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Shift Operators on the Base a $(a > 0, \neq 1)$ and Pseudo-Polynomials of Fractional Order

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Abstract

The aim of the present paper is to introduce and use the generalized exponential shift operators, operators on the base a $(a > 0, \neq 1)$, to deal with the families of pseudo-Kampé de Fériet polynomials, which can be viewed as the natural complement for the theory of fractional derivatives and partial fractional differential equations of evolutive type. We show that these families allow the possibility of treating a large variety of exponential operators, operators on the base a $(a > 0, \neq 1)$, providing generalized fractional forms of shift operators.

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1 Introduction

In what follows, we consider analytic function $f(x)$ so that the corresponding Taylor expansion

$$f(x+\lambda) \; = \; \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} f^{(k)}(x),$$

converges to corresponding value of f in a suitable neighborhood of x. we have shown that the action of generalized exponential operators [10] and [11].
In 2003, Dattoli et al. [2] discussed the exponential operators, the operators on the natural base e. In the present paper we generalize the exponential operators [10] on the base a ($a > 0, \neq 1$), as follows

$$\widehat{A}_m = a^{\lambda\left(\frac{\partial}{\partial x}\right)^m} \tag{1.1}$$

In the case when $m = 1$, it reduces to the ordinary shift operator, while for $m = 2$ it can be identified with the operatorial version of the Gauss transform

$$a^{\lambda\left(\frac{\partial}{\partial x}\right)} f(x) = f(x + \lambda \ln(a)) \tag{1.2}$$

Making use of the following identity, we have

$$e^{b^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2 + 2b\xi} d\xi,$$

we find

$$a^{\lambda\left(\frac{\partial^2}{\partial x^2}\right)} f(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left[-\xi^2 + 2\sqrt{\pi\lambda\ln(a)}\xi\frac{\partial}{\partial x}\right] f(\xi) d\xi$$

$$= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2} f(x + 2\xi\sqrt{\lambda\ln(a)}) d\xi$$

or

$$a^{\lambda\left(\frac{\partial^2}{\partial x^2}\right)} f(x) = \frac{1}{2\sqrt{\pi\lambda\ln(a)}} \int_{-\infty}^{\infty} e^{\frac{(x-\xi)^2}{4\lambda\ln(a)}} f(\xi) d\xi. \tag{1.3}$$

after a suitable change of variables.
Both the eqs. (1.2) and (1.3) are solution of the partial differential equation:

$$\begin{cases} \dfrac{\partial}{\partial \lambda} F(x, \lambda\ln(a)) = \ln(a) \left(\dfrac{\partial}{\partial x}\right)^m F(x, \lambda\ln(a)), \\ F(x, 0) = f(x), \quad m = 1, 2. \end{cases} \tag{A}$$

In case when $m > 2$, the exponential operator $\widehat{A}_m = a^{\lambda\left(\frac{\partial}{\partial x}\right)^m}$ provides formal solution for the generalized heat equation. It does not seem possible to associate it to any transformation of the Gauss type. We must, however, emphasize that the Hermite-Kampé de Fériet polynomials [13] of the type

$$H_n^{(m)}(x, y\ln(a)) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{x^{n-2r}(y\ln(a))^r}{(n-2r)!r!} = g_n^m(x, y\ln(a))$$

or equivalently the Gould-Hopper polynomials [8, p. 76, eq. (1.9) (6)]:

$$g_n^m(x, y\ln(a)) = \sum_{r=0}^{[\frac{n}{2}]} \frac{n!}{r!(n-2r)!} x^{n-2r} (y\ln(a))^r = H_n^{(m)}(x, y\ln(a))$$

are a solution of

$$\begin{cases} \frac{\partial}{\partial\lambda} F(x, \lambda\ln(a)) = \ln(a) \left(\frac{\partial}{\partial x}\right)^m F(x, \lambda\ln(a)), \\ F(x, 0) = x^n. \end{cases} \qquad (A')$$

or in other words[10] and [11]

$$a^{\lambda\left(\frac{\partial}{\partial x}\right)^m} x^n = H_n^{(m)}(x, y\ln(a))$$

This last result is particularly important, since it allows the conclusion that if $f(x)$ is an analytic function defined by the series expansion

$$f(x) = \sum_n c_n x^n$$

then, by Taylor Theorem, we write

$$a^{\lambda\left(\frac{\partial}{\partial x}\right)^m} f(x) = \sum_n c_n H_n^{(m)}(x, y\ln(a))$$

The polynomials $H_n^{(m)}(x, y\ln(a))$ will be said to be the polynomials of index n and order m.
Particular case: The replacement of a with e into the equations of this section give raise to the eqs. given in the first section of Dattoli et al. [2].

2 Exponential Operators on the Base a $(a > 0, \neq 1)$

In 2003, extensive uses of exponential operators on the natural base e were used by Dattoli et al. [2]. Let us see the applications of the exponential shift operators on the base a $(a > 0, \neq 1)$ used in [10] and [11], which play an important role in problems concerning pure and applied Mathematics [3].

$$\widehat{A} = a^{\lambda q(x)\frac{d}{dx}} \qquad (2.1)$$

For $a = e$, the properties of the generalized shift operator are similar to that of discussed in ref. [14-15] and their importance for the solution of generalized difference equations are similar to that of stressed in ref [5]. The action of $\widehat{A}$ on a given function $f(x)$ has been shown to be provided by [10] and [11]

$$\widehat{A}f(x) = f[F^{-1}(\lambda \ln(a) + F(x))], \tag{2.2}$$

where

$$F(x) \ = \ \int^x \frac{d\xi}{q(\xi)} \ \text{ or } \ \frac{d}{dx}F(x) \ = \ \frac{1}{q(x)} \ \text{ or } \ q(x)\frac{d}{dx}F(x) \ = \ 1 \tag{2.3}$$

defines the associated characteristic function of the generalized shift operator and $F^{-1}(\cdot)$ is its inverse. The proof of the above identity can be easily given, by Taylor Theorem noting that

$$\widehat{A}F(x) \ = \ F(x) + \lambda \ln(a)$$

only if [10] and [11]

$$\left[q(x)\frac{d}{dx}, F(x)\right] \ = \ 1,$$

where $[\cdot, \cdot]$ denotes commutation brackets, that is:

$$q(x)\frac{d}{dx}F(x) \ = \ 1.$$

More generally, we can always write [10] and [11]

$$\widehat{A}f(x) = f[F^{-1}(\lambda \ln(a) + F(x))],$$

It is evident that for $q(x) = 1$, $\widehat{A}$ reduces to the ordinary shift operator, when we put $q(x) = x$ we find $F(x) = \ln(x)$ and $F^{-1}(x) = e^x$, we have

$$a^{\lambda x \frac{d}{dx}} f(x) \ = \ f(a^\lambda x) \tag{2.4}$$

It is evident that the operators [10] and [11]

$$\widehat{\mathcal{T}}_x \ = \ q(x)\frac{d}{dx} \tag{2.5}$$

can be viewed as an ordinary derivative, although $F(x)$ is a function, $[F(x)]^n$ behaves, under the action of $\widehat{\mathcal{T}}_x$, as an ordinary monomial, we obtain indeed

$$\widehat{\mathcal{T}}_x[F(x)]^n \ = \ n[F(x)]^{n-1}, \tag{2.6}$$

we can take advantage from this trivial property to discuss the rule associated with the use of operators like

$$\widehat{A}_m \ = \ a^{\lambda\left(\widehat{\mathcal{T}}_x\right)^m} \ = \ a^{\lambda\left(q(x)\frac{d}{dx}\right)^m}, \tag{2.7}$$

for m. integer or real.
According to the conclusion of the introductory section and to these last relations, we can introduce the polynomials

$$h_n^{(m)}(x, y\ln(a)) = H_n^{(m)}(F(x), y\ln(a)), \tag{2.8}$$

which satisfy the recurrences

$$\left[F(x) + my\ln(a)\left(\frac{\partial}{\partial x}\right)^{m-1}\right] H_n^{(m)}(F(x), y\ln(a))$$

$$= n! \sum_{r=0}^{[\frac{n+1}{m}]} \frac{(n+1-mr)[F(x)]^{n+1-mr}}{(n+1-mr)!} \frac{(y\ln(a))^r}{r!}$$
$$+m\left[n! \sum_{r=1}^{[\frac{n}{m}]} \frac{[F(x)]^{n+1-m(r+1)}}{(n+1-m(r+1))!} \frac{(r+1)(y\ln(a))^{r+1}}{(r+1)!}\right]$$
$$= n! \sum_{r=0}^{[\frac{n+1}{m}]} \frac{(n+1-mr)[F(x)]^{n+1-mr}}{(n+1-mr)!} \frac{(y\ln(a))^r}{r!} + m\left[n! \sum_{r=0}^{[\frac{n+1}{m}]} \frac{[F(x)]^{n+1-mr}}{(n+1-mr)!} \frac{r(y\ln(a))^r}{r!}\right]$$
$$= (n+1)n! \sum_{r=0}^{[\frac{n+1}{m}]} \frac{(y\ln(a))^r [F(x)]^{n+1-mr}}{r!(n+1-mr)!}$$

or

$$\left[F(x) + my\ln(a)\left(\frac{\partial}{\partial x}\right)^{m-1}\right] H_n^{(m)}(F(x), y\ln(a)) = H_{n+1}^{(m)}(F(x), y\ln(a)),$$

by using the eq. (2.8), we have

$$\left.\begin{array}{l} \left[F(x) + my\ln(a)\left(\widehat{\mathcal{T}}_x\right)^{m-1}\right] h_n^{(m)}(x, y\ln(a)) = h_{n+1}^{(m)}(x, y\ln(a)), \\ \text{and} \\ \widehat{\mathcal{T}}_x[h_n^{(m)}(x, y\ln(a))] = nh_{n-1}^{(m)}(x, y\ln(a)). \end{array}\right\} \tag{2.9}$$

Clearly $h_n^{(m)}(x, y\ln(a))$ are functions satisfying polynomial type identities and will therefore be called pseudo H.K.d.F..
It becomes also evident that identities of the following type

$$\left[F(x) + 2y\ln(a)\left(\widehat{\mathcal{T}}_x\right)\right]^n = \sum_{s=0}^{n} \binom{n}{s} H_{n-s}(F(x), y\ln(a)) \left(2y\ln(a)(\widehat{\mathcal{T}}_x)\right)^s. \tag{2.10}$$

We show that eq. (2.10) follows from the Weyl identity. Note that since $F(x)$ and $2y\ln(a)(\widehat{\mathcal{T}}_x)$ do not commute, therefore the use of the Newton binomial formula is not allowed. Multiplying the left-hand side of eq. (2.10) by $\frac{t^n}{n!}$ and summing over n, we find

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \left[F(x) + 2y\ln(a)\left(\widehat{\mathcal{T}}_x\right)\right]^n = e^{t\left(F(x)+2y\ln(a)\widehat{\mathcal{T}}_x\right)},$$

by using the Weyl identity, where $\widehat{P} = tF(x)$ and $\widehat{Q} = 2yt\ln(a)\widehat{\mathcal{T}}_x$

since
$$[[\widehat{P}, \widehat{Q}], \widehat{P}] = [[\widehat{P}, \widehat{Q}], \widehat{Q}] = 0$$

further noting that

$$[\widehat{P}, \widehat{Q}] = \widehat{P}\widehat{Q} - \widehat{Q}\widehat{P} = -2y\ln(a)t^2,$$

and

$$e^{\widehat{P}+\widehat{Q}} = e^{\widehat{P}} e^{\widehat{Q}} e^{-\frac{1}{2}[\widehat{P},\widehat{Q}]}$$

therefore, we can write

$$e^{t(F(x)+2yt\ln(a)\widehat{\mathcal{T}}_x)} = e^{tF(x)+y\ln(a)t^2} e^{2yt\ln(a)\widehat{\mathcal{T}}_x}$$

By expanding the exponential function, we obtain

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(F(x), y\ln(a)) \sum_{s=0}^{\infty} \frac{t^s}{s!} (2y\ln(a)\widehat{\mathcal{T}}_x)^s = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} \frac{t^{n+s}}{n!s!} H_n(F(x), y\ln(a))(2y\ln(a)\widehat{\mathcal{T}}_x)^s.$$

Setting $k = n + s$ and inverting summations, we find

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} (F(x)+2y\ln(a)\widehat{\mathcal{T}}_x)^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{s=0}^{k} \binom{k}{s} H_{k-s}(F(x), y\ln(a))(2y\ln(a)\widehat{\mathcal{T}}_x)^s.$$

Therefore, (2.10) follows from the comparison of the coefficients of $\frac{t^k}{k!}$ in the last equation. By using the eq. (2.8) we find the following result

$$\left[F(x) + 2y\ln(a)\left(\widehat{\mathcal{T}}_x\right)\right]^n = \sum_{s=0}^{n} (2y\ln(a))^s \binom{n}{s} h_{n-s}^{(2)}(x, y\ln(a)) \left(\widehat{\mathcal{T}}_x\right)^s \quad (2.11)$$

and

$$a^{y(\widehat{\mathcal{T}}_x)^m} f(F(x)) = e^{y\ln(a)(\widehat{\mathcal{T}}_x)^m} f(F(x)) = f(F(x) + my\ln(a)\left(\widehat{\mathcal{T}}_x\right)^{m-1}) a^{y(\widehat{\mathcal{T}}_x)^m} \quad (2.12)$$

which realize an extension of the ordinary Burchnall and Crofton [4, 14] identities valid for $q(x) = 1$.

It is evident that all the wealth of properties of H.K.d.F. can be extended fairly straightforwardly to the functions $h_n^{(m)}(x, y\ln(a))$. The use of the previously

discussed rules may greatly simplify the application of different types of exponential polynomials.
To give some examples, we note e.g. that

$$a^{y\left(x\frac{\partial}{\partial x}\right)^m}(x^n) = e^{y\ln(a)\left(x\frac{\partial}{\partial x}\right)^m}(x^n) \;=\; e^{y\ln(a)\left(x\frac{\partial}{\partial x}\right)^m} e^{n\ln(x)}$$

$$= \sum_{r=0}^{\infty} \frac{n^r}{r!} H_r^{(m)}(\ln(x), y\ln(a)), \tag{2.13}$$

and since

$$\sum_{r=0}^{\infty} \frac{t^r}{r!} H_r^{(m)}(x, y\ln(a)) \;=\; \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{k=0}^{[\frac{r}{m}]} r! \frac{x^{r-mk}(y\ln(a))^k}{(r-mk)!k!} \;=\; \sum_{r=0}^{\infty} \sum_{k=0}^{[\frac{r}{m}]} \frac{x^{r-mk}(y\ln(a))^k t^r}{(r-mk)!k!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(xt)^n}{n!} \frac{(y\ln(a)t^m)^k}{k!} \;=\; \sum_{n=0}^{\infty} \frac{(xt)^n}{n!} \sum_{k=0}^{\infty} \frac{(y\ln(a)t^m)^k}{k!} \;=\; e^{xt+y\ln(a)t^m}$$

or

$$\sum_{r=0}^{\infty} \frac{t^r}{r!} H_r^{(m)}(x, y\ln(a)) \;=\; e^{xt+y\ln(a)t^m} \tag{2.14}$$

or, equivalently,

$$\sum_{r=0}^{\infty} \frac{t^r}{r!} g_r^{(m)}(x, y\ln(a)) = e^{xt+y\ln(a)t^m}, \tag{2.15}$$

by replacing t and x with n and $\ln(x)$ respectively, for the aforementioned Gould-Hopper polynomials [8, p. 86, eq. (1.11) (27)], we find that

$$a^{y\left(x\frac{\partial}{\partial x}\right)^m}(x^n) = e^{y\ln(a)\left(x\frac{\partial}{\partial x}\right)^m}(x^n) \;=\; x^n e^{y\ln(a)n^m} \;=\; x^n a^{yn^m}. \tag{2.16}$$

It is now evident that if $f(x)$ is specified by any analytic function $(f(x) = \sum_n c_n x^n)$, then

$$a^{y\left(x\frac{\partial}{\partial x}\right)^m} f(x) = \sum_n c_n x^n a^{yn^m}, \tag{2.17}$$

provided that the last series is convergent. Further comments on this last result will be presented in the concluding section (4).
A further example of exponential operator is provided by the case

$$q(x) = (x-b)^2, \qquad F(x) \;=\; \int^x \frac{d\xi}{(\xi-b)^2} \;=\; -\frac{1}{(x-b)},$$

for which we find

$$a^{y\left[(x-b)^2\frac{\partial}{\partial x}\right]^2}\left[\frac{x-b}{x}\right] \;=\; e^{y\ln(a)\left[(x-b)^2\frac{\partial}{\partial x}\right]^2}\left[\frac{x-b}{x}\right]$$

$$= e^{y\ln(a)\left[(x-b)^2\frac{\partial}{\partial x}\right]^2}\left[\frac{1}{1+\frac{b}{x-b}}\right]$$
$$= e^{y\ln(a)\left[(x-b)^2\frac{\partial}{\partial x}\right]^2}\sum_{s=0}^{\infty}\left(-\frac{b}{x-b}\right)^s$$
$$= \sum_{s=0}^{\infty}(b)^s e^{y\ln(a)\left[(x-b)^2\frac{\partial}{\partial x}\right]^2}\left(-\frac{1}{x-b}\right)^s$$

let $-\frac{1}{x-b} = t$, therefore $\frac{\partial t}{\partial x} = \frac{1}{(x-b)^2}$ and $\frac{\partial}{\partial x} = \frac{\partial}{\partial t}\frac{\partial t}{\partial x} = \frac{1}{(x-b)^2}\frac{\partial}{\partial t}$ or $\frac{\partial}{\partial t} = (x-b)^2\frac{\partial}{\partial x}$. Now

$$\sum_{s=0}^{\infty}(b)^s e^{y\ln(a)\left(\frac{\partial}{\partial t}\right)^2}t^s = \sum_{s=0}^{\infty}(b)^s H_s\left(t, y\ln(a)\right)$$

or

$$a^{y\left[(x-b)^2\frac{\partial}{\partial x}\right]^2}\left[\frac{x-b}{x}\right] = \sum_{s=0}^{\infty}(b)^s H_s\left(-\frac{1}{x-b}, y\ln(a)\right). \tag{2.18}$$

Particular case: The results determined by Dattpli et al. [2; in 2 section] can be obtained with the substitution $a = e$, into the equations of this section.

3 Exponential Operators of Fractional Order on the Base a $(a > 0, \neq 1)$

In this section we will discuss fractional shift operators of the type

$$\widehat{A}_\mu = a^{\left(q(x)\frac{d}{dx}\right)^\mu} \tag{3.1}$$

with $q(x) = 1$ and μ any real number such that $0 < \mu < 1$.
Before entering into the main body of the discussion, we recall that the Riemann-Liouville derivative of fractional order m is defined by (see [9]; see also [3, p. 286, eq. (5.1) (8)])

$$\left(\frac{d}{dx}\right)^\nu f(x) = \frac{1}{\Gamma(m-\nu)}\frac{d^m}{dx^m}\int_0^x (x-t)^{m-\nu-1}f(t)dt, \tag{3.2}$$

where m is a positive integer such that $m-1 < \nu < m$. Accordingly, we get

$$a^{\left(\frac{\partial}{\partial x}\right)^\mu}x^n = H_n^{(\mu)}(x, y\ln(a)) \tag{3.3}$$

where $H_n^{(\mu)}(x, y\ln(a))$ are H.K.d.F. pseudo-polynomials of fractional order, defined by

$$H_n^{(\mu)}(x, y\ln(a)) = n!\sum_{r=0}^{\infty}\frac{x^{n-\mu r}(y\ln(a))^r}{\Gamma(n-\mu r+1)!r!}, \tag{3.4}$$

whose validity can be easily checked by direct expansion of the operator and by the fact that [9]

$$\left[\left(\frac{d}{dx}\right)^{\mu}\right]^{k} x^n = \left(\frac{d}{dx}\right)^{k\mu} x^n = \frac{\Gamma(n+1)x^{n-\mu r}}{\Gamma(n-\mu r+1)r!}, \qquad (\mu < n+1). \tag{3.5}$$

According to the previous discussions $H_n^{(\mu)}(x, y\ln(a))$ is the natural solution of the fractional Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial y}u(x, y\ln(a)) = \ln(a)\left[\dfrac{\partial}{\partial x}\right]^{\mu} u(x, y\ln(a)), \\ u(x,0) = x^n. \end{cases} \tag{B}$$

We must underline that the function $H_n^{(\mu)}(x, y\ln(a))$ is an extension of the ordinary H.K.d.F. or Gould-Hopper polynomials. More generally, we can solve the problem (B) with the general condition

$$u(x,0) = f(x) = \sum_n c_n x^n \tag{3.6}$$

according to the following relation

$$u(x, y\ln(a)) = f(x) = \sum_n c_n H_n^{(\mu)}(x, y\ln(a)). \tag{3.7}$$

It is obvious that we can combine relevant to generalized shift operators exponential operators of the type

$$\widehat{A}_{\mu} = a^{\left(q(x)\frac{d}{dx}\right)^{\mu}}.$$

With this purpose in view, we consider the problem

$$\begin{cases} \dfrac{\partial}{\partial y}u(x, y\ln(a)) = \ln(a)\left[q(x)\dfrac{\partial}{\partial x}\right]^{\mu} u(x, y\ln(a)), \\ u(x,0) = (F(x))^n, \end{cases} \tag{C}$$

where

$$F(x) = \int^{x} \frac{d\xi}{q(\xi)}.$$

It is fairly natural to write the solution of (C) as follows:

$$u(x, y\ln(a)) = H_n^{(\mu)}(F(x), y\ln(a))(3.8)$$

This last result completes the purposes of the present paper aimed at providing a general framework for the families of exponential operators.
Particular case: The substitution $a = e$, reduces the eqs. (3.1) to (3.8) to the results due to Dattoli et al. [2; p. 220-221, section 3].

4 Concluding Remark

In the previous section we have seen that the theory of exponential operators can be conveniently complemented by the use of functions satisfying recurrences of quasi monomial nature. In these concluding remarks we will discuss the introduction of a family of functions which can be viewed as a fairly natural consequence of the so far developed formalism.
We consider indeed the case of logarithmic Bessel functions, whose generating function can be cast in the form

$$G(x,\vartheta)=x^{i\sin(\vartheta)}=\sum_{n=-\infty}^{\infty}e^{in\vartheta}J_n(\ln(x)), \tag{4.1}$$

where $J_n(x)$ denote the first kind cylinder Bessel functions, that is

$$J_n(z) \;=\; \left(\frac{1}{2}z\right)^n\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{4}z^2\right)^k}{k!(n+k)!}.$$

It is evident that we can take advantage from the discussion of the previous sections, to consider the following problem

$$a^{y\left(x\frac{\partial}{\partial x}\right)^2}x^{i\sin(\vartheta)}=e^{y\ln(a)\left(x\frac{\partial}{\partial x}\right)^2}x^{i\sin(\vartheta)} \;=\; e^{i\sin(\vartheta)\left(\ln(x)+2y\ln(a)x\frac{\partial}{\partial x}\right)}. \tag{4.2}$$

The exponential can be decoupled by means of the Weyl rule

$$e^{\widehat{P}+\widehat{Q}}=e^{\widehat{P}}e^{\widehat{Q}}e^{-\frac{1}{2}[\widehat{P},\widehat{Q}]},$$

where

$$[\widehat{P},\widehat{Q}]=\widehat{P}\widehat{Q}-\widehat{Q}\widehat{P},$$

by setting indeed

$$\left.\begin{array}{l}\widehat{P}=i\sin(\vartheta)\ln(x),\\ \widehat{Q}=2iy\ln(a)(\sin(\vartheta))x\frac{\partial}{\partial x},\end{array}\right\} \tag{4.3}$$

we find

$$[\widehat{P},\widehat{Q}]=2y\ln(a)[\sin(\vartheta)]^2, \tag{4.4}$$

thus getting

$$a^{y\left(x\frac{\partial}{\partial x}\right)^2}x^{i\sin(\vartheta)}=x^{i\sin(\vartheta)}e^{-y\ln(a)[\sin(\vartheta)]^2}. \tag{4.5}$$

Which is the generating function of a two-variable Bessel function, namely

$$x^{i\sin(\vartheta)}e^{-y\ln(a)[\sin(\vartheta)]^2} \;=\; \sum_{n=-\infty}^{\infty}e^{in\vartheta}(\,{}_hJ_n(x,y\ln(a))). \tag{4.6}$$

$${}_hJ_n(x,y\ln(a)) \;=\; \sum_{r=0}^{\infty}\frac{(-1)^rH_{n+2r}(\ln(x),y\ln(a))}{2^{n+2r}r!(n+r)!}. \tag{4.7}$$

It is evident that we ended up with a Bessel type function generalizing those of Hermite nature discussed in ref. [7]. It is worth emphasizing that the above equations satisfy a partial differential equation of the type

$$\begin{cases} \dfrac{\partial}{\partial y}({}_hJ_n(x, y\ln(a))) = \ln(a)\left(x\dfrac{\partial}{\partial x}\right)^2 ({}_hJ_n(x, y\ln(a))), \\ {}_hJ_n(x, 0) = J_n(\ln(x)). \end{cases} \tag{4.8}$$

It is evident that the above considerations can be extended to any generating function of the type [2]

$$a^{iF(x)\sin(\vartheta)}. \tag{4.9}$$

Before concluding we will show how the combined use of integral transform and the previous formalism allows the derivation of further important relations. According to the identity [16]

$$a^{\lambda \widehat{P}^2} = e^{\lambda\ln(a)\widehat{P}^2} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi^2+2\xi\sqrt{\lambda\ln(a)}\widehat{P}}d\xi. \tag{4.10}$$

we can easily conclude that the polynomials $h_n^{(2)}(x, y)$ can also be realized in terms of the integral representation

$$a^{y\left[q(x)\frac{\partial}{\partial x}\right]^2}[F(x)]^n = e^{y\ln(a)\left[q(x)\frac{\partial}{\partial x}\right]^2}[F(x)]^n = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi^2}[F(x)+2\sqrt{y\ln(a)}\xi]^n d\xi. \tag{4.11}$$

Going back to eq. (2.17) and specializing for $m = 2$, the use of the above relations allows to state the following identity

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^{-yn^2}}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n e^{-y\ln(a)n^2}}{n!} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi^2}e^{-\cos(2\sqrt{y}\xi)}\cos(\sin(2\sqrt{y}\xi))d\xi.$$

Finally since [6]

$$e^{-yd} = \frac{y}{2\sqrt{\pi}}\int_0^{\infty}\frac{1}{t\sqrt{t}}e^{-\frac{y^2}{4t}}e^{-d^2t}dt \tag{4.12}$$

by replacing d with $(\ln(a)\mathcal{T}_x)^{\frac{1}{2}}$ and by using the previously discussed rules we find

$$e^{-y(\ln(a)\mathcal{T}_x)^{\frac{1}{2}}} = \frac{y}{2\sqrt{\pi}}\int_0^{\infty}\frac{1}{t\sqrt{t}}e^{-\frac{y^2}{4t}}e^{-(\ln(a)\mathcal{T}_x)t}dt = \frac{y}{2\sqrt{\pi}}\int_0^{\infty}\frac{1}{t\sqrt{t}}e^{-\frac{y^2}{4t}}a^{-(\mathcal{T}_x)t}dt$$

now

$$e^{-y(\ln(a)\mathcal{T}_x)^{\frac{1}{2}}}f(x) = \frac{y}{2\sqrt{\pi}}\int_0^{\infty}\frac{1}{t\sqrt{t}}e^{-\frac{(y)^2}{4t}}a^{-(\mathcal{T}_x)t}f(x)dt$$

or

$$e^{-y(\ln(a)\mathcal{T}_x)^{\frac{1}{2}}}f(x) = \frac{y}{2\sqrt{\pi}}\int_0^\infty \frac{1}{t\sqrt{t}}e^{-\frac{y^2}{4t}}f(F^{-1}(F(x)-t\ln(a)))dt. \tag{4.13}$$

Which realizes the transform providing the action of a fractional generalized shift operator on a given function, the validity of both (3.11) and (3.12) is limited to the case in which the integral converges.

Particular case: The substitution of $a = e$ into the eqs. (4.1) to (4.13) give raise to the eqs. (15) to (26) due to Dattoli et al. [2].

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