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A Common Fixed Point Theorem in Menger Probabilistic Metric Spaces Using Compatibility

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Abstract

In this paper, we show that a result of Servet Kutukcu and Sushil Sharma of [6] is not valid and provide also modifications Theorem 2.6 and corollary 2.7. For this we introduce the notion of a strict Menger space. We also give supporting examples. Two open problems also are given.

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1 Introduction

In this paper we show that Corollary 1 and hence Theorem 1 of [6] is not valid, through an example and obtains necessary modifications to the above theorem.
We start with

Definition 1.1 :[4] A function $F : \mathbb{R} \to [0, 1]$ is called a distribution function if

(i) F is non-decreasing,
(ii) F is left continuous,
(iii) $\inf_{x\in\mathbb{R}} F(x) = 0$ *and* $\sup_{x\in\mathbb{R}} F(x) = 1$

Definition 1.2: [2] A triangular norm $* : [0,1] \times [0,1] \to [0,1]$ is a function satisfying the following conditions

(i) $a * 1 = a \quad \forall\, a \in [0,1]$
(ii) $a * b = b * a \;\forall\, a, b \in [0,1]$
(iii) $c * d \geq a * b \;\; \forall\, a, b, c, d \in [0,1]$ *with* $c \geq a$ *and* $d \geq b$
(iv) $(a * b) * c = a * (b * c) \quad \forall\, a, b, c \in [0,1]$

A triangular norm is also denoted by t-norm.
For any $a, b \in [0,1]$, if we define $a * b = min\{a, b\}$, then $*$ is a t-norm and is denoted by 'min'.
We observe that if $a * a \geq a \;\; \forall\, a \in [0,1]$, then * is min t-norm.

Definition 1.3: [4] Let X be a non-empty set and let $F : X \times X \to \mathfrak{D}$ (The set of distribution functions). For $p,\ q \in$ X, we denote the image of the pair (p,q) by $F_{p,q}$ which is a distribution function so that $F_{p,q}(x) \in [0,1]$, for every real x.
Suppose F satisfies:

(i) $F_{p,q}(x) = 1$ for all $x > 0$ if and only if $p = q$
(ii) $F_{p,q}(0) = 0$
(iii) $F_{p,q}(x) = F_{q,p}(x)$
(iv) If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$ then $F_{p,r}(x + y) = 1$ where $p, q, r \in X$.

Then (X, F) is called a probabilistic metric space.

Definition 1.4: [2] Let X be a non empty set, $*$ be a t-norm and $F : X \times X \to \mathfrak{D}$ be a function satisfying

(i) $F_{p,q}(0) = 0$
(ii) $F_{p,q}(x) = 1$ for all $x > 0$ if and only if $p = q$
(iii) $F_{p,q}(x) = F_{q,p}(x)$
(iv) $F_{p,r}(x + y) \geq F_{p,q}(x) * F_{q,r}(y)$ for all $x, y \geq 0$ and $p, q, r \in X$.

Then the triplet $(X, F, *)$ is called a Menger space.

Definition 1.5:[5]

(i) Let $(X, F, *)$ be a Menger space and $p \in$ X. For $\epsilon > 0,\ 0 < \lambda < 1$, the (ϵ, λ) -neighborhood of p is defined as Up $(\epsilon, \lambda) = \{q \in$ X: Fp,q $(\epsilon) >$ 1- λ }.
It may be observed that, if $*$ is continuous then the topology induced by the family {Up $(\varepsilon,\ \lambda\ )$: $p \in$ X, $\epsilon > 0,\ 0 < \lambda < 1\}$ is a Hausdorff topology on X and is known as the (ϵ, λ) -topology.

(ii) A sequence $\{x_n\}$ in X is said to converge to $p \in$ X in the (ϵ, λ) -topology, if for any $\epsilon > 0$ and $0 < \lambda < 1$ there exists a positive integer N = N (ϵ, λ) such that $F_{x_n,p}\ (\epsilon) > 1 - \lambda$ where $n > N$.

(iii) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence in the (ϵ, λ) -topology, if for $\epsilon > 0$ and $0 < \lambda < 1$ there exists a positive integer N = N (ε, λ) such that $F_{x_m,x_n}(\epsilon) > 1 - \lambda$ for all $m,\ n >$ N.

(iv) A Menger space $(X, F, *)$ where $*$ is continuous, is said to be complete if every Cauchy sequence in X is convergent in (ϵ, λ) -topology.

Definition 1.6:[6] Two self mappings A and B of a Menger space $(X, F, *)$ are said to be

(i) compatible of type (P) if

$F_{ABx_n,BBx_n}(t) \to 1$ and $F_{BAx_n,AAx_n}(t) \to 1$ for all $t > 0$

where $\{x_n\}$ is a sequence in X such that $Ax_n,\ Bx_n \to z$ for some z in X as $n \to \infty$.

(ii) compatible of type (P_1) if $F_{ABx_n,BBx_n}(t) \to 1$ for all $t > 0$

where $\{x_n\}$ is a sequence in X such that $Ax_n,\ Bx_n \to z$ for some z in X as $n \to \infty$.

(iii) compatible of type (P_2) if $F_{BAx_n,AAx_n}(t) \to 1$ for all $t > 0$

where $\{x_n\}$ is a sequence in X such that $Ax_n,\ Bx_n \to z$ for some z in X as $n \to \infty$.

Lemma 1.7:[3] Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in a Menger space $(X, F, *)$ with Hadzic-type t-norm $*$. If there exists $k \in (0, 1)$ such that

$$F_{x_n,x_{n+1}}(kt) \geq F_{x_{n-1},x_n}(t) \text{ for all } t > 0,$$

then $\{x_n\}$ is a Cauchy sequence.

We observe that 'min' t-norm is of Hadzic type.

Lemma 1.8:[7] Let $(X, F, *)$ be a Menger space. If there exists $k \in (0, 1)$ such that $F_{x,y}(kt) \geq F_{x,y}(t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.

2 Main Results

Now we state below Theorem 1 of [6], which we show to be not valid (Example 2.3) and obtain modifications (Theorem 2.6 and Corollary 2.7).

Theorem 2.1 ([6], Theorem 1): Let A, B, P, Q, S and T be self mappings of a complete Menger space $(X, F, *)$ with continuous t-norm $*$ such that $t*t \geq t\ \forall\ t \in [0, 1]$ (i.e. $*$ is the 'min' t-norm), satisfying:

(i) P(X) $\subseteq$ ST(X), Q(X) $\subseteq$ AB(X)

(ii) there exists a constant $k \in (0, 1)$ such that

$$F_{Px,Qy}(kt) \geq F_{ABx,STy}(t) * F_{Px,ABx}(t) * F_{Qy,STy}(t) * F_{Px,STy}(\alpha t) * F_{Qy,ABx}((2-\alpha)t)$$

for all $x, y \in X,\ t > 0$ and $\alpha \in (0, 2)$

(iii) either P or C is continuous

(iv) the pairs (P,C) and (Q,R) are both compatible of type (P_1) or type (P_2)

(v) $AB = BA, ST = TS, PB = BP, QT = TQ$

Then A, B, P, Q, S and T have a unique common fixed point. The following Corollary of Theorem 2.1 is given in [6].

Corollary 2.2 ([6], Corollary 1): Let P, Q be self maps on Menger space $(X, F, *)$ with continuous t-norm and $t * t \geq t \ \ \forall\ t \in [0, 1]$. If there exists a constant $k \in (0, 1)$ such that

$$F_{Px,Qy}(kt) \geq F_{x,y}(t) * F_{Px,x}(t) * F_{Qy,y}(t) * F_{Px,y}(\alpha t) * F_{Qy,x}((2-\alpha)t)$$

for all $x,\ y \in X,\ t > 0$ *and* $\alpha \in (0,\ 2)$,
then P and Q have a unique common fixed point.
Corollary 2.2 is not valid in view of the following example.
Example 2.3: Let $X = Z^+$ and the function F by for any $m, n \in Z^+$

$$F_{m,n}(t) = \begin{cases} 0 & if\ t \leq \max\{m, n\} \\ 1 & if\ t > \max\{m, n\} \end{cases}$$

Define the functions P and Q on Z^+ by P$n = n{+}1$ and Q$n = 1$ for all $n \in Z^+$.
Let $k \in (0, 1)$ and take $\alpha = k$
Then for every $x, y \in X,\ t > 0$, we have

$$F_{x+1,1}(kt) \geq F_{x,y}(t) * F_{x+1,x}(t) * F_{1,y}(t) * F_{x+1,y}(\alpha t) * F_{1,x}((2-\alpha)t) \ .... \ (1)$$

This is so, because $F_{x+1,1}(kt) = 1$ if $kt > x + 1$ and (1) is satisfied and if $kt < x + 1$, then $F_{x+1,1}(kt) = 0$ and $F_{x+1,1}(\alpha t) = 0$ ($\because \alpha = k$)
So that (1) is again satisfied. In either case the inequality (1) holds, but P and Q fail to have a common fixed point.
Hence Theorem 2.1 needs modification which we give below. In this connection it may be observed that $F_{m,n}(t)$ is not strictly increasing in $(0, \infty)$.We make use of this observation in obtaining the necessary modifications to Theorem 2.1

Definition 2.4: Let $(X, F, *)$ be a Menger space such that $F_{x,y}(t)$ is strictly increasing in t when ever $x \neq y$. Then $(X, F, *)$ is called a strict Menger space.
Example 2.5: Let (X, d) be a metric space. Define $F_{x,y}(t) = \frac{t}{t+d(x,y)}\ \forall\ t > 0$ *and* $x, y \in X$. If t-norm $*$ is $a * b = \min\{a, b\}\ \ \forall\ \ a, b \in [0, 1]$, then $(X, F, *)$ is a strict Menger space.

The following Theorem 2.6 and Corollary 2.7 may be regards as modifications to Theorem 2.1.
Theorem 2.6: Let P, Q, R and C be self mappings of a complete strict Menger space $(X, F, *)$ where $*$ is the min t-norm, satisfying:

(i) P(X) $\subseteq$ R(X), Q(X) $\subseteq$ C(X)

(ii) there exists a constant $k \in (0, 1)$ such that

$$F_{Px,Qy}(kt) \geq F_{Cx,Ry}(t) * F_{Px,Cx}(t) * F_{Qy,Ry}(t) * F_{Px,Ry}(2t) * F_{Qy,Cx}(2t)$$

for all $x, y \in X, \quad t > 0$

(iii) either P or C is continuous

(iv) the pairs (P,C) and (Q,R) are both compatible of type (P_1) or type (P_2)

Then P, Q, R and C have a unique common fixed point.

Proof: Let $x_0 \in X$. By (i), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$Px_{2n} = Rx_{2n+1} = y_{2n}$ and $Qx_{2n+1} = Cx_{2n+2} = y_{2n+1}$for $n =$ 0,1,2 ...

By taking $x = x_{2n}, y = x_{2n+1}$ for all $t > 0$ in (ii), we get

$$F_{Px_{2n},Qx_{2n+1}}(kt) \geq F_{Cx_{2n},Rx_{2n+1}}(t) * F_{Px_{2n},Cx_{2n}}(t) * F_{Qx_{2n+1},Rx_{2n+1}}(t) * F_{Px_{2n},Rx_{2n+1}}(2t) * F_{Qx_{2n+1},Cx_{2n}}(2t)$$

$$\begin{aligned} \Rightarrow F_{y_{2n},y_{2n+1}}(kt) &\geq F_{y_{2n-1},y_{2n}}(t) * F_{y_{2n},y_{2n-1}}(t) * F_{y_{2n+1},y_{2n}}(t) * F_{y_{2n},y_{2n}}(2t) * F_{y_{2n+1},y_{2n-1}}(2t) \\ &\geq F_{y_{2n-1},y_{2n}}(t) * F_{y_{2n+1},y_{2n}}(t) * F_{y_{2n+1},y_{2n-1}}(2t) \\ &\geq F_{y_{2n-1},y_{2n}}(t) * F_{y_{2n+1},y_{2n}}(t) * F_{y_{2n+1},y_{2n}}(t) * F_{y_{2n},y_{2n-1}}(t) \\ &\geq F_{y_{2n-1},y_{2n}}(t) * F_{y_{2n+1},y_{2n}}(t) \end{aligned}$$

Similarly, we can prove that $F_{y_{2n+1},y_{2n+2}}(kt) \geq F_{y_{2n},y_{2n+1}}(t) * F_{y_{2n+1},y_{2n+2}}(t)$

Hence $F_{y_n,y_{n+1}}(kt) \geq F_{y_{n-1},y_n}(t) * F_{y_n,y_{n+1}}(t)$

Consequently $F_{y_n,y_{n+1}}(kt) \geq F_{y_{n-1},y_n}(t)$ for all $t > 0,\ k \in (0,1),\ n \in N$ which is obvious if $y_n = y_{n+1}$ and follows from the strictly increasing nature of $F_{y_n,y_{n+1}}(t)$ when $y_n \neq y_{n+1}$.

By Lemma [1.7], $\{y_n\}$ is a Cauchy sequence.

Since $(X, F, *)$ is complete, it converges to a point z in X. Also its sub sequences $\{Px_{2n}\} \to z,\ \{Cx_{2n}\} \to z,\ \{Qx_{2n+1}\} \to z$ and $\{Rx_{2n+1}\} \to z$

Case (i): C is continuous, (P,C) and (Q,R) are compatible of type (P_2)

$CCx_{2n} \to Cz$, $CPx_{2n} \to Cz$ ($\because$ C is continuous)

$PPx_{2n} \to Cz$ ($\because$ (P,C) is compatible of type (P_2))

Now by taking $x = Px_{2n},\ y = x_{2n+1}$in (ii), we get $Cz = z$.

Similarly by taking $x = z,\ y = x_{2n+1}$ in (ii), we get $Pz = z$.

Since P(X) ⊆ R(X), there exists $w \in X$ such that $z = Pz = Rw$

By taking $x = x_{2n},\ y = w$ in (ii), we get $Qw = z$

$\therefore Rw = Qw = z$

Since (Q,R) is compatible of type (P_2), we have $RQw = QQw$.

Therefore $Rz = Qz$.

Now by taking $x = x_{2n},\ y = z$ in (ii), we get $Qz = z$.

$\therefore Pz = Qz = Cz = Rz = z$.

i.e. z is a common fixed point for P, Q, R and C.

Case (ii): P is continuous and (P,C), (Q,R) are both compatible of type (P_2)

$PPx_{2n} \to Pz$, $PCx_{2n} \to Pz$ ($\because P$ is continuous)

$CPx_{2n} \to Pz$ ($\because$ (P,C) is compatible of type (P_2))

By taking $x = Px_{2n},\ y = y_{2n+1}$in (ii), we get P$z = z$.

we have $z =$ Q$z =$ Rz.

Since Q(X) ⊆ C(X), there exists $w \in X$ such that $z = Qz = Cw$.
By taking $x = w,\ y = x_{2n+1}$ in (ii), we get z = Pw.
Since $z = Qz = Cw$, hence $Pw = Cw$.
(P,C) is compatible of type (P_2), we have $CPw = PPw$
i.e. $Cz = Pz$.
$\therefore z = Pz = Cz = Qz = Rz$
i.e. z is a common fixed point for P, Q, R and C.
$\therefore z$ is a common fixed point for P, Q, R and C when C is continuous(or P is continuous) and (P,C), (Q,R) are compatible of type P_2 (or P_1)
For uniqueness, let v be a common fixed point for P, Q, R and C.
Take $x = z,\ y = v$ in the condition (ii), then we get $v = z$.
Therefore P, Q, R and C have a unique common fixed point.

Corollary 2.7: Let A, B, P, Q, S and T be self mappings of a complete strict Menger space $(X, F, *)$ where $*$ is the min t-norm, satisfying:
(i) $P(X) \subseteq ST(X)$,$Q(X) \subseteq AB(X)$
(ii) there exists a constant $k \in (0, 1)$ such that
$F_{Px,Qy}(kt) \geq F_{ABx,STy}(t) * F_{Px,ABx}(t) * F_{Qy,STy}(t) * F_{Px,STy}(2t) * F_{Qy,ABx}(2t)$
for all $x, y \in X, \quad t > 0$
(iii) either P or ST is continuous
(iv) the pairs (P,ST) and (Q,AB) are both compatible of type (P_1) or type (P_2)
(v) $AB = BA, ST = TS, PB = BP, QT = TQ$
Then A, B, P, Q, S and T have a unique common fixed point.
Proof: Write $C = STandR = AB$
Then, by Theorem 2.6, there exists $z \in X$ such that $z = Pz = Rz = Qz = Cz$.
Hence $z = Pz = Rz = STz = Qz = Cz = ABz$.
Now $STz = z \Rightarrow T\,(STz) = Tz \Rightarrow TSTz = Tz \Rightarrow STTz = Tz$
$\therefore Tz$ is a fixed point for ST.
Since $ABz = z\ \Rightarrow BABz = Bz\ \Rightarrow ABBz = Bz$
i.e. Bz is a fixed point for AB.
Similarly, $ABz = z\ \Rightarrow AABz = Az\ \Rightarrow ABAz = Az$
i.e. Az is a fixed point for AB.
Therefore $AzandBz$ are fixed points for AB.
Now $Pz = z \Rightarrow BPz = Bz\ \Rightarrow PBz = Bz$
i.e. Bz is a fixed point for P.
Since $Qz = z\ \Rightarrow TQz = Tz\ \Rightarrow QTz = Tz$
i.e. Tz is a fixed point for Q.
By taking $x = Bz,\ y = Tz$ in (b), we get Bz = Tz
$\therefore Bz$ is a common fixed point for P, Q, AB, ST.
By Theorem 2.6,$Bz = z = Tz$ is a common fixed point for P, Q, AB, ST.
Since $ABz = z\ \Rightarrow Az = z$ and $STz = z\ \Rightarrow Sz = z$

$\therefore z$ is a common fixed point for A, B, S, T, P and Q.
For uniqueness, let v be a common fixed point for A, B, S, T, P and Q.
By taking $x = z,\ y = v$ in condition (ii), we get $z = v$.

We provide the following example in support of Theorem 2.6.
Example 2.8: $(\mathbb{R}, F, *)$ is a strict Menger space where $\mathbb{R}$ is a real line with the usual metric, and $F : \mathbb{R} \to [0, 1]$ is defined by $F_{x,y}(t) = \frac{t}{t+d(x,y)}\ \forall\ x, y \in \mathbb{R}$ and $*$ is the min t-norm, i.e. $* = \min\{a, b\}$.
Let P, Q, R and C be the self maps on $\mathbb{R}$, defined by $Px = \begin{cases} 0 & if\ x \leq 3 \\ 1 & if\ x > 3 \end{cases}$
$Qx = 0, Rx = x^2\ and\ Cx = I\ \ \forall\ x \in \mathbb{R}$.

Then clearly P, Q, R and C satisfy the hypothesis of Theorem 2.6. It is clear that 0 is the only common fixed point of P, Q, R and C.
It may be noted that P is discontinuous and C is continuous.

The following example is supporting Corollary 2.7.
Example 2.9: Let $(\mathbb{R}, F, *)$ be as in Example 2.8.
Let A, B, P, Q, S and T be the self maps on $\mathbb{R}$, defined by $Px = \begin{cases} 0 & if\ x \leq 3 \\ 1 & if\ x > 3 \end{cases}$
$Qx = 0, Ax = x^4, Bx = \sqrt{|x|}\ \ and\ \ Sx = -x = Tx\ \ \forall\ x \in \mathbb{R}$.

Clearly A, B, P, Q, S and T satisfy the hypothesis of Corollary 2.7. It is clear that 0 is the only common fixed point of A, B, P, Q, S and T.

We conclude the paper with two open problems.
Open problem 2.10: Is Theorem 2.6 valid if $2t$ in condition (ii) is replaced by αt where $\alpha \in (1, 2)$?

Open problem 2.11: Is Theorem 2.6 valid if $(X, F, *)$is not necessarily strict?

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