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On Some Common Fixed Point Theorems for f-Contraction Mappings in Cone Metric Spaces

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Abstract

In this paper, we generalized the contraction mappings which were given in [3].We obtained some common fixed point results for two Banach pairs of mappings which satisfy $f-contraction$ condition on cone metric spaces without the assumption of normality condition of the cone.

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1 Introduction

In [5] Huang and Zhang generalized the concept of metric spaces, replacing the set of real numbers by an ordered Banach space, hence they have defined the cone metric spaces. They also described the convergence of sequences and introduced the notion of completeness in cone metric spaces. They have proved some fixed point theorems of contractive mappings on complete cone metric space with the assumption of normality of a cone. Subsequently, various authors have generalized the results of Huang and Zhang and have studied fixed point theorems for normal and non-normal cones.There exist a lot of works involving fixed points used the Banach contraction principle. This principle has been extended kind of contraction mappings by various authors. A new generalization of contraction mapping has been introduced and called $f-contraction$ mappings on metric spaces which are depending on another function by Beiranvand [2]. Afterwards; in [10] authors have proved some common fixed point theorems for a Banach pair of mappings satisfying $f-$Hardy-Rogers type contraction condition in cone metric spaces. Also, Morales and Rojas [8-9] have extended the notion of $f-contraction$ mappings to cone metric spaces by proving fixed point theorems for $f-Kannan$, $f-Zamfirescu$, $f-$ weakly contraction mappings. In the sequel, we need a definition which was introduced and called Banach operator of type k by Subrahmanyam [11]. This notion was extended to Banach operator pair by Chen and Li and they have proved various best approximation results using common fixed theorems for $f-$ nonexpansive mappings [4].

In this paper our main goal is to obtain common fixed point theorems for two Banach pairs of mappings which satisfy $f-contraction$ condition on a complete cone metric space without assumption of normality condition of the cone. Also we get some consequences related with these theorems and we give a corollary which depends on P property of a map.

2 Preliminary Notes

We give some facts and definitions which we need them in the sequel.

Definition 2.1 *A self mapping f of a metric space (X,d) is said to be a contraction mapping, if there exists a real number $k\in[0,1)$ such that*

$$d\,(fx,fy)\leq kd\,(x,y) \tag{1}$$

for all $x,y\in X$.

The following definition is given by Beiranvand et al.[2].

Definition 2.2 *Let (X,d)be a metric space and $T,f:X\rightarrow X$ be two functions. A mapping T said to be $f-contraction$ if there exists $k\in[0,1)$ such that*

$$d\,(fTx,fTy)\leq d\,(fx,fy) \tag{2}$$

for all $x,y\in X$.

If we take f as identity map, we get the inequality (2). We will give an example to show that if a map is $f-contraction$ then it need not to be contraction.

Example 2.3 *Let $X=[0,\infty)$ be with the usual metric. Let define two mappings as*

$$T,f:X\rightarrow X$$

$$Tx = \beta x, \beta > 1$$

$$fx = \frac{\alpha}{x^2}, \alpha \in R.$$

It is clear that, T is not a contraction but it is $f-$contraction since,

$$d\left(fTx, fTy\right) = \left|\frac{\alpha}{\beta^2 x^2} - \frac{\alpha}{\beta^2 y^2}\right| = \frac{1}{\beta^2}\left|fx - fy\right|.$$

Definition 2.4 *Let (X, d) be a metric space.*

1. *A mapping $T, f : X \to X$ is said to be sequentially convergent, if the sequence $\{y_n\}$ in X is convergent whenever $\{Ty_n\}$ is convergent.*

2. *$T, f : X \to X$ is said to be sub-sequentially convergent, if the sequence $\{y_n\}$ has a convergent subsequence whenever $\{Ty_n\}$ is convergent.*

Definition 2.5 *Let f be a self mapping of a normed space X. Then f is called a Banach operator of type k if*

$$\left\|f^2 x - fx\right\| \leq k \left\|fx - x\right\|$$

for some $k \geq 0$ and all $x \in X$.

This concept was introduced by Subrahmanyam [11], then Chen and Li [4] extended this as following:

Definition 2.6 *Let f and T be self mappings of a nonempty subset M of a normed linear space X. Then (f, T) is a Banach operator pair, if any one of the following conditions is satisfied:*

1. *$f\left[F\left(T\right)\right] \subseteq F\left(T\right),$*

2. *$Tfx = fx$ for each $x \in F\left(T\right),$*

3. *$fTx = Tfx$ for each $x \in F\left(T\right),$*

4. *$\left\|fTx - Tx\right\| \leq k\left\|Tx - x\right\|$ for some $k \geq 0$.*

Let E be a real Banach space and K be a subset of E. K is called a cone if and only if

1. K is closed, nonempty and $K \neq \{0\}$,

2. $a, b \in R; a, b \geq 0; x, y \in K \Rightarrow ax + by \in K$,

3. $x \in K$ and $-x \in K \Rightarrow x = 0$.

Given a cone $K \subset E$, we define a partial ordering $\leq$ with respect to K by $x \leq y$ if and only if $y - x \in K$. We write $x < y$ if $x \leq y$ but $x \neq y$; $x \ll y$ if $y - x \in intK$, where $intK$ is the interior of K.There exists two kinds of cones which are normal and non normal cones. A cone K is called normal if there is a number $M > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \Rightarrow \|x\| \leq M \|y\| . \tag{3}$$

The least positive number satisfying (3) is called the normal constant of K.

Definition 2.7 *Let X be nonempty set, E be a real Banach space and $K \subset E$ be a cone. Suppose the mapping $d : X \times X \to E$ satisfies*

d1. $0 < d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

d2. $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

d3. $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. It is obvious that the concept of a cone metric space is more general than a metric space.

Definition 2.8 *Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$,*

1. *there is $N \in N$ such that for all $n > N$, $d(x_n, x) \ll c$ then x_n is said to be convergent,*

2. *there is $N \in N$ such that for all $n, m > N$, $d(x_n, x_m) \ll c$ then x_n is called a Cauchy sequence in X.*

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. If K is a normal cone with normal constant$M$ then the limit of a convergent sequence is unique.

Lemma 2.9 *([6]) Let (X, d) be a cone metric space, $u, v, w \in X$. Then*

i. *If $u \ll v$ and $v \ll w$, then $u \ll w$,*

ii. *If $u \leq v$ and $v \ll w$, then $u \ll w$,*

iii. *If $0 \leq u \ll c$ for each $c \in intK$, then $u = 0$,*

iv. *If $c \in intK$, $0 \leq a_n$ and $a_n \to 0$, then there exists n_0 such that for all $n > n_0$, it follows that $a_n \ll c$.*

Throughout the paper we assume that E is a real Banach space and K is a cone in E with int$K \neq \emptyset$. By this way, we uniquely determine the limit of a sequence. Now, we will give our main results as following:

3 Main Results

The results which we will give, are generalization of theorem 2.1 and 2.3 of [3].

Theorem 3.1 *Let f, T and S be continuous self mappings of a complete cone metric space (X, d). Assume that f is a injective mapping. If the mappings f, T and S satisfy*

$$d(fSx, fTy) \leq \alpha d(fx, fy) + \beta(d(fx, fTy) + d(fy, fSx)) \tag{4}$$

for all $x, y \in X$ and $\alpha, \beta \in [0, 1)$ with $\alpha + 2\beta < 1$. Then T and S have a unique common fixed point. Moreover, if (f, T) and (f, S) are Banach pairs, then f, T and S have a unique common fixed point in X.

Proof. Take $x_0 \in X$ as an arbitrary element and define the sequences $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for each $n \geq 0$. Then, by using (4) and triangle inequality

$$\begin{aligned} d(fx_{2n+1}, fx_{2n}) &= d(fSx_{2n}, fTx_{2n-1}) \\ &\leq \alpha d(fx_{2n}, fx_{2n-1}) + \beta(d(fx_{2n}, fTx_{2n-1}) + d(fx_{2n-1}, fSx_{2n})) \\ &= \alpha d(fx_{2n}, fx_{2n-1}) + \beta(d(fx_{2n}, fx_{2n}) + d(fx_{2n-1}, fx_{2n+1})) \\ &\leq \alpha d(fx_{2n}, fx_{2n-1}) + \beta(d(fx_{2n-1}, fx_{2n}) + d(fx_{2n}, fx_{2n+1})) \\ d(fx_{2n+1}, fx_{2n}) &= \frac{\alpha + \beta}{1 - \beta} d(fx_{2n}, fx_{2n-1}). \end{aligned}$$

Similarly,

$$\begin{aligned} d(fx_{2n+3}, fx_{2n+2}) &= d(fSx_{2n+2}, fTx_{2n+1}) \\ &\leq \alpha d(fx_{2n+2}, fx_{2n+1}) + \beta(d(fx_{2n+2}, fTx_{2n+1}) + d(fx_{2n+1}, fSx_{2n+2})) \\ &= \alpha d(fx_{2n+2}, fx_{2n+1}) + \beta(d(fx_{2n+2}, fx_{2n+2}) + d(fx_{2n+1}, fx_{2n+3})) \\ &\leq \alpha d(fx_{2n+2}, fx_{2n+1}) + \beta(d(fx_{2n+1}, fx_{2n+2}) + d(fx_{2n+2}, fx_{2n+3})) \\ d(fx_{2n+3}, fx_{2n+2}) &= \frac{\alpha + \beta}{1 - \beta} d(fx_{2n+2}, fx_{2n+1}). \end{aligned}$$

Thus,

$$d(fx_{n+1}, fx_n) \leq \lambda d(fx_n, fx_{n-1}) \leq ... \leq \lambda^n d(fx_1, fx_0)$$

for all $n \geq 0$ where $\lambda = \frac{\alpha+\beta}{1-\beta} < 1$. Now, for $n > m$ we have

$$\begin{aligned} d(fx_n, fx_m) &\leq d(fx_n, fx_{n-1}) + d(fx_{n-1}, fx_{n-2}) + ... + d(fx_{m+1}, fx_m) \\ &\leq \left(\lambda^{n-1} + \lambda^{n-2} + ... + \lambda^m\right) d(fx_1, fx_0) \leq \frac{\lambda^m}{1 - \lambda} d(fx_1, fx_0). \end{aligned}$$

Let $0 \ll c$ be given. Choose $\delta > 0$ such that $c + N_\delta(0) \subseteq K$, where

$$N_\delta(0) = \{y \in E : \|y\| < \delta\}.$$

Also, choose a natural number N_1 such that $\frac{\lambda^m}{1-\lambda} d(fx_1, fx_0) \in N_\delta(0)$, for all $m \geq N_1$. Then $\frac{\lambda^m}{1-\lambda} d(fx_1, fx_0) \ll c$, for all $m \geq N_1$. Thus,

$$d(fx_n, fx_m) \leq \frac{\lambda^m}{1-\lambda} d(fx_1, fx_0)$$

and

$$\frac{\lambda^m}{1-\lambda} d(fx_1, fx_0) \ll c$$

for all $m > n$.Then we get $d(fx_n, fx_m) \ll c$ for all $n > m$. Therefore, $\{fx_n\}$ is a Cauchy sequence in (X, d). As X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} fx_n = z$. Since f is sub-sequentially convergent, $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that $\lim_{n\to\infty} x_m = u$. As f is continuous,

$$\lim_{m\to\infty} fx_m = fu.$$

By the uniqueness of the limit, $z = fu$. Since T and S is continuous, $\lim_{m\to\infty} Tx_m = Tu$ and $\lim_{n\to\infty} Sx_m = Su$. Again since f is continuous, $\lim_{m\to\infty} fTx_m = fTu$ and $\lim_{m\to\infty} fSx_m = fSu$. Therefore, if m is odd, then

$$\lim_{n\to\infty} fTx_{2n+1} = fTu.$$

Choose a natural number N_2 such that $d(fx_{2n+1}, fu) \ll \left[\frac{c}{2}\left(\frac{\alpha+\beta}{1-\beta}\right)\right]$ for all $n \geq N_2$. Now consider,

$$\begin{aligned}
d(fu, fTu) &\leq d(fu, fx_{2n+1}) + d(fx_{2n+1}, fTu) \\
&= d(fu, fSx_{2n}) + d(fSx_{2n}, fTu) \\
&\leq d(fu, fSx_{2n}) + \alpha d(fx_{2n}, fu) \\
&\quad +\beta(d(fx_{2n}, fTu) + d(fu, fSx_{2n})) \\
&\leq d(fu, fx_{2n+1}) + \alpha d(fx_{2n}, fu) \\
&\quad + \beta(d(fx_{2n}, fu) + d(fu, fTu) + d(fu, fx_{2n+1})) \\
&= (\alpha+\beta) d(fx_{2n}, fu) \\
&\quad + (1+\beta) d(fu, fx_{2n+1}) + \beta d(fu, fTu).
\end{aligned}$$

So,

$$d(fu, fTu) \leq \left[\frac{\alpha+\beta}{1-\beta}\right] d(fx_{2n}, fu) + \left[\frac{1+\beta}{1-\beta}\right] d(fu, fx_{2n+1}) \ll c$$

for all $n \geq N_2$. Therefore, $d(fu, fTu) \ll \frac{c}{i}$ for all $i \geq 1$. Hence, $\frac{c}{i} - d(fu, fTu) \in K$ for all $i \geq 1$. Since K is closed, $-d(fu, fTu) \in K$ and so $d(fu, fTu) = 0$. Hence, $fu = fTu$. As f is injective, $u = Tu$. Thus u is the fixed point of T. And if m is even, then we have

$$\lim_{n \to \infty} fSx_{2n} = fSu.$$

Now, by using [4] and triangle inequality we have

$$d(fSu, fu) \leq \left[\frac{\alpha + \beta}{1 - \beta}\right] d(fx_{2n+1}, fu) + \left[\frac{1 + \beta}{1 - \beta}\right] d(fu, fx_{2n+2}) \ll c$$

for all $n \geq N_2$. Therefore, $d(fSu, fu) \ll \frac{c}{i}$ for all $i \geq 1$. Hence, $\frac{c}{i} - d(fSu, fu) \in K$ for all $i \geq 1$. Since K is closed, $-d(fSu, fu) \in K$ and so $d(fSu, fu) = 0$. Hence, $fu = fSu$. As f is injective, $u = Su$ i.e. u is the fixed point of S, too. For the uniqueness suppose that u^* is another common fixed point of T and S.

$$\begin{aligned} d(fu, fu^*) &= d(fSu, fTu^*) \\ &\leq \alpha d(fu, fu^*) + \beta(d(fu, fTu^*) + d(fu^*, fSu)) \\ d(fu, fu^*) &\leq (\alpha + 2\beta) d(fu, fu^*) \end{aligned}$$

Since $\alpha + 2\beta < 1$, $d(fu, fu^*) = 0$ which implies that $fu = fu^*$. We know that f is injective, $u = u^*$ is the unique common fixed point of T and S.Since we have assumed that (f, S) and (f, T) are Banach pairs; $\{f, S\}$ and $\{f, T\}$ commutes at the fixed point of S and T, respectively. This implies that $fSu = Sfu$ for $u \in F(S)$. So, $fu = Sfu$ which gives that fu is another fixed point of S. It is true for T, too. By the uniqueness of fixed point of S, $fu = u$. Hence $u = fu = Su = Tu$, u is unique common fixed point of f, S and T in X.

Corollary 3.2 *Let f, T and S be continuous self mappings of a complete cone metric space (X, d). Assume that f is a injective mapping. If the mappings f, T and S satisfy*

$$d(fSx, fTy) \leq \alpha d(fx, fy) + \beta d(fy, fSx) \tag{5}$$

for all $x, y \in X$ and $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. Then T and S have a unique common fixed point. Moreover, if (f, T) and (f, S) are Banach pairs, then f, T and S have a unique common fixed point in X.

Proof. The proof of this corollary is similar with Theorem 3.1, so we omit it.

Corollary 3.3 *Let (X, d) be a complete cone metric space and f and T be two continuous self mappings which satisfy*

$$d(fTx, fTy) \leq \alpha d(fx, fy) + \beta d(fx, fTy) + \gamma d(fy, fTx) \tag{6}$$

for all $x, y \in X$ and $\alpha, \beta\gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$ where f is a injective mapping. Then T has a unique common fixed point in X. Moreover, if (f, T) is a Banach pair , then f and T have a unique common fixed point in X.

Proof. If we take $S = T$ and $\gamma = \beta$ in Theorem 3.1 we get the proof.

The next result is about P property of T. Now, first we give the definition of this property.

Definition 3.4 *If a map satisfies $F(T) = F(T^n)$ for each $n \in N$, then it is said to have property P, where $F(T)$ is the set of fixed points of the mapping T.*

Corollary 3.5 *f and T be two continuous self mappings which satisfy the inequality (6) for all $x, y \in X$ and $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$ where f is a injective mapping in complete cone metric space (X, d). Then T has property P.*

Proof. Let $u \in F(T^n)$. Then;

$$d(fTu, fu) = d(fT^{n+1}u, fT^nu)$$

$$\leq \alpha d(fT^nu, fT^{n-1}u) + \beta d(fT^nu, fT^nu)$$

$$+ \gamma d(fT^{n-1}u, fT^{n+1}u)$$

$$d(fT^{n+1}u, fT^nu) \leq \left[\frac{\alpha+\gamma}{1-\gamma}\right] d(fT^nu, fT^{n-1}u)$$

$$d(fTu, fu) = d\left(fT^{n+1}u, fT^nu\right) \leq \lambda d\left(fT^nu, fT^{n-1}u\right) \leq ... \leq \lambda^n d(fTu, fu)$$

where $\lambda = \frac{\alpha+\gamma}{1-\gamma} < 1$. Let $0 \ll c$ be given. Since $\lambda^n \to 0$ as $n \to \infty$, we get that $d(fTu, fu) \ll c$. Therefore, $d(fTu, fu) \ll \frac{c}{m}$ for all $m \geq 1$. Hence, $\frac{c}{m} - d(fTu, fu) \in K$ for all $m \geq 1$. Since K is closed, $-d(fTu, fu) \in K$ and so $fTu = fu$. By the injectivity of f, $Tu = u$ and so we conclude that the mapping T has P property.

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