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The Bessel Wavelet Transform

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Abstract

In this paper continuity of the Bessel wavelet transform B_ψ of function ϕ in terms of a mother wavelet ψ is investigated on certain distribution spaces when the Hankel transform of ψ defined by $\widehat{\psi}(x,y) \in C^\infty(\mathbf{R}_+^\mathbf{2})$. A sobolev space boundedness result is obtained.

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1 Introduction

The Hankel transformation is usually defined by

$$\widehat{\phi}(x) = (h_\mu \phi)(x) = \int_0^\infty (xy)^{1/2} J_\mu(xy)\phi(y)dy, x \in \mathbf{R}_+ = (0,\infty) \tag{1.1}$$

whereJ_μ represents the Bessel function of the first kind and order μ. We shall assume throughout this paper that $\mu \geq -1/2$ and $\phi \in L^1(\mathbf{R}_+), \mathbf{R}_+ = (0,\infty)$. The inversion formula for (1.1) [2, p.239] is given by

$$\phi(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy)(h_\mu \phi)(x)dx,\ y \in \mathbf{R}_+ \tag{1.2}$$

The above transformation has been extended to distributions by Zemanian [6]. For every $\mu \in \mathbf{R}_+$, he introduced the space $H_\mu(\mathbf{R}_+)$ consisting of all infinitely differentiable function ϕ defined on $\mathbf{R}_+ = (0, \infty)$, such that $\forall\ m, k \in \mathbf{N}_0$ the quantities

$$\gamma^{\mu}_{m,k} = \sup_{x \in R_+} \left| x^m (x^{-1} d/dx)^k x^{-\mu-1/2} \phi(x) \right| < \infty. \tag{1.3}$$

Using theory of H_μ space of Zemanian [6], Pathak and Dixit [3] investigated the Bessel wavelet transform B_ψ defined as follows:

$$(B_\psi \phi)(b, a) := \int_0^\infty (bu)^{1/2} J_\mu(bu) \widehat{\phi}(u) \overline{\widehat{\psi}(au)} du, \tag{1.4}$$

where $\widehat{\phi}(u) = (h_\mu \phi)(u)$.

Let us assume that for any real number $\rho, \widehat{\psi}$ satisfies

$$(1+x)^l \left| (x^{-1} d/dx)^\alpha (y^{-1} d/dy)^\beta \widehat{\psi}(xy) \right| \le C_{\alpha,\beta,l} (1+y)^{\rho-\beta}, \forall \alpha, \beta, l \in \mathbf{N}_0 \tag{1.5}$$

where $C_{\alpha,\beta,l} > 0$ is a constant and $\widehat{\psi}$ denotes the Hankel transform of the basic wavelet ψ. The class of all such wavelet $\widehat{\psi}$ is denoted by H^ρ.

This permits us to define the Hankel transform with respect to x of $\widehat{\psi}(ax)$

$$h_\mu \left[\overline{(h_\mu(\psi))} \right] (a\xi) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi) \overline{(h_\mu \psi)}(ax) dx. \tag{1.6}$$

We shall use some notation and terminology of [2,4,5,7]. The differential operator $S_{\mu,x} (or \triangle_{\mu,x})$ is defined by

$$S_{\mu,x} = \frac{d^2}{dx^2} + \frac{(1-4\mu^2)}{4x^2}. \tag{1.7}$$

From [2,4] we know that for any $\phi \in H_\mu$,

$$h_\mu(S_\mu \phi) = -y^2 h_\mu \phi, \tag{1.8}$$

$$(x^{-1} d/dx)^k (\psi\phi) = \sum_{\nu=0}^{k} \binom{k}{\nu} (x^{-1} d/dx)^\nu \phi (x^{-1} d/dx)^{k-\nu} \psi \tag{1.9}$$

and from [2] also we have

$$S^r_{\mu,x} \phi(x) = \sum_{j=0}^{r} b_j x^{2j+\mu+1/2} (x^{-1} d/dx)^{r+j} (x^{-\mu-1/2} \phi(x)), \tag{1.10}$$

where b_j are constants depending only on μ.

Definition 1.1 *A tempered distribution* $\phi \in H'_\mu(\mathbf{R}_+)$ *is said to belong to the Sobolev space* $G^{s,p}_\mu, s, \mu \in \mathbf{R}, 1 \leq p < \infty$, *if its Hankel transform* $h_\mu\phi$ *corresponds to a locally integrable function over* $\mathbf{R}_+ = (0, \infty)$ *such that*

$$\| \phi \|_{G^{s,p}_\mu(\mathbf{R}_+)} = \left(\int_0^\infty \mid (1+\xi^2)^s (h_\mu\phi)(\xi) \mid^p d\xi \right)^{1/p}. \tag{1.11}$$

2 The Bessel Wavelet Transform

Lee [1] has defined the space $B_{\mu,b}$ and $\Upsilon^{2q}_{\mu,b}$ as follows:

Definition 2.1 $\phi \in B_{\mu,b}$ *if* ϕ *is smooth function* $\phi(x) = 0$ *for* $x > b$ *and*

$$\alpha^\mu_{b,k}(\phi) = \sup_{x \in \mathbf{R}_+} \left| (x^{-1}d/dx)^k x^{-\mu-1/2} \phi(x) \right| < \infty, k = 0, 1, 2, ... \tag{2.1}$$

where $b > 0$ *is a constant and* μ *is real number.*

Definition 2.2 *For each q=1,2,3,...,* $\Phi \in \Upsilon^{2q}_{\mu,b}$, *if* $z^{-\mu-1/2}\Phi$ *is an even entire function and*

$$\beta^{\mu,2q}_{b,k}(\Phi) = \sup_{z=x+iy} \left| e^{-by^{2q}} z^{2k-\mu-1/2} \Phi(z) \right| < \infty, k = 0, 1, 2, ... \tag{2.2}$$

where $\Phi = (h_\mu\phi), b > 0$ *is constant and* μ *is real number.*

The topology of the spaces $B_{\mu,b}$ and $\Upsilon^{2q}_{\mu,b}$ are generated by the seminorms $\left\{\alpha^\mu_{b,k}\right\}^\infty_{k=0}$ and $\left\{\beta^{\mu,2q}_{b,k}\right\}^\infty_{k=0}$ respectively. It follows from Definition 2.1 and 2.2 that $B_{\mu,b}$ and $\Upsilon^{2q}_{\mu,b}$ are Frechet spaces. We define

$$\sigma^\mu_{b,k}(\phi) = \max_{0 \leq \nu \leq k} \alpha^\mu_{b,\nu}(\phi); \ \rho^{\mu,2q}_{b,k}(\phi) = \max_{0 \leq \nu \leq k} \beta^{\mu,2q}_{b,\nu}(\phi). \tag{2.3}$$

Then $\sigma^\mu_{b,k}$ and $\rho^{\mu,2q}_{b,k}$ define a norm on the space $B_{\mu,b}$ and $\Upsilon^{2q}_{\mu,b}$ respectively. Following technique of Zemanian [7, p.141], we can write

$$\begin{aligned} x^\nu (x^{-1}d/dx)^n x^{-\mu-1/2} h_\mu\phi(x) &= \int_0^\infty y^{2\mu+2n+\nu+1} (y^{-1}d/dy)^\nu (y^{-\mu-1/2}\phi(y)) \\ &\times \ (xy)^{-(\mu+n)} J_{\mu+\nu+n}(xy) dy. \end{aligned} \tag{2.4}$$

Theorem 2.3 *The Bessel wavelet transform* B_ψ *is a continuous linear mapping of* $B_{\mu,b}$ *into* $\Upsilon^{2q}_{\mu,b}$.

Proof: Let $z = x + iy$ and $\mu \geq -1/2$, the Bessel wavelet transform B_ψ has the representation

$$(B_\psi\phi)(z,a) = \int_0^b (zu)^{1/2} J_\mu(zu)(h_\mu\phi)(u)\overline{(h_\mu\psi)(au)}du, \mu \geq -1/2$$

with $b > 0$ and $(h_\mu\phi)(u)\overline{(h_\mu\psi)(au)} \in L^2(0,b)$ if and only if $(B_\psi\phi)(z,a) \in L^2(0,\infty)$, $z^{-\mu-1/2}(B_\psi\phi)(z,a)$ is an even entire function of z and there exists a constant C such that

$$|(B_\psi\phi)(z,a)| \leq C \ \ exp(b\,|y|), \ \forall \ z.$$

Let $\phi \in B_{\mu,b}$, then

$$\begin{aligned}(B_\psi\phi)(z,a) &= \int_0^b (zu)^{1/2} J_\mu(zu)(h_\mu\phi)(u)\overline{(h_\mu\psi)(au)}du, \mu \geq -1/2 \\ &= h_\mu\left[(h_\mu\phi)(u)\overline{(h_\mu\psi)(au)}\right](z).\end{aligned}$$

Applying the technique of the Zemanian for fixed a, from (2.4),

$$\begin{aligned}z^{2k-\mu-1/2}(B_\psi\phi)(z,a) &= \int_0^b u^{2k+2\mu+1}\left[(u^{-1}D)^{2k}u^{-\mu-1/2} \ \ \overline{(h_\mu\psi)(au)}(h_\mu\phi)(u)\right] \\ &\times \ \left[(zu)^{-\mu}J_{\mu+2k}(zu)\right] du.\end{aligned}$$

So that

$$\begin{aligned}&\left|e^{-by^{2q}} z^{2k-\mu-1/2}(B_\psi\phi)(z,a)\right| \\ &\qquad\leq \int_0^b \left|u^{2k+2\mu+1}\left[(u^{-1}D)^{2k}u^{-\mu-1/2} \ \ \overline{(h_\mu\psi)(au)}(h_\mu\phi)(u)\right]\right| \\ &\qquad\times \left|\left[(zu)^{-\mu}J_{\mu+2k}(zu)\right] e^{-by^{2q}}\right| du\end{aligned}$$

$$\begin{aligned}&\leq \int_0^b \left|u^{2k+2\mu+1}\sum_{s=0}^{2k}\binom{2k}{s}(u^{-1}D)^s \ \ \overline{(h_\mu\psi)(au)}(u^{-1}D)^{2k-s}\right. \\ &\times \ \left. u^{-\mu-1/2}(h_\mu\phi)(u)\right| \sup_{z,u}\left|\left[(zu)^{-\mu}J_{\mu+2k}(zu)\right] e^{-by^{2q}}\right| du \\ &\leq \int_0^b \sum_{s=0}^{2k}\binom{2k}{s}\sup_u\left|(1+u)^{2k+2\mu+1}(u^{-1}D)^s \ \overline{\widehat{\psi}(au)}\right| \\ &\times \ \sup_u\left|(u^{-1}D)^{2k-s}u^{-\mu-1/2}(h_\mu\phi)(u)\right| \sup_{z,u}\left|\left[(zu)^{-\mu}J_{\mu+2k}(zu)\right] e^{-by^{2q}}\right| du.\end{aligned}$$

Applying inequalities (1.5) and (2.1), then from the above, we have

$$\int_0^b \sum_{s=0}^{2k} \binom{2k}{s} (1+u)^{2k+2\mu+1} C_s (1+u)^{\rho}\ \alpha^{\mu}_{b,2k-s}(h_\mu\phi) \times \sup_{z,u} \left| \left[(zu)^{-\mu} J_{\mu+2k}(zu)\right] e^{-by^{2q}} \right| du$$

$$\leq \sum_{s=0}^{2k} \binom{2k}{s} C_s\ \alpha^{\mu}_{b,2k-s}(h_\mu\phi) \sup_{z,u} \left|(zu)^{-\mu} J_{\mu+2k}(zu) e^{-by^{2q}}\right| \times \int_0^b (1+u)^{2k+2\mu+\rho+1} du. \tag{2.5}$$

We note that for all z such that$|z| \leq 1$

$$\left|z^{-\mu} J_{\mu+2k}(z)\right| \leq \frac{2^{-\mu}\ e}{\mu!}$$

and for $|z| > 1$

$$\left|z^{-\mu} J_{\mu+2k}(z)\right| \leq C(\pi/2)^{-1} \left|z\right|^{-\mu-1/2} exp(|Imz|).$$

Since $|y| \leq |y|^{2q}$, if $|y| \geq 1$ and $|y| \geq |y|^{2q}$, if $|y| < 1$, then

$\left|(zu)^{-\mu} J_{\mu+2k}(zu) e^{-by^{2q}}\right| \leq \frac{2^{-\mu}\ e}{\mu!} e^{-by^{2q}}, |y| \leq 1$

$\leq C(\pi/2)^{-1} \left|z\right|^{-\mu-1/2} e^{-b\left(|y|^{2q}-|y|\right)}, |y > 1|.$

Thus $\left|(zu)^{-\mu} J_{\mu+2k}(zu) e^{-by^{2q}}\right| \leq L_1$ for $\mu + 1/2 \geq 0$.

Therefore

$$\left|e^{-by^{2q}} z^{2k-\mu-1/2} (B_\psi\phi)(z,a)\right| \leq L_1 \sum_{s=0}^{2k} \binom{2k}{s} C_s\ \ \alpha^{\mu}_{b,2k-s}(h_\mu\phi).$$

This completes the proof of the theorem.

3 The Sobolev Type Space

We have already defined the Sobolev space $G^{s,p}_{\mu}(\mathbf{R}_+)$ by (1.11). In the following, we shall make use of the following norm on $G^{s,p}_{\mu}(\mathbf{R}_+ \times \mathbf{R}_+)$ in the proof of the boundedness result

$$\parallel \phi \parallel_{G^{s,p}_{\mu}(\mathbf{R}_+\times\mathbf{R}_+)} = \left(\int_0^\infty \int_0^\infty \mid (1+\xi^2)^s (1+\eta^2)^s\ \overline{(h_\mu\phi)(\xi,\eta)} \mid^p d\xi d\eta\right)^{1/p}, \quad \phi \in H^{'}_{\mu}(\mathbf{R}_+ \times \mathbf{R}_+).$$

Lemma 3.1 *Let us assume that for any positive real number* ρ, $\widehat{\psi}(x)$ *satisfies*

$$\left|\left(x^{-1}\frac{d}{dx}\right)^{l} x^{-\mu-1/2}\widehat{\psi}(x)\right| \leq C_{l,\rho}(1+x)^{\rho-l}, \forall\ l \in \mathbf{N}_0 \tag{3.1}$$

then there exists a positive constant $C^{'}$ *such that*

$$\left|h_\mu\left[\overline{(h_\mu\psi)}\right](a\xi)\right| \leq C^{'}(1+a)^{\rho+2l+\mu+1/2}(1+\xi^2)^{-l}. \tag{3.2}$$

Proof: From the definition (1.6), we know that

$$h_\mu\left[\overline{(h_\mu\psi)}\right](a\xi) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(h_\mu\psi)(ax)dx.$$

So that from [4],

$$(1+\xi^2)^l\ h_\mu\left[\overline{(h_\mu\psi)}\right](a\xi) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(1-\Delta_{\mu,x})^l(h_\mu\psi)(ax)dx, \tag{3.3}$$

$\forall\ \xi, a \in \mathbf{R}_+, l \in \mathbf{N}_0$ and $\Delta_{\mu,x}$ defined as (1.7).
Now,

$$(1-\Delta_{\mu,x})^l(h_\mu\psi)(ax) = \sum_{r=0}^{l}\binom{l}{r}(-1)^r\Delta_{\mu,x}^r(h_\mu\psi)(ax)$$

$$= \sum_{r=0}^{l}\binom{l}{r}(-1)^r\sum_{j=0}^{r} b_j x^{2j+\mu+1/2}\left(x^{-1}\frac{d}{dx}\right)^{r+j}\left(x^{-\mu-1/2}(h_\mu\psi)(ax)\right). \tag{3.4}$$

Hence by (3.3) and (3.4), and inequality (3.1), we have

$$\begin{aligned}\left|h_\mu\left[\overline{(h_\mu\psi)}\right](a\xi)\right| &\leq Q_\mu(1+a)^{\rho+2l+\mu+1/2}(1+\xi^2)^{-l}\ \sum_{r=0}^{l}\sum_{j=0}^{r} b_j\ C_{r+j,\rho}\binom{l}{r} \\ &\times \int_0^\infty (1+x)^{2j+2\mu+\rho+1-2l}dx.\end{aligned}$$

Choosing $l-j > \mu+\rho/2+1$, we conclude that

$$\left|h_\mu\left[\overline{(h_\mu\psi)}\right](a\xi)\right| \leq C^{'}(1+a)^{\rho+2l+\mu+1/2}(1+\xi^2)^{-l}.$$

Definition 3.2 *Let $(h_\mu\psi)(a\xi)$ be a wavelet in H^ρ defined by (1.5), Then the Bessel wavelet transform B_ψ defined by*

$$(B_\psi\phi)(y,x) = \int_0^\infty (y\eta)^{1/2} J_\mu(y\eta)\overline{(h_\mu\psi)(x\eta)}(h_\mu\phi)(\eta)d\eta \tag{3.5}$$

exists for $\phi \in H_\mu(\mathbf{R}_+)$.

Theorem 3.3 *For any wavelet $h_\mu\psi \in H^\rho$, the Bessel wavelet transform $(B_\psi\phi)(y,x)$ admits the representation*

$$(B_\psi\phi)(y,x) = \int_0^\infty \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(y\eta)^{1/2} J_\mu(y\eta)\overline{(h_\mu\psi)(\xi\eta)}(h_\mu\phi)(\eta)d\xi d\eta, \tag{3.6}$$

where $\phi \in H_\mu(\mathbf{R}_+)$.

Proof: From definition (3.5)

$$\begin{aligned}
(B_\psi\phi)(y,x) &= \int_0^\infty (y\eta)^{1/2} J_\mu(y\eta)\overline{(h_\mu\psi)(x\eta)}(h_\mu\phi)(\eta)d\eta \\
&= \int_0^\infty (y\eta)^{1/2} J_\mu(y\eta)\left[\int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)\overline{(h_\mu\widehat{\psi})(\xi\eta)}d\xi\right](h_\mu\phi)(\eta)d\eta \\
&= \int_0^\infty \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(y\eta)^{1/2} J_\mu(y\eta)\overline{(h_\mu\widehat{\psi})(\xi\eta)}(h_\mu\phi)(\eta)d\xi d\eta.
\end{aligned}$$

The last integral exists because $(h_\mu\phi)(\eta) \in H_\mu(\mathbf{R}_+)$ and $\overline{(h_\mu\widehat{\psi})(\xi\eta)}$ satisfies the inequality (3.2).

Corollary 3.4 *For any basic wavelet $h_\mu\psi \in H^\rho$, the Bessel wavelet transform $h_\mu[B_\psi\phi](\xi,\eta)$ admits the representation*

$$h_\mu[B_\psi\phi](\xi,\eta) = \overline{(h_\mu\widehat{\psi})(\xi\eta)}(h_\mu\widehat{\phi})(\eta), \tag{3.7}$$

where $\phi \in H_\mu(\mathbf{R}_+)$.

Proof: The right hand side of (3.7)

$$\begin{aligned}
\overline{(h_\mu\widehat{\psi})(\xi\eta)}(h_\mu\widehat{\phi})(\eta) &= h_\mu\left[\overline{(h_\mu\psi)(\xi\eta)}\right] h_\mu\left[(h_\mu\phi)(\eta)\right] \\
&= \left[\int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)\overline{(h_\mu\psi)(x\eta)}dx\right] \\
&\times \left[\int_0^\infty (y\eta)^{1/2} J_\mu(y\eta)(h_\mu\phi)(y)dy\right] \\
&= \int_0^\infty \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(y\eta)^{1/2} J_\mu(y\eta)\overline{(h_\mu\psi)(x\eta)} \\
&\times (h_\mu\phi)(y)dxdy \\
&= h_\mu[B_\psi\phi](\xi,\eta).
\end{aligned}$$

Theorem 3.5 *Let $h_\mu\psi \in H^\rho$ and $(B_\psi\phi)(x,y)$ be the Bessel wavelet transform then there exists $D > 0$ such that for $\rho \in \mathbf{R}_+$ and $l \in \mathbf{N}_0$,*

$$\| (B_\psi\phi) \|_{G_\mu^{s,p}(\mathbf{R}_+\times\mathbf{R}_+)} \leq D \| h_\mu\phi \|_{G_\mu^{s+(\rho+2l+\mu+1/2)/2,p}} (\mathbf{R}_+), \forall\phi \in H_\mu(\mathbf{R}_+).$$

Proof: Using Lemma 3.1, we have

$$\begin{aligned}
&\| (B_\psi\phi) \|_{G_\mu^{s,p}(\mathbf{R}_+\times\mathbf{R}_+)} \\
=& \left(\int_0^\infty\int_0^\infty \left|(1+\xi^2)^s(1+\eta^2)^s\,(h_\mu[B_\psi\phi](\xi,\eta))\right|^p d\xi d\eta\right)^{1/p} \\
=& \left(\int_0^\infty\int_0^\infty \left|(1+\xi^2)^s(1+\eta^2)^s\,\overline{(h_\mu\widehat{\psi})\,(\xi\eta)}(h_\mu\widehat{\phi})(\eta)\right|^p d\xi d\eta\right)^{1/p} \\
\leq& \left(\int_0^\infty\int_0^\infty \left|(1+\xi^2)^s(1+\eta^2)^s\right|^p \left|h_\mu\overline{(h_\mu\psi)\,(\xi\eta)}\right|^p |(h_\mu\widehat{\phi})(\eta)|^p d\xi d\eta\right)^{1/p} \\
\leq& \left(\int_0^\infty\int_0^\infty \right|(1+\xi^2)^s(1+\eta^2)^s\; C'(1+\eta)^{\rho+2l+\mu+1/2} \\
\times& \left.(1+\xi^2)^{-l}(h_\mu\widehat{\phi})(\eta)\right|^p d\xi d\eta\Big)^{1/p}.
\end{aligned}$$

We note that

$$(1+\eta)^{\rho+2l+\mu+1/2} \leq 2^{(\rho+2l+\mu+1/2)/2}\left(1+\eta^2\right)^{(\rho+2l+\mu+1/2)/2}, \rho \geq 0$$

and

$$(1+\eta)^{\rho+2l+\mu+1/2} \leq \left(1+\eta^2\right)^{(\rho+2l+\mu+1/2)/2}, \rho < 0.$$

Therefore

$$(1+\eta)^{\rho+2l+\mu+1/2} \leq max\left(1, 2^{(\rho+2l+\mu+1/2)/2}\right)\left(1+\eta^2\right)^{(\rho+2l+\mu+1/2)/2}.$$

Hence $\| (B_\psi\phi) \|_{G_\mu^{s,p}(\mathbf{R}_+\times\mathbf{R}_+)}$

$$\begin{aligned}
\leq& \; C'\left(\int_0^\infty\int_0^\infty \right|(1+\xi^2)^{s-l}max(1, 2^{(\rho+2l+\mu+1/2)/2})(1+\eta^2)^{s+(\rho+2l+\mu+1/2)/2} \\
\times& \left.(h_\mu\widehat{\phi})(\eta)\right|^p d\xi d\eta\Big)^{1/p} \\
\leq& \; C''\left(\int_0^\infty \left|(1+\eta^2)^{s+(\rho+2l+\mu+1/2)/2}\,(h_\mu\widehat{\phi})(\eta)\right|^p d\eta\right)^{1/p} \\
\times& \left(\int_0^\infty \left|(1+\xi^2)^{s-l}\right|^p d\xi\right)^{1/p},
\end{aligned}$$

where C'' is certain constant. The ξ integral is convergent as l can be chosen large enough so that

$$\| (B_\psi\phi) \|_{G_\mu^{s,p}(\mathbf{R}_+\times\mathbf{R}_+)} \leq D \| h_\mu\phi \|_{G_\mu^{s+(\rho+2l+\mu+1/2)/2,p}(\mathbf{R}_+)}.$$

4 Product of two Bessel Wavelet Transforms

Let B_{ψ_1} and B_{ψ_2} be two Bessel Wavelet Transforms of $\phi \in H_\mu(\mathbf{R}_+)$ defined as follows:

$$(B_{\psi_1}\phi)(b,a) := B_1(b,a) := \int_0^\infty (bu)^{1/2} J_\mu(bu) \overline{\widehat{\psi_1}(au)} \widehat{\phi}(u) du, \tag{4.1}$$

and

$$(B_{\psi_2}\phi)(d,c) := B_2(d,c) := \int_0^\infty (du)^{1/2} J_\mu(du) \overline{\widehat{\psi_2}(cu)} \widehat{\phi}(u) du, \tag{4.2}$$

Then, their product $B_{\psi_1} o B_{\psi_2} (or B_1 o B_2)$ is defined by

$$\begin{aligned} B(b,a,c) &= (B_1 o B_2)(b,a,c) \\ &= \int_0^\infty (bu)^{1/2} J_\mu(bu) \overline{\widehat{\psi_1}(au)} h_\mu [B_{\psi_2}\phi](u,c) du \end{aligned} \tag{4.3}$$

$$= \int_0^\infty (bu)^{1/2} J_\mu(bu) \overline{\widehat{\psi_1}(au)}\ \overline{\widehat{\psi_2}(cu)} \widehat{\phi}(u) du \tag{4.4}$$

$$= \int_0^\infty (bu)^{1/2} J_\mu(bu) \chi(a,c,u) \widehat{\phi}(u) du,$$

where $\widehat{\phi}$ denotes the Hankel transformation of ϕ and $\chi(a,c,u) = \overline{\widehat{\psi_1}(au)}\ \overline{\widehat{\psi_2}(cu)}$, provided the integral is convergent.

Theorem 4.1 *Let $\overline{\widehat{\psi_1}(au)} \in H^{\rho_1}$ and $\overline{\widehat{\psi_2}(cu)} \in H^{\rho_2}$, then for certain constant $D > 0$ and $\rho_1, \rho_2 \in \mathbf{R}_+$,*

$$\| (B_{\psi_1} B_{\psi_2}\phi)(b,a,c) \|_{G_\mu^{s,p}(\mathbf{R}_+ \times \mathbf{R}_+)} \le D \| \widehat{\phi} \|_{G_\mu^{s+(\rho_1+\rho_2)/2,p}(\mathbf{R}_+)} .$$

Proof: By definition (4.3), we have

$$(B_{\psi_1} B_{\psi_2}\phi)(b,a,c) = \int_0^\infty (bu)^{1/2} J_\mu(bu) \overline{\widehat{\psi_1}(au)}\ h_\mu [B_{\psi_2}\phi](u,c) du.$$

From (4.4), it follows that $(B_{\psi_1} B_{\psi_2}\phi)$ has Hankel transform equal to $\left[\overline{\widehat{\psi_1}(au)}\ \overline{\widehat{\psi_2}(au)} \widehat{\phi}(u) du\right]$.
Therefore

$$\| \, (B_{\psi_1} B_{\psi_2} \phi) \, (b,a,c) \, \|_{G_\mu^{s,p}(\mathbf{R}_+ \times \mathbf{R}_+)}$$
$$= \left(\int_0^\infty \int_0^\infty | \, (1+u^2)^s (1+a^2)^s \overline{\widehat{\psi_1}(au)} \; \overline{\widehat{\psi_2}(au)} \widehat{\phi}(u) du \, |^p \, da du \right)^{1/p}.$$

Since from (1.5)

$$\begin{aligned} \left| \widehat{\psi_1}(au) \right| &\leq C_{\rho_1,l}(1+a)^{-l}(1+u)^{\rho_1} \\ &\leq C_{\rho_1,l} \max\left(1, 2^{-l/2}\right) \left(1+a^2\right)^{-l/2} \max\left(1, 2^{\rho_1/2}\right) \left(1+u^2\right)^{\rho_1/2} \end{aligned}$$

and

$$\begin{aligned} \left| \widehat{\psi_2}(au) \right| &\leq C_{\rho_2,l}(1+a)^{-l}(1+u)^{\rho_2} \\ &\leq C_{\rho_2,l} \max\left(1, 2^{-l/2}\right) \left(1+a^2\right)^{-l/2} \max\left(1, 2^{\rho_2/2}\right) \left(1+u^2\right)^{\rho_2/2}. \end{aligned}$$

Therefore

$$\begin{aligned} &\| \, (B_{\psi_1} B_{\psi_2} \phi) \, \|_{G_\mu^{s,p}(\mathbf{R}_+ \times \mathbf{R}_+)} \\ &\qquad \leq \left(\int_0^\infty \int_0^\infty \left| (1+u^2)^s (1+a^2)^s C_{\rho_1,l} C'_{\rho_2,l} (1+u^2)^{(\rho_1+\rho_2)/2} \right. \right. \\ &\qquad \times \left. \left. (1+a^2)^{-l} \, \widehat{\phi}(u) \right|^p da du \right)^{1/p} \end{aligned}$$

$$\leq C_{\rho_1,\rho_2,l} \left(\int_0^\infty \left| (1+a^2)^{s-l} \right|^p da \right)^{1/p} \left(\int_0^\infty \left| (1+u^2)^{s+(\rho_1+\rho_2)/2} \, \widehat{\phi}(u) \right|^p du \right)^{1/p},$$

where $C_{\rho_1,\rho_2,l}$ is certain positive constant.
The a-integral can be made convergent by choosing l sufficiently large, so that

$$\| \, (B_{\psi_1} B_{\psi_2} \phi) \, \|_{G_\mu^{s,p}(\mathbf{R}_+ \times \mathbf{R}_+)} \leq D \, \| \, \widehat{\phi} \, \|_{G_\mu^{s+(\rho_1+\rho_2)/2,p}(\mathbf{R}_+)},$$

where D is positive constant.

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