The Bessel Wavelet Transform

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Abstract

In this paper continuity of the Bessel wavelet transform $B_{\psi}$ of function $\phi$ in terms of a mother wavelet $\psi$ is investigated on certain distribution spaces when the Hankel transform of $\psi$ defined by $\hat{\psi}(x, y) \in C^\infty(R^2_+)$. A sobolev space boundedness result is obtained.

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1 Introduction

The Hankel transformation is usually defined by

$$\hat{\phi}(x) = (h_\mu \phi)(x) = \int_0^\infty (xy)^{1/2} J_\mu(xy) \phi(y) dy, x \in R_+ = (0, \infty) \quad (1.1)$$

where $J_\mu$ represents the Bessel function of the first kind and order $\mu$. We shall assume throughout this paper that $\mu \geq -1/2$ and $\phi \in L^1(R_+), R_+ = (0, \infty)$. The inversion formula for (1.1) [2, p.239] is given by

$$\phi(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy)(h_\mu \phi)(x) dx, \ y \in R_+ \quad (1.2)$$
The above transformation has been extended to distributions by Zemanian [6]. For every \( \mu \in \mathbb{R}_+ \), he introduced the space \( H_\mu(\mathbb{R}_+) \) consisting of all infinitely differentiable function \( \phi \) defined on \( \mathbb{R}_+ = (0, \infty) \), such that \( \forall \ m, k \in \mathbb{N}_0 \) the quantities

\[
\gamma_{m,k}^\mu = \sup_{x \in \mathbb{R}_+} \left| x^m (x^{-1}d/dx)^k x^{-\mu-1/2} \phi(x) \right| < \infty. \tag{1.3}
\]

Using theory of \( H_\mu \) space of Zemanian [6], Pathak and Dixit [3] investigated the Bessel wavelet transform \( B_\psi \) defined as follows:

\[
(B_\psi \phi)(b, a) := \int_0^\infty (bu)^{1/2} J_\mu (bu) \widehat{\phi}(u) \overline{\psi}(au) du, \tag{1.4}
\]

where \( \widehat{\phi}(u) = (h_\mu \phi)(u) \).

Let us assume that for any real number \( \rho \), \( \widehat{\psi} \) satisfies

\[
(1 + x)^l \left| (x^{-1}d/dx)^\alpha (y^{-1}d/dy)^\beta \widehat{\psi}(xy) \right| \leq C_{\alpha,\beta,l} (1 + y)^{\rho - \beta}, \forall \alpha, \beta, l \in \mathbb{N}_0 \tag{1.5}
\]

where \( C_{\alpha,\beta,l} > 0 \) is a constant and \( \widehat{\psi} \) denotes the Hankel transform of the basic wavelet \( \psi \). The class of all such wavelet \( \widehat{\psi} \) is denoted by \( H^\rho \).

This permits us to define the Hankel transform with respect to \( x \) of \( \widehat{\psi}(ax) \)

\[
h_\mu \left[ (h_\mu(\psi)) \right](a\xi) = \int_0^\infty (x\xi)^{1/2} J_\mu (x\xi) (h_\mu \psi)(ax) dx. \tag{1.6}
\]

We shall use some notation and terminology of [2,4,5,7]. The differential operator \( S_{\mu,x} \) is defined by

\[
S_{\mu,x} = \frac{d^2}{dx^2} + \frac{(1 - 4\mu^2)}{4x^2}. \tag{1.7}
\]

From [2,4] we know that for any \( \phi \in H_\mu \),

\[
h_\mu (S_\mu \phi) = -y^2 h_\mu \phi, \tag{1.8}
\]

and from [2] also we have

\[
S_{\mu,x}^r \phi(x) = \sum_{j=0}^r b_j x^{2j+\mu+1/2} (x^{-1}d/dx)^j (x^{-\mu-1/2} \phi(x)), \tag{1.10}
\]

where \( b_j \) are constants depending only on \( \mu \).
Definition 1.1 A tempered distribution $\varphi \in H'_\mu(R_+)$ is said to belong to the Sobolev space $G^s_{\mu,p}$, $s, \mu \in \mathbb{R}, 1 \leq p < \infty$, if its Hankel transform $h_\mu \varphi$ corresponds to a locally integrable function over $R_+ = (0, \infty)$ such that

$$\| \varphi \|_{G^s_{\mu,p}(R_+)} = \left( \int_0^\infty | (1 + \xi^2)^s (h_\mu \varphi)(\xi) |^p d\xi \right)^{1/p}. \quad (1.11)$$

2 The Bessel Wavelet Transform

Lee [1] has defined the space $B_{\mu,b}$ and $\Upsilon_{\mu,b}^{2q}$ as follows:

Definition 2.1 $\varphi \in B_{\mu,b}$ if $\varphi$ is smooth function $\varphi(x) = 0$ for $x > b$ and

$$\alpha_{\mu,b,k}^\mu(\varphi) = \sup_{x \in \mathbb{R}_+} \left| (x^{-1}d/dx)^k x^{-\mu-1/2} \varphi(x) \right| < \infty, k = 0, 1, 2, ... \quad (2.1)$$

where $b > 0$ is a constant and $\mu$ is real number.

Definition 2.2 For each $q=1,2,3,...$, $\Phi \in \Upsilon_{\mu,b}^{2q}$, if $z^{-\mu-1/2} \Phi$ is an even entire function and

$$\beta_{b,k}^{\mu,2q}(\Phi) = \sup_{z=x+iy} \left| e^{-by^{2q}} z^{2k-\mu-1/2} \Phi(z) \right| < \infty, k = 0, 1, 2, ... \quad (2.2)$$

where $\Phi = (h_\mu \varphi), b > 0$ is constant and $\mu$ is real number.

The topology of the spaces $B_{\mu,b}$ and $\Upsilon_{\mu,b}^{2q}$ are generated by the seminorms $\{\alpha_{\mu,b,k}^\mu\}_{k=0}^\infty$ and $\{\beta_{b,k}^{\mu,2q}\}_{k=0}^\infty$ respectively. It follows from Definition 2.1 and 2.2 that $B_{\mu,b}$ and $\Upsilon_{\mu,b}^{2q}$ are Frechet spaces. We define

$$\sigma_{\mu,b,k}^\mu(\varphi) = \max_{0 \leq \nu \leq k} \alpha_{\mu,b,\nu}^\mu(\varphi); \quad \rho_{b,k}^{\mu,2q}(\Phi) = \max_{0 \leq \nu \leq k} \beta_{b,\nu}^{\mu,2q}(\Phi). \quad (2.3)$$

Then $\sigma_{\mu,b,k}^\mu$ and $\rho_{b,k}^{\mu,2q}$ define a norm on the space $B_{\mu,b}$ and $\Upsilon_{\mu,b}^{2q}$ respectively.

Following technique of Zemanian [7, p.141], we can write

$$x^\nu (x^{-1}d/dx)^n x^{-\mu-1/2} h_\mu \varphi(x) = \int_0^\infty y^{2\mu+2n+\nu+1} (y^{-1}d/dy)^\nu (y^{-\mu-1/2} \varphi(y)) \times (xy)^{-(\mu+n)} J_{\mu+\nu+n}(xy) dy. \quad (2.4)$$

Theorem 2.3 The Bessel wavelet transform $B_{\psi}$ is a continuous linear mapping of $B_{\mu,b}$ into $\Upsilon_{\mu,b}^{2q}$. 
Proof: Let \( z = x + iy \) and \( \mu \geq -1/2 \), the Bessel wavelet transform \( B_\psi \) has the representation

\[
(B_\psi \phi)(z, a) = \int_0^b (zu)^{1/2} J_\mu(zu)(h_\mu \phi)(u)(h_\mu \psi)(au) du, \mu \geq -1/2
\]

with \( b > 0 \) and \( (h_\mu \phi)(u)(h_\mu \psi)(au) \in L^2(0, b) \) if and only if \( (B_\psi \phi)(z, a) \in L^2(0, b) \), \( z^{-\mu - 1/2}(B_\psi \phi)(z, a) \) is an even entire function of \( z \) and there exists a constant \( C \) such that

\[
| (B_\psi \phi)(z, a) | \leq C \exp (b |y|), \quad \forall z.
\]

Let \( \phi \in B_{\mu, b} \), then

\[
(B_\psi \phi)(z, a) = \int_0^b (zu)^{1/2} J_\mu(zu)(h_\mu \phi)(u)(h_\mu \psi)(au) du, \mu \geq -1/2
\]

\[
= h_\mu \left[ (h_\mu \phi)(u)(h_\mu \psi)(au) \right] (z).
\]

Applying the technique of the Zemanian for fixed \( a \), from (2.4),

\[
z^{2k-\mu - 1/2}(B_\psi \phi)(z, a) = \int_0^b u^{2k+2\mu + 1} \left[ (u^{-1} D)^{2k} u^{-\mu - 1/2} \left( h_\mu \psi \right)(au)(h_\mu \phi)(u) \right]
\]

\[
\times \left[ (zu)^{-\mu} J_{\mu+2k}(zu) \right] du.
\]

So that

\[
| e^{-by^2} z^{2k-\mu - 1/2}(B_\psi \phi)(z, a) |
\]

\[
\leq \int_0^b \left| u^{2k+2\mu + 1} \left[ (u^{-1} D)^{2k} u^{-\mu - 1/2} \left( h_\mu \psi \right)(au)(h_\mu \phi)(u) \right] \right|
\]

\[
\times \left| \left[ (zu)^{-\mu} J_{\mu+2k}(zu) \right] e^{-by^2} \right| du
\]

\[
\leq \int_0^b \sum_{s=0}^{2k} \binom{2k}{s} (u^{-1} D)^s \left( h_\mu \psi \right)(au)(u^{-1} D)^{2k-s}
\]

\[
x u^{-\mu - 1/2}(h_\mu \phi)(u) \sup_{z,u} \left| \left[ (zu)^{-\mu} J_{\mu+2k}(zu) \right] e^{-by^2} \right| du
\]

\[
\leq \int_0^b \sum_{s=0}^{2k} \binom{2k}{s} \sup_u (1 + u)^{2k+2\mu + 1} (u^{-1} D)^s \bar{\psi}(au)
\]

\[
\times \sup_{u} \left| (u^{-1} D)^{2k-s} u^{-\mu - 1/2}(h_\mu \phi)(u) \sup_{z,u} \left| \left[ (zu)^{-\mu} J_{\mu+2k}(zu) \right] e^{-by^2} \right| \right| du.
\]
Applying inequalities (1.5) and (2.1), then from the above, we have
\[
\int_0^b \sum_{s=0}^{2k} \left( \frac{2k}{s} \right) (1 + u)^{2k+2\mu+1} C_s(1 + u)^\rho \alpha_{b,2k-s}(h_\mu \phi) \times \sup_{z,u} \left| (zu)^{-\mu} J_{\mu+2k}(zu) e^{-by^{2q}} \right| du 
\leq \sum_{s=0}^{2k} \left( \frac{2k}{s} \right) C_s \alpha_{b,2k-s}(h_\mu \phi) \sup_{z,u} \left| (zu)^{-\mu} J_{\mu+2k}(zu) e^{-by^{2q}} \right| 
\times \int_0^b (1 + u)^{2k+2\mu+1} du. \tag{2.5}
\]
We note that for all \( z \) such that \( |z| \leq 1 \)
\[
|z^{-\mu} J_{\mu+2k}(z)| \leq \frac{2^{-\mu} e}{\mu!}
\]
and for \( |z| > 1 \)
\[
|z^{-\mu} J_{\mu+2k}(z)| \leq C(\pi/2)^{-1} |z|^{-\mu-1/2} \exp(|Imz|).
\]
Since \( |y| \leq |y|^{2q} \), if \( |y| \geq 1 \) and \( |y| \geq |y|^{2q} \), if \( |y| < 1 \), then
\[
\left| (zu)^{-\mu} J_{\mu+2k}(zu) e^{-by^{2q}} \right| \leq \frac{2^{-\mu} e^{-by^{2q}}}{\mu!}, |y| \leq 1
\leq C(\pi/2)^{-1} |z|^{-\mu-1/2} e^{-b(|y|^{2q}-|y|)}, |y| > 1|.
\]
Thus \( \left| (zu)^{-\mu} J_{\mu+2k}(zu) e^{-by^{2q}} \right| \leq L_1 \) for \( \mu + 1/2 \geq 0 \).

Therefore
\[
\left| e^{-by^{2q}} z^{2k-\mu-1/2} (B_\psi \phi)(z,a) \right| \leq L_1 \sum_{s=0}^{2k} \left( \frac{2k}{s} \right) C_s \alpha_{b,2k-s}(h_\mu \phi).
\]
This completes the proof of the theorem.

3 The Sobolev Type Space

We have already defined the Sobolev space \( G_{\mu}^{s,p}(\mathbb{R}+) \) by (1.11). In the following, we shall make use of the following norm on \( G_{\mu}^{s,p}(\mathbb{R}+ \times \mathbb{R}+) \) in the proof of the boundedness result

\[
\| \phi \|_{G_{\mu}^{s,p}(\mathbb{R}+ \times \mathbb{R}+)} = \left( \int_0^\infty \int_0^\infty \left| (1 + \xi^2)^s (1 + \eta^2)^s (h_\mu \phi)(\xi, \eta) \right|^p d\xi d\eta \right)^{1/p},
\]

\( \phi \in H_{\mu}'(\mathbb{R}+ \times \mathbb{R}+) \).
Lemma 3.1 Let us assume that for any positive real number $\rho$, $\hat{\psi}(x)$ satisfies

$$\left| \left( x^{-1} \frac{d}{dx} \right)^l x^{-\mu - 1/2} \hat{\psi}(x) \right| \leq C_{l,\rho}(1 + x)^{\rho - l}, \forall \ l \in \mathbb{N}_0 \tag{3.1}$$

then there exists a positive constant $C'$ such that

$$\left| h_\mu \left[ \overline{(h_\mu \psi)} \right] (a\xi) \right| \leq C'(1 + a)^{\rho + 2l + \mu + 1/2}(1 + \xi^2)^{-l}. \tag{3.2}$$

**Proof:** From the definition (1.6), we know that

$$h_\mu \left[ \overline{(h_\mu \psi)} \right] (a\xi) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(h_\mu \psi)(ax) \, dx.$$  

So that from [4],

$$(1 + \xi^2)^l h_\mu \left[ \overline{(h_\mu \psi)} \right] (a\xi) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(1 - \Delta_{\mu,x})^l(h_\mu \psi)(ax) \, dx, \tag{3.3}$$

$\forall \ \xi, a \in \mathbb{R}_+, l \in \mathbb{N}_0$ and $\Delta_{\mu,x}$ defined as (1.7).

Now,

$$(1 - \Delta_{\mu,x})^l(h_\mu \psi)(ax) = \sum_{r=0}^l \binom{l}{r} (-1)^{r} \Delta_{\mu,x}^r(h_\mu \psi)(ax)$$

$$= \sum_{r=0}^l \binom{l}{r} (-1)^{r} \sum_{j=0}^r b_j x^{2j+\mu+1/2} \left( x^{-1} \frac{d}{dx} \right)^{r+j} \left( x^{-\mu - 1/2} (h_\mu \psi)(ax) \right). \tag{3.4}$$

Hence by (3.3) and (3.4), and inequality (3.1), we have

$$\left| h_\mu \left[ \overline{(h_\mu \psi)} \right] (a\xi) \right| \leq Q_\mu(1 + a)^{\rho + 2l + \mu + 1/2}(1 + \xi^2)^{-l} \sum_{r=0}^l \sum_{j=0}^r b_j C_{r+j,\rho} \binom{l}{r}$$

$$\times \int_0^\infty \left( 1 + x \right)^{2j+2\mu+\rho + 1-2l} \, dx.$$  

Choosing $l - j > \mu + \rho/2 + 1$, we conclude that

$$\left| h_\mu \left[ \overline{(h_\mu \psi)} \right] (a\xi) \right| \leq C'(1 + a)^{\rho + 2l + \mu + 1/2}(1 + \xi^2)^{-l}.$$
**Definition 3.2** Let \((h, \psi)(a, \xi)\) be a wavelet in \(H^\rho\) defined by (1.5). Then the Bessel wavelet transform \(B_\psi\) defined by
\[
(B_\psi \phi)(y, x) = \int_0^\infty \chi_{\eta}^{\frac{1}{2}} J_\mu(\eta \chi) \overline{(h, \psi)(\eta \chi)} (h, \phi)(\eta) \, d\eta
\]
exists for \(\phi \in H_\mu(R_+)\).

**Theorem 3.3** For any wavelet \(h, \psi \in H^\rho\), the Bessel wavelet transform \((B_\psi \phi)(y, x)\) admits the representation
\[
(B_\psi \phi)(y, x) = \int_0^\infty \int_0^\infty (x^2)^{\frac{1}{2}} J_\mu(x^2)(\eta)^{\frac{1}{2}} J_\mu(\eta)(h, \psi)(\eta)(h, \phi)(\eta) \, d\eta \, d\xi,
\]
where \(\phi \in H_\mu(R_+)\).

**Proof:** From definition (3.5)
\[
(B_\psi \phi)(y, x) = \int_0^\infty \chi_{\eta}^{\frac{1}{2}} J_\mu(\eta \chi) \overline{(h, \psi)(\eta \chi)} (h, \phi)(\eta) \, d\eta
\]
\[
= \int_0^\infty \chi_{\eta}^{\frac{1}{2}} J_\mu(\eta \chi) \left[ \int_0^\infty (x^2)^{\frac{1}{2}} J_\mu(x^2)(\eta)^{\frac{1}{2}} J_\mu(\eta) \, d\xi \right] (h, \phi)(\eta) \, d\eta
\]
\[
= \int_0^\infty \int_0^\infty (x^2)^{\frac{1}{2}} J_\mu(x^2)(\eta)^{\frac{1}{2}} J_\mu(\eta)(h, \psi)(\eta)(h, \phi)(\eta) \, d\eta \, d\xi.
\]
The last integral exists because \((h, \phi)(\eta) \in H_\mu(R_+)\) and \(\overline{(h, \psi)(\eta)}\) satisfies the inequality (3.2).

**Corollary 3.4** For any basic wavelet \(h, \psi \in H^\rho\), the Bessel wavelet transform \(h, [B_\psi \phi](\xi, \eta)\) admits the representation
\[
h, [B_\psi \phi](\xi, \eta) = \overline{(h, \psi)(\xi \eta)} (h, \phi)(\eta),
\]
where \(\phi \in H_\mu(R_+)\).

**Proof:** The right hand side of (3.7)
\[
(h, \psi)(\xi \eta)(h, \phi)(\eta) = h, \left[ (h, \psi)(\xi \eta) \right] h, [(h, \phi)(\eta)]
\]
\[
= \left[ \int_0^\infty (x^2)^{\frac{1}{2}} J_\mu(x^2)(\eta) \, dx \right]
\]
\[
\times \left[ \int_0^\infty (y^2)^{\frac{1}{2}} J_\mu(y^2)(\eta) \, dy \right]
\]
\[
= \int_0^\infty \int_0^\infty (x^2)^{\frac{1}{2}} J_\mu(x^2)(\eta)^{\frac{1}{2}} J_\mu(\eta)(\overline{(h, \psi)(\eta)}) \, d\eta \, d\xi
\]
\[
\times (h, \phi)(\eta) \, d\eta,
\]
\[
= h, [B_\psi \phi](\xi, \eta).
\]
Theorem 3.5 Let \( h_\mu \psi \in H^\rho \) and \((B_\psi \phi)(x,y)\) be the Bessel wavelet transform then there exists \( D > 0 \) such that for \( \rho \in \mathbb{R}_+ \) and \( l \in \mathbb{N}_0 \),

\[
\| (B_\psi \phi) \|_{G^s_{\mu,\rho}^p(\mathbb{R}_+ \times \mathbb{R}_+)} \leq D \| h_\mu \phi \|_{G^{s+(\rho+2l+\mu+1/2)/2}_\mu(\mathbb{R}_+)} , \forall \phi \in H_\mu(\mathbb{R}_+) .
\]

Proof: Using Lemma 3.1, we have

\[
\| (B_\psi \phi) \|_{G^s_{\mu,\rho}^p(\mathbb{R}_+ \times \mathbb{R}_+)} = \left( \int_0^\infty \int_0^\infty (1+\xi^2)^s (1+\eta^2)^s |(h_\mu[B_\psi \phi](\xi,\eta)|^p \, d\xi \, d\eta \right)^{1/p}
\]

\[
= \left( \int_0^\infty \int_0^\infty (1+\xi^2)^s (1+\eta^2)^s |(h_\mu[B_\psi \phi](\xi,\eta)|^p \, d\xi \, d\eta \right)^{1/p}
\]

\[
\leq \left( \int_0^\infty \int_0^\infty (1+\xi^2)^s (1+\eta^2)^s \left| h_\mu(h_\mu \psi)(\xi,\eta) \right|^p \, d\xi \, d\eta \right)^{1/p}
\]

\[
\leq \left( \int_0^\infty \int_0^\infty (1+\xi^2)^s (1+\eta^2)^s \, d\xi \, d\eta \right)^{1/p}.
\]

We note that

\[
(1+\eta)^{\rho+2l+\mu+1/2} \leq 2^{\rho+2l+\mu+1/2} (1+\eta^2)^{(\rho+2l+\mu+1/2)/2} , \rho \geq 0
\]

and

\[
(1+\eta)^{\rho+2l+\mu+1/2} \leq (1+\eta^2)^{\rho+2l+\mu+1/2} , \rho < 0 .
\]

Therefore

\[
(1+\eta)^{\rho+2l+\mu+1/2} \leq \max \left( 1, 2^{(\rho+2l+\mu+1/2)/2} (1+\eta^2)^{(\rho+2l+\mu+1/2)/2} \right)
\]

Hence

\[
\| (B_\psi \phi) \|_{G^s_{\mu,\rho}^p(\mathbb{R}_+ \times \mathbb{R}_+)} \leq C' \left( \int_0^\infty \int_0^\infty (1+\xi^2)^s \max(1, 2^{(\rho+2l+\mu+1/2)/2} (1+\eta^2)^{(\rho+2l+\mu+1/2)/2}) \right.
\]

\[
\times (h_\mu \psi)(\eta)|^p \, d\xi \, d\eta \right)^{1/p}
\]

\[
\leq C' \left( \int_0^\infty (1+\eta^2)^{(\rho+2l+\mu+1/2)/2} (h_\mu \psi)(\eta)|^p \, d\eta \right)^{1/p}
\]

\[
\times \left( \int_0^\infty (1+\xi^2)^{(s-l)} \, d\xi \right)^{1/p},
\]

where \( C' \) is certain constant. The \( \xi \) integral is convergent as \( l \) can be chosen large enough so that

\[
\| (B_\psi \phi) \|_{G^s_{\mu,\rho}^p(\mathbb{R}_+ \times \mathbb{R}_+)} \leq D \| h_\mu \phi \|_{G^{s+(\rho+2l+\mu+1/2)/2}_\mu(\mathbb{R}_+)} .
\]
4 Product of two Bessel Wavelet Transforms

Let $B_{\psi_1}$ and $B_{\psi_2}$ be two Bessel Wavelet Transforms of $\phi \in H_\mu(R_+)$ defined as follows:

\[(B_{\psi_1}\phi)(b,a) := B_1(b,a) : = \int_0^\infty (bu)^{1/2}J_\mu(bu)\hat{\psi}_1(au)\hat{\phi}(u)du,\]  \hfill (4.1)

and

\[(B_{\psi_2}\phi)(d,c) := B_2(d,c) : = \int_0^\infty (du)^{1/2}J_\mu(du)\hat{\psi}_2(au)\hat{\phi}(u)du,\]  \hfill (4.2)

Then, their product $B_{\psi_1}oB_{\psi_2}$ (or $B_1oB_2$) is defined by

\[B(b,a,c) = (B_1oB_2)(b,a,c)\]
\[= \int_0^\infty (bu)^{1/2}J_\mu(bu)\hat{\psi}_1(au)h_\mu [B_{\psi_2}\phi](u,c)du\]  \hfill (4.3)

\[= \int_0^\infty (bu)^{1/2}J_\mu(bu)\hat{\psi}_2(au)\hat{\phi}(u)du\]  \hfill (4.4)

\[= \int_0^\infty (bu)^{1/2}J_\mu(bu)\chi(a,c,u)\hat{\phi}(u)du,\]

where $\hat{\phi}$ denotes the Hankel transformation of $\phi$ and $\chi(a,c,u) = \hat{\psi}_1(au)\hat{\psi}_2(au)$, provided the integral is convergent.

**Theorem 4.1** Let $\hat{\psi}_1(au) \in H^{\rho_1}$ and $\hat{\psi}_2(au) \in H^{\rho_2}$, then for certain constant $D > 0$ and $\rho_1, \rho_2 \in R_+$,

\[\| (B_{\psi_1}B_{\psi_2}\phi)(b,a,c) \|_{G^{0,p}((R_+)^2)} \leq D \| \hat{\phi} \|_{G^{\rho_1+\rho_2}((R_+)^2)} .\]

**Proof:** By definition (4.3), we have

\[(B_{\psi_1}B_{\psi_2}\phi)(b,a,c) = \int_0^\infty (bu)^{1/2}J_\mu(bu)\hat{\psi}_1(au)h_\mu [B_{\psi_2}\phi](u,c)du.\]

From (4.4), it follows that $(B_{\psi_1}B_{\psi_2}\phi)$ has Hankel transform equal to $\int_0^\infty \hat{\psi}_1(au)\hat{\psi}_2(au)\hat{\phi}(u)du$. Therefore
\[ \| (B_{\psi_1} B_{\psi_2} \phi) (b, a, c) \|_{G^{s,p}_{\mu}(\mathbb{R}_+ \times \mathbb{R}_+)} = \left( \int_0^\infty \int_0^\infty |(1 + u^2)^s (1 + a^2)^s \hat{\psi}_1(au) \hat{\psi}_2(au) \hat{\phi}(u) du |^p \, da \right)^{1/p}. \]

Since from (1.5)
\[ \left| \hat{\psi}_1(au) \right| \leq C_{\rho_1,l} (1 + a)^{-l} (1 + u)^{\rho_1} \]
\[ \leq C_{\rho_1,l} \max \left( 1, 2^{-l/2} \right) (1 + a^2)^{-l/2} \max \left( 1, 2^{\rho_1/2} \right) (1 + u^2)^{\rho_1/2} \]
and
\[ \left| \hat{\psi}_2(au) \right| \leq C_{\rho_2,l} (1 + a)^{-l} (1 + u)^{\rho_2} \]
\[ \leq C_{\rho_2,l} \max \left( 1, 2^{-l/2} \right) (1 + a^2)^{-l/2} \max \left( 1, 2^{\rho_2/2} \right) (1 + u^2)^{\rho_2/2}. \]

Therefore
\[ \| (B_{\psi_1} B_{\psi_2} \phi) \|_{G^{s,p}_{\mu}(\mathbb{R}_+ \times \mathbb{R}_+)} \leq \left( \int_0^\infty \int_0^\infty |(1 + u^2)^s (1 + a^2)^s C_{\rho_1,l} C_{\rho_2,l} (1 + u^2)^{(\rho_1 + \rho_2)/2} \times (1 + a^2)^{-l} \hat{\phi}(u) |^p \, da \right)^{1/p} \]
\[ \leq C_{\rho_1,\rho_2,l} \left( \int_0^\infty |(1 + a^2)^{s-l} |^p \, da \right)^{1/p} \left( \int_0^\infty \left| (1 + u^2)^{s+(\rho_1 + \rho_2)/2} \hat{\phi}(u) \right|^p \, du \right)^{1/p}, \]

where \( C_{\rho_1,\rho_2,l} \) is certain positive constant.

The a-integral can be made convergent by choosing \( l \) sufficiently large, so that
\[ \| (B_{\psi_1} B_{\psi_2} \phi) \|_{G^{s,p}_{\mu}(\mathbb{R}_+ \times \mathbb{R}_+)} \leq D \| \hat{\phi} \|_{G^{s+(\rho_1 + \rho_2)/2,p}_{\mu}(\mathbb{R}_+)}, \]

where \( D \) is positive constant.

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**References**


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