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On Generating Relations of Some Triple Hypergeometric Functions

Fadhle B. F. Mohsen and Gamal A. Qashash

Department of Mathematics, Faculty of Education
Zingibar –Aden University, Aden, Yemen
mfazalmohsen@yahoo.com

Abstract

In this paper we aim at presenting certain generating relations, involving the Exton's functions X_2, X_4, X_7, X_8 *and* X_{12}, where every one can be represented by Laplace integral formulae.

Some of known results are obtained as special cases of our main results.

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1. Introduction

Exton in [2] defined and gave integral representations of some hypergeometric functions of three variables, and denoted them by $X_1\ , X_2\ ,..., X_{20}$.

The definitions and integral representations of X_2, X_4, X_7, X_8 *and* X_{12}, are as follows (cf. [2]).

$$X_2\left(\alpha,\beta;\gamma,\delta,\lambda;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{2m+2n+p}\ (\beta)_p}{(\gamma)_m\ (\delta)_n\left(\lambda\right)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\ , \tag{1.1}$$

$$X_4\left(\alpha,\beta;\gamma,\delta,\lambda;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{2m+n+p}\ (\beta)_{n+p}}{(\gamma)_m\ (\delta)_n\left(\lambda\right)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\ , \tag{1.2}$$

$$X_7(\alpha,\beta_1,\beta_2;\gamma,\delta;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{2m+n+p}\ (\beta_1)_n\left(\beta_2\right)_p}{(\gamma)_m\ (\delta)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad (1.3)$$

$$X_8(\alpha,\beta_1,\beta_2;\gamma,\delta,\lambda;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{2m+n+p}\ (\beta_1)_n\left(\beta_2\right)_p}{(\gamma)_m\ (\delta)_n\left(\lambda\right)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad (1.4)$$

$$X_{12}(\alpha,\beta;\gamma,\delta,\lambda;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{2m+n}\ (\beta)_{n+2p}}{(\gamma)_m\ (\delta)_n\left(\lambda\right)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad (1.5)$$

$$X_2(\alpha,\beta;\gamma,\delta,\lambda;x,y,z)=\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-s}s^{\alpha-1}\,{}_0F_1\left(-;\gamma;xs^2\right)$$

$$\times\,{}_0F_1\left(-;\delta;ys^2\right)\,{}_1F_1\left(\beta;\lambda;zs\right)ds, \quad (1.6)$$

$$X_4(\alpha,\beta;\gamma,\delta,\lambda;x,y,z)=\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-s}s^{\alpha-1}\,{}_0F_1\left(-;\gamma;xs^2\right)$$

$$\times\,\Psi_2\left(\beta;\delta,\lambda;ys,zs\right)ds, \quad (1.7)$$

$$X_7(\alpha,\beta_1,\beta_2;\gamma,\delta;x,y,z)=\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-s}s^{\alpha-1}\,{}_0F_1\left(-;\gamma;xs^2\right)$$

$$\times\,\Phi_2\left(\beta_1,\beta_2;\delta;ys,zs\right)ds, \quad (1.8)$$

$$X_8(\alpha,\beta_1,\beta_2;\gamma,\delta,\lambda;x,y,z)=\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-s}s^{\alpha-1}\,{}_0F_1\left(-;\gamma;xs^2\right)$$

$$\times\,{}_1F_1\left(\beta_1;\delta;ys\right)\,{}_1F_1\left(\beta_2;\lambda;zs\right)ds, \quad (1.9)$$

$$X_{12}(\alpha,\beta;\gamma,\delta,\lambda;x,y,z)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty\int_0^\infty e^{-s-t}s^{\alpha-1}t^{\beta-1}\,{}_0F_1\left(-;\gamma;xs^2\right)$$

$$\times\,{}_0F_1\left(-;\delta;ys\ t\right)\,{}_0F_1\left(-;\lambda;zt^2\right)ds\ dt, \quad (1.10)$$

where $\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0$, and the functions in the integrand are Humbert functions
(cf. [4], p.58)

2. Results

We have established the following new generating functions

$$\sum_{n=0}^{\infty}\frac{1}{n!}X_2\left(\alpha+n,\beta;\gamma,\delta,\lambda;x^2,y^2,z\right)w^n$$

$$=(1-z)^{-\beta}\sum_{n=0}^{\infty}\frac{w^n}{n!}F_4\left(\frac{\alpha+n}{2},\frac{\alpha+n+1}{2};\gamma,\delta;4x^2,4y^2\right), \qquad (2.1)$$

$$\sum_{n=0}^{\infty}\frac{1}{n!}X_2\left(\alpha+n,\beta;\gamma,\delta,\lambda;x^2,y^2,z\right)w^n$$

$$=(2x+2y+1)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{2x+2y+1}\right)^n$$

$$\times F_A^{(3)}\left(\alpha+n,\beta,\gamma-\frac{1}{2},\delta-\frac{1}{2};\lambda,2\gamma-1,2\delta-1;\frac{z}{2x+2y+1},\frac{4x}{2x+2y+1},\frac{4y}{2x+2y+1}\right), \qquad (2.2)$$

$$\sum_{n=0}^{\infty}\frac{1}{n!}X_2\left(\alpha+n,\beta;\gamma,\delta,\lambda;x^2,y,z\right)w^n$$

$$=(2x+1)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{2x+1}\right)^n$$

$$\times X_8\left(\alpha+n,\gamma-\frac{1}{2},\beta;\delta,2\gamma-1,\lambda;\frac{y}{(2x+1)^2},\frac{4x}{2x+1},\frac{z}{2x+1}\right), \qquad (2.3)$$

$$\sum_{n=0}^{\infty}\frac{1}{n!}X_2\left(\alpha+n,\beta;\gamma,\delta,\lambda;x,y,z\right)w^n=(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-z}\right)^n$$

$$\times X_2\left(\alpha+n,\lambda-\beta;\gamma,\delta,\lambda;\frac{x}{(1-z)^2},\frac{y}{(1-z)^2},\frac{z}{1-z}\right), \qquad (2.4)$$

$$\sum_{n=0}^{\infty}\frac{1}{n!}X_4\left(\alpha+n,\beta;\gamma,\delta,\lambda;x^2,y\ \ ,z\right)w^n=\left(2x+1\right)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{2x+1}\right)^n$$

$$\times F_E\left(\alpha+n,\alpha+n,\alpha+n,\gamma-\frac{1}{2},\beta,\beta;2\gamma-1,\delta,\lambda;\frac{4x}{2x+1},\frac{y}{2x+1},\frac{z}{2x+1}\right), \quad (2.5)$$

$$\sum_{n=0}^{\infty}\frac{1}{n!}X_7\left(\alpha+n,\beta_1,\beta_2;\gamma,\delta;x^2,y\ \ ,z\right)w^n=\left(2x+1\right)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{2x+1}\right)^n$$

$$\times F_G\left(\alpha+n,\alpha+n,\alpha+n,\gamma-\frac{1}{2},\beta_1,\beta_2;2\gamma-1,\delta,\delta;\frac{4x}{2x+1},\frac{y}{2x+1},\frac{z}{2x+1}\right), \quad (2.6)$$

$$\sum_{n=0}^{\infty}\frac{1}{n!}X_8\left(\alpha+n,\beta_1,\beta_2;\gamma,\delta,\lambda;x^2,y\ \ ,z\right)w^n=\left(2x+1\right)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{2x+1}\right)^n$$

$$\times F_A^{(3)}\left(\alpha+n,\gamma-\frac{1}{2},\beta_1,\beta_2;2\gamma-1,\delta,\lambda;\frac{4x}{2x+1},\frac{y}{2x+1},\frac{z}{2x+1}\right), \quad (2.7)$$

$$\sum_{n=0}^{\infty}\frac{1}{n!}X_{12}\left(\alpha+n,\beta+n;\gamma,\delta,\lambda;x^2,y\ \ ,z^2\right)w^n=\left(2x+2z+1\right)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{2x+2z+1}\right)^n$$

$$\times F_A^{(3)}\left(\alpha+n,\beta+n,\gamma-\frac{1}{2},\lambda-\frac{1}{2};\delta,2\gamma-1,2\lambda-1;\frac{y}{2x+2z+1},\frac{4x}{2x+2z+1},\frac{4z}{2x+2z+1}\right), \quad (2.8)$$

where F_4 is the Appell's function (cf. [4]), $F_A^{(3)}$ is the three variables Lauricella function (cf,[4],p.60), and F_E , F_G are Saran's functions of three variables (cf.[4],p.67).

3. Proofs of results

For proving the above results, we need the following formulae (cf. [1], p.137, 217 and [3], p.517)

$(i)\quad {}_1F_1(a;c;x)=e^x\ {}_1F_1(c-a;c;-x)$

$(ii)\quad {}_0F_1(—;c;x^2)=e^{-2x}\ {}_1F_1\left(c-\frac{1}{2};2c-1;4x\right)$

$(iii)\quad (\lambda)_{2n}=2^{2n}\left(\frac{\lambda}{2}\right)_n\left(\frac{\lambda+1}{2}\right)_n \qquad ,n=0,1,2,3,...$

$(iv)\ L\left\{t^{\gamma-1}\ {}_1F_1(\alpha;\gamma;\lambda t)\right\}=\Gamma(\gamma)\,p^{\alpha-\gamma}\,(p-\lambda)^{-\alpha}, \qquad \mathrm{Re}(\gamma)>0\ ,\mathrm{Re}(p)>0\ ,\mathrm{Re}(\lambda)>0$

$(v)\ L\left\{t^{\nu}\right\}=\Gamma(\nu+1)\ p^{-\nu-1}, \qquad \mathrm{Re}(\nu)>-1\ ,\mathrm{Re}(p)>0\ .$

$(vi)\ L\left\{x^{\mu}\ {}_1F_1(a,b;\sigma x)\ {}_1F_1(c,d;\omega y)\right\}=\Gamma(\mu+1)\,p^{-\mu-1}F_2\left(\mu+1,a,c;b,d;\frac{\omega}{p},\frac{\sigma}{p}\right)$

$\mathrm{Re}(\mu)>-1\ ,\ \mathrm{Re}(p-\sigma),\mathrm{Re}(p-\omega),\mathrm{Re}(p-\sigma-\omega)>0.$

Where L is the Laplace transform, and F_2 is Appell's function.

Proof of (2.1)

Let us denote the left hand side of (2.1) by I, using (1.6)

$$I=\frac{w^n}{\Gamma(\alpha+n)n!}\int_0^{\infty}e^{-s}s^{\alpha+n-1}\ {}_0F_1(—;\gamma;x^2s^2)\ \ {}_0F_1(—;\delta;y^2s^2)\ {}_1F_1(\beta;\lambda;zs)\,ds$$

Now, writing the second of two functions ${}_0F_1$ in the integrand in its series form, and interchanging the order of the summation and integral sign which is permissible here, we get

$$I=\sum_{n,p,q=0}^{\infty}\frac{w^n x^{2p}y^{2q}}{\Gamma(\alpha+n)(\gamma)_p(\delta)_q\,n!\,p!\,q!}\int_0^{\infty}e^{-s}s^{(\alpha+n+2p+2q-1)}\ {}_1F_1(\beta;\lambda;zs)\,ds\ , \qquad (3.1)$$

Also using (iv) , we get

$$I=\sum_{n,p,q=0}^{\infty}\frac{w^n x^{2p}y^{2q}}{\Gamma(\alpha+n)(\gamma)_p(\delta)_q\,n!\,p!\,q!}\,\frac{\Gamma(\alpha+n+2p+2q)}{(1-z)^{\beta}}\ ,$$

$$=\sum_{n,p,q=0}^{\infty}\frac{(\alpha+n)_{2p+2q}\,w^n x^{2p}y^{2q}}{(\gamma)_p(\delta)_q\,n!\,p!\,q!}\,\frac{1}{(1-z)^{\beta}}\ , \qquad (3.2)$$

Now in (3.2), simplified by using series manipulation (iii), considering the definition of the Appell's function F_4 ,we will get the right hand side of (2.1), which complete the Proof of (2.1).
The proof of (2.2) to (2.8) runs in the same way, considering the appropriate integral representation and Laplace transform during the proof.

4. Special cases

Some generating relations, which believed to be new, can be established as special cases as follow:

In (2.1), choosing x = 0, we get the following relation.

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_4\left(\alpha+n,\beta;\delta,\lambda;y^2,z\right)w^n$$
$$=(1-z)^{-\beta}\sum_{n=0}^{\infty}\frac{w^n}{n!}\,{}_2F_1\left(\frac{\alpha+n}{2},\frac{\alpha+n+1}{2};\delta;4y^2\right), \quad (4.1)$$

In (2.2), put z = 0 , we get

$$\sum_{n=0}^{\infty} \frac{1}{n!} F_4\left(\frac{\alpha+n}{2},\frac{\alpha+n+1}{2};\gamma,\delta;4x^2,4y^2\right)w^n$$
$$=(2x+2y+1)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{2x+2y+1}\right)^n$$
$$\times F_2\left(\alpha+n,\gamma-\frac{1}{2},\delta-\frac{1}{2};2\gamma-1,2\delta-1;\frac{4x}{2x+2y+1},\frac{4y}{2x+2y+1}\right), \quad (4.2)$$

In (2.3), put y = 0, we get

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_4\left(\alpha+n,\beta;\delta,\lambda;x^2,z\right)w^n$$

$$=(2x+1)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{2x+1}\right)^n F_2\left(\alpha+n,\gamma-\frac{1}{2},\beta;2\gamma-1,\lambda;\frac{4x}{2x+1},\frac{z}{2x+1}\right), \quad (4.3)$$

From (2.1) and (2.2), we have

$$(1-z)^{-\beta}\ \sum_{0}^{\infty}\frac{w^n}{n!}\ F_4\left(\frac{\alpha+n}{2},\frac{\alpha+n+1}{2};\gamma,\delta;4x^2,4y^2\right)$$

$$=(2x+2y+1)^{-\alpha}\ \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{2x+2y+1}\right)^n$$

$$\times F_A^{(3)}\left(\alpha+n,\beta,\gamma-\frac{1}{2},\delta-\frac{1}{2};\lambda,2\gamma-1,2\delta-1;\frac{z}{2x+2y+1},\frac{4x}{2x+2y+1},\frac{4y}{2x+2y+1}\right),\ (4.4)$$

In (4.4), put z = 0, we get

$$\sum_{0}^{\infty}\frac{w^n}{n!}\ F_4\left(\frac{\alpha+n}{2},\frac{\alpha+n+1}{2};\gamma,\delta;4x^2,4y^2\right)$$

$$=(2x+2y+1)^{-\alpha}\ \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{2x+2y+1}\right)^n$$

$$\times F_2\ \left(\alpha+n,\gamma-\frac{1}{2},\delta-\frac{1}{2};2\gamma-1,2\delta-1;\frac{4x}{2x+2y+1},\frac{4y}{2x+2y+1}\right),\qquad (4.5)$$

In (2.5), put z = 0, we get

$$\sum_{n=0}^{\infty}\ \frac{1}{n!}\ H_4\ \left(\alpha+n,\beta;\gamma,\delta;x^2,y\right)w^n=(2x+1)^{-\alpha}\ \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{2x+1}\right)^n$$

$$\times F_2\ \left(\alpha+n,\gamma-\frac{1}{2},\beta;2\gamma-1,\delta;\frac{4x}{2x+1},\frac{y}{2x+1}\right),\qquad (4.6)$$

In (2.6) and (2.7), put y = 0 , we have

$$\sum_{n=0}^{\infty}\ \frac{1}{n!}\ H_4\ \left(\alpha+n,\beta_2;\gamma,\delta;x^2,z\right)w^n=(2x+1)^{-\alpha}\ \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{2x+1}\right)^n$$

$$\times F_2\ \left(\alpha+n,\gamma-\frac{1}{2},\beta_2;2\gamma-1,\delta;\frac{4x}{2x+1},\frac{z}{2x+1}\right),\qquad (4.7)$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_4\left(\alpha+n,\beta_2;\gamma,\lambda;x^2,z\right) w^n = \left(2x+1\right)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{w}{2x+1}\right)^n$$

$$\times F_2\left(\alpha+n,\gamma-\frac{1}{2},\beta_2;2\gamma-1,\lambda;\frac{4x}{2x+1},\frac{z}{2x+1}\right), \qquad (4.8)$$

In (2.8), put x = 0, we get

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_4\left(\beta+n,\alpha+n;\lambda,\delta;z^2,y\right) w^n = \left(2z+1\right)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{w}{2z+1}\right)^n$$

$$\times F_2\left(\alpha+n,\beta+n,\lambda-\frac{1}{2};\delta,2\lambda-1;\frac{y}{2z+1},\frac{4z}{2z+1}\right), \qquad (4.9)$$

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