b\} \forall a, b \in [0, 1]$, then $(X, F, \ast)$ is a strict Menger space.

The following Theorem 2.6 and Corollary 2.7 may be regards as modifications to Theorem 2.1.

Theorem 2.6: Let $P$, $Q$, $R$ and $C$ be self mappings of a complete strict Menger space $(X, F, \ast)$ where $\ast$ is the min $t$-norm, satisfying:

(i) $P(X) \subseteq R(X), Q(X) \subseteq C(X)$

(ii) there exists a constant $k \in (0, 1)$ such that

$$F_{P,x,Qy}(kt) \geq F_{C,x,Ry}(t) \ast F_{P,x,Cx}(t) \ast F_{Q,y,Ry}(t) \ast F_{P,x,Ry}(2t) \ast F_{Q,y,Cx}(2t)$$

for all $x, y \in X, t > 0$

(iii) either $P$ or $C$ is continuous
(iv) the pairs (P,C) and (Q,R) are both compatible of type (P₁) or type (P₂)

Then P, Q, R and C have a unique common fixed point.

**Proof:** Let \( x_0 \in X \). By (i), there exist sequences \( \{x_n\} \) and \( \{y_n\} \) in X such that

\[
P_{x_{2n}} = R_{x_{2n+1}} = y_{2n} \quad \text{and} \quad Q_{x_{2n+1}} = C_{x_{2n+2}} = y_{2n+1} \quad \text{for} \quad n = 0, 1, 2, \ldots
\]

By taking \( x = x_{2n}, \ y = x_{2n+1} \) for all \( t > 0 \) in (ii), we get

\[
F_{P_{x_{2n}}, Q_{x_{2n+1}}}(kt) \geq F_{C_{x_{2n}}, R_{x_{2n+1}}}(t) * F_{P_{x_{2n}}, C_{x_{2n}}}(t) * F_{Q_{x_{2n+1}}, R_{x_{2n+1}}}(t) *
\]

\[
F_{P_{x_{2n}}, R_{x_{2n+1}}}(2t) * F_{Q_{x_{2n+1}}, C_{x_{2n}}}(2t)
\]

\[
\Rightarrow F_{y_{2n}, y_{2n+1}}(kt) \geq F_{y_{2n-1}, y_{2n}}(t) * F_{y_{2n}, y_{2n-1}}(t) * F_{y_{2n-1}, y_{2n}}(2t) *
\]

\[
F_{y_{2n-1}, y_{2n}}(2t)
\]

Similarly, we can prove that \( F_{y_{2n+1}, y_{2n+2}}(kt) \geq F_{y_{2n}, y_{2n+1}}(t) * F_{y_{2n+1}, y_{2n+2}}(t) \)

Hence \( F_{y_n, y_{n+1}}(kt) \geq F_{y_{n-1}, y_n}(t) * F_{y_n, y_{n+1}}(t) \)

Consequently \( F_{y_n, y_{n+1}}(kt) \geq F_{y_{n-1}, y_n}(t) \) for all \( t > 0, \ k \in (0, 1), \ n \in N \)

which is obvious if \( y_n = y_{n+1} \) and follows from the strictly increasing nature of \( F_{y_n, y_{n+1}}(t) \) when \( y_n \neq y_{n+1} \).

By Lemma [1.7], \( \{y_n\} \) is a Cauchy sequence.

Since \( (X, F, *) \) is complete, it converges to a point \( z \) in X. Also its sub sequences

\( \{P_{x_{2n}}\} \rightarrow z, \ \{C_{x_{2n}}\} \rightarrow z, \ \{Q_{x_{2n+1}}\} \rightarrow z \) and \( \{R_{x_{2n+1}}\} \rightarrow z \)

Case (i): C is continuous, (P,C) and (Q,R) are compatible of type (P₂)

\[
CC_{x_{2n}} \rightarrow Cz, \ CP_{x_{2n}} \rightarrow Cz \quad (\because \ C \text{ is continuous})
\]

\[
PP_{x_{2n}} \rightarrow Cz \quad (\because \ (P,C) \text{ is compatible of type (P₂)})
\]

Now by taking \( x = P_{x_{2n}}, \ y = x_{2n+1} \) in (ii), we get \( Cz = z \).

Similarly by taking \( x = z, \ y = x_{2n+1} \) in (ii), we get \( Pz = z \).

Since \( P(X) \subseteq R(X) \), there exists \( w \in X \) such that \( z = Pz = Rw \)

By taking \( x = x_{2n}, \ y = w \) in (ii), we get \( Qw = z \)

\[
\therefore Rw = Qw = z
\]

Since \( (Q,R) \) is compatible of type \( (P₂) \), we have \( RQw = QQw \).

Therefore \( Rz = Qz \).

Now by taking \( x = x_{2n}, \ y = z \) in (ii), we get \( Qz = z \).

\[
\therefore Pz = Qz = Cz = Rz = z.
\]

i.e. \( z \) is a common fixed point for P, Q, R and C.

Case (ii): P is continuous and \( (P,C), (Q,R) \) are both compatible of type (P₂)

\[
PP_{x_{2n}} \rightarrow Pz, \ PC_{x_{2n}} \rightarrow Pz \quad (\because \ P \text{ is continuous})
\]

\[
CP_{x_{2n}} \rightarrow Pz \quad (\because \ (P,C) \text{ is compatible of type (P₂)})
\]

By taking \( x = P_{x_{2n}}, \ y = y_{2n+1} \) in (ii), we get \( Pz = z \).

we have \( z = Qz = Rz \).
Since $Q(X) \subseteq C(X)$, there exists $w \in X$ such that $z = Qz = Cw$.

By taking $x = w$, $y = x_{2n+1}$ in (ii), we get $z = Pw$.

Since $z = Qz = Cw$, hence $Pw = Cw$.

($P, C$) is compatible of type ($P_2$), we have $CPw = PPw$

i.e. $Cz = Pz$.

:. $z = Pz = Cz = Qz = Rz$

i.e. $z$ is a common fixed point for $P$, $Q$, $R$ and $C$.

:. $z$ is a common fixed point for $P$, $Q$, $R$ and $C$ when $C$ is continuous (or $P$ is continuous) and ($P, C$), ($Q, R$) are compatible of type $P_2$ (or $P_1$).

For uniqueness, let $v$ be a common fixed point for $P$, $Q$, $R$ and $C$.

Take $x = z$, $y = v$ in the condition (ii), then we get $v = z$.

Therefore $P$, $Q$, $R$ and $C$ have a unique common fixed point.

**Corollary 2.7:** Let $A$, $B$, $P$, $Q$, $S$ and $T$ be self mappings of a complete strict Menger space $(X, F, \ast)$ where $\ast$ is the min $t$-norm, satisfying:

(i) $P(X) \subseteq ST(X), Q(X) \subseteq AB(X)$

(ii) there exists a constant $k \in (0, 1)$ such that

\[ F_{P_x, Q_y}(kt) \geq F_{AB_x, ST_y}(t) * F_{P_x, AB_x}(t) * F_{Q_y, ST_y}(2t) * F_{P_x, ST_y}(2t) * F_{Q_y, AB_x}(2t) \]

for all $x, y \in X$, $t > 0$

(iii) either $P$ or $ST$ is continuous

(iv) the pairs ($P, ST$) and ($Q, AB$) are both compatible of type ($P_1$) or type ($P_2$)

(v) $AB = BA, ST = TS, PB = BP, QT = TQ$

Then $A$, $B$, $P$, $Q$, $S$ and $T$ have a unique common fixed point.

**Proof:** Write $C = ST$ and $R = AB$

Then, by Theorem 2.6, there exists $z \in X$ such that $z = Pz = Rz = Qz = Cz$.

Hence $z = Pz = Rz = STz = Qz = Cz = ABz$.

Now $STz = z \Rightarrow T(STz) = Tz \Rightarrow TSTz = Tz \Rightarrow STTz = Tz$

:. $Tz$ is a fixed point for $ST$.

Since $ABz = z \Rightarrow BABz = Bz \Rightarrow ABBz = Bz$

i.e. $Bz$ is a fixed point for $AB$.

Similarly, $ABz = z \Rightarrow AABz = Az \Rightarrow ABAz = Az$

i.e. $Az$ is a fixed point for $AB$.

Therefore $Az$ and $Bz$ are fixed points for $AB$.

Now $Pz = z \Rightarrow BPz = Bz \Rightarrow PBz = Bz$

i.e. $Bz$ is a fixed point for $P$.

Since $Qz = z \Rightarrow TQz = Tz \Rightarrow QTz = Tz$

i.e. $Tz$ is a fixed point for $Q$.

By taking $x = Bz$, $y = Tz$ in (b), we get $Bz = Tz$

:. $Bz$ is a common fixed point for $P$, $Q$, $AB$, $ST$.

By Theorem 2.6, $Bz = z = Tz$ is a common fixed point for $P$, $Q$, $AB$, $ST$.

Since $ABz = z \Rightarrow Az = z$ and $STz = z \Rightarrow Sz = z$
Common fixed point theorem

∴ z is a common fixed point for A, B, S, T, P and Q.
For uniqueness, let v be a common fixed point for A, B, S, T, P and Q.
By taking \( x = z, y = v \) in condition (ii), we get \( z = v \).

We provide the following example in support of Theorem 2.6.
**Example 2.8:** \((\mathbb{R}, F, *)\) is a strict Menger space where \(\mathbb{R}\) is a real line with the usual metric, and \(F : \mathbb{R} \rightarrow [0, 1]\) is defined by \(F_{x,y}(t) = \frac{t}{t+d(x,y)}\) \(\forall x, y \in \mathbb{R}\) and * is the min t-norm, i.e. \(* = \min\{a, b\}\).

Let \(P, Q, R\) and \(C\) be the self maps on \(\mathbb{R}\), defined by
\[
P x = \begin{cases} 
0 & \text{if } x \leq 3 \\
1 & \text{if } x > 3
\end{cases}
\]
\(Q x = 0, R x = x^2\) and \(C x = I\) \(\forall x \in \mathbb{R}\).

Then clearly \(P, Q, R\) and \(C\) satisfy the hypothesis of Theorem 2.6. It is clear that 0 is the only common fixed point of \(P, Q, R\) and \(C\).

It may be noted that \(P\) is discontinuous and \(C\) is continuous.

The following example is supporting Corollary 2.7.
**Example 2.9:** Let \((\mathbb{R}, F, *)\) be as in Example 2.8.

Let \(A, B, P, Q, S\) and \(T\) be the self maps on \(\mathbb{R}\), defined by
\[
P x = \begin{cases} 
0 & \text{if } x \leq 3 \\
1 & \text{if } x > 3
\end{cases}
\]
\(Q x = 0, A x = x^4, B x = \sqrt{|x|}\) and \(S x = -x = T x\) \(\forall x \in \mathbb{R}\).

Clearly \(A, B, P, Q, S\) and \(T\) satisfy the hypothesis of Corollary 2.7. It is clear that 0 is the only common fixed point of \(A, B, P, Q, S\) and \(T\).

We conclude the paper with two open problems.
**Open problem 2.10:** Is Theorem 2.6 valid if \(2t\) in condition (ii) is replaced by \(\alpha t\) where \(\alpha \in (1, 2)\)?

**Open problem 2.11:** Is Theorem 2.6 valid if \((X, F, *)\) is not necessarily strict?

References:


[3]. **K.P.R. Sastry, G.V.R. Babu and M.L. Sandhya:** Weak contractions


Received: July, 2010