Int. Journal of Math. Analysis, Vol. 5, 2011, no. 2, 65 - 80

A Generalized Fixed Point Theorem in Intuitionistic Menger Spaces and its Application to Integral Equations

M. Goudarzi

Dept. of Mathematics, Faculty of Sciences
Yasouj University ,Yasouj, Iran
goudarzi@mail.yu.ac.ir

Abstract

In this paper we introduce a definition of contraction which will be the generalization of some old definitions and we prove a result in generalized fixed point theory, on intuitionitic Menger spaces. As its application, we present an extended version of a fixed point theorem in metric spaces. Finally, under some conditions, we use the definition of contraction and fixed point theory to show the existence and the uniqueness of solution for some kinds of Uryson's integral equations in an intuitionistic Menger normed spaces.

Mathematics Subject Classification: 46A55; 46B99

Keywords: Contraction map, intuitionistic probabilistic metric space, intuitionistic Menger normed spaces, Multivalued maps, Fixed point theorem

1 Introduction

The definition of an intuitionistic probabilistic metric space, an intuitionistic Menger space and an intuitionistic Menger normed space recently has been introduced in ([10]). We will to introduce a generalized case of contractions in an intuitionistic Menger space. Intuitively we expect that this generalization can help to extend some corresponding cases in metric spaces. In the last part of section 3 we see a notable result. At the end we see an interesting application of fixed point theorem to find the existence and the uniqueness solution of some kinds of Uryson's integral equation in an intuitionistic Menger normed space.

Definition 1.1 ([9]) A triangular norm(t-norm) $* = T$ on $[0, 1]$ is defined as an increasing, commutative and associative mapping $T : [0, 1]^2 \to [0, 1]$ satisfying $T(0, x) = 0 * x = x$, for all $x \in [0, 1]$.

Definition 1.2 ([9]) A binary operation $S : [0,1] \times [0,1] \longrightarrow [0,1]$ is a t-conorm if S is increasing, commutative, associative and $S(a,0) = a, \forall a \in [0,1]$. For any continuous t-norm T, a binary operation T^* on [0,1] which is related to T by $T^*(a,b) = 1 - T(1-a, 1-b)$, for all $a, b \in [0,1]$, is called the t-conorm of T.

Throughout this paper we let, $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}^+ = [0, +\infty)$.

Definition 1.3 ([10]) A distance distribution function is a non-decreasing and left continuous mapping $F : \mathbb{R} \to \mathbb{R}^+$ with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$. We will denote by D the family of all distance distribution functions and by $\mathcal{H}$ the special element of D defined by

$$\mathcal{H}(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0. \end{cases}$$

If X is a nonempty set, $F : X \times X \longrightarrow D$ is called a probabilistic distance on X and $F(x,y)$ is usually denoted by F_{xy}.

Definition 1.4 ([17]) A Menger space (probabilistic metric space) is an order pair $(X,\ F)$ where X is a nonempty set and F is a probabilistic distance on X, satisfying the following condition, for all $p, q, r \in X$ and $x, y \in \mathbb{R}$.
i) $F_{pq}(x) = 1,\ \forall x > 0$ if and only if $p = q$
ii) $F_{pq}(x) = F_{qp}(x)$
iii) If $F_{pq}(x) = 1, F_{qr}(y) = 1$ then $F_{pr}(x+y) = 1$.
A generalized Menger space is a triple $(X,\ F,\ T)$ where $(X,\ F)$ is a probabilistic metric space, T is a t-norm and the following inequality holds

$$F_{pr}(x+y) \geq T(F_{pq}(x), F_{qr}(y)),\ \forall p, q, r \in X \text{ and } x, y \in \mathbb{R}.$$

Definition 1.5 Let $(X,\ F,\ T)$ be a generalized Menger space. A mapping $f : X \longrightarrow X$ is called :
i) a C-contraction ([8]) if there exists $k \in (0,1)$ such that for all $p, q \in X$ the following implication holds

$$F_{pq}(t) > 1 - t \Rightarrow F_{f(p)f(q)}(kt) > 1 - kt,\ \forall t \in (0, +\infty).$$

ii) a weak Hicks-contraction ([11]) if there exists $k \in (0,1)$ such that for all $p, q \in X$ the following implication holds

$$F_{pq}(t) > 1 - t \Rightarrow F_{f(p)f(q)}(kt) > 1 - kt,\ \forall t \in (0,1).$$

iii) a $(\varepsilon - \lambda)$-probabilistic contraction ([12]) if there exists $k \in (0,1)$ such that for all $p, q \in X$

$$F_{pq}(\varepsilon) > 1 - \lambda \Rightarrow F_{f(p)f(q)}(k\varepsilon) > 1 - k\lambda,$$

for $\varepsilon > 0$ and $\lambda \in (0,1)$.
iv) a (k_1, k_2) contraction of (ε, λ)-type $(k_1, k_2 \in (0,1))$ ([5]) if

$$F_{pq}(\varepsilon) > 1 - \lambda \Rightarrow F_{f(p)f(q)}(k_1\varepsilon) > 1 - k_2\lambda,$$

for $\varepsilon > 0$ and $\lambda \in (0,1)$.
v) a strict B-contraction ([14]) if for some $k \in (0,1)$

$$F_{f(x)f(y)}(kt) \geq \frac{F_{xy}(t)}{F_{xy}(t) + k(1 - F_{xy}(t))}, \ \forall x, y \in X, \ \forall t > 0.$$

vi) a probabilistic contraction of type $r \in (0,1)$ ([15]) if there exists $k \in (0,1)$ such that

$$F_{f(x)f(y)}(k^r t) \geq \frac{F_{xy}(t)}{F_{xy}(t) + k^{1-r}(1 - F_{xy}(t))}, \ \forall x, y \in X, \ \forall t > 0.$$

Definition 1.6 ([10]) A non-distance distribution function is a non-increasing and right continuous mapping $L : \mathbb{R} \to \mathbb{R}^+$ with $\sup_{t\in\mathbb{R}} L(t) = 0$ and $\inf_{t\in\mathbb{R}} L(t) = 1$. We will denote by E the family of all non-distance distribution functions and by $\mathcal{G}$ the special element of E defined by

$$\mathcal{G}(t) = \begin{cases} 1 & t \leq 0 \\ 0 & t > 0. \end{cases}$$

If X is a nonempty set, $L : X \times X \longrightarrow E$ is called a probabilistic non-distance on X and $L(x, y)$ is usually denoted by L_{xy}.

Definition 1.7 ([10]) A triple $(X,\ F,\ L)$ is said to be an intuitionistic probabilistic metric space if X is a nonempty set, F is a probabilistic distance and L is a probabilistic non-distance on X satisfying the following conditions, for $x, y, z \in X$ and $s, t \geq 0$.
a) $F_{xy}(t) + L_{xy}(t) \leq 1$
b) $F_{xy}(0) = 0$
c) $F_{xy}(t) = \mathcal{H}(t)$ if and only if $x = y$
d) $F_{xy}(t) = F_{yx}(t)$
e) If $F_{xz}(t) = 1$ and $F_{zy}(s) = 1$, then $F_{xy}(t + s) = 1$
f) $L_{xy}(0) = 1$
g) $L_{xy}(t) = \mathcal{G}(t)$ if and only if $x = y$
h) $L_{xy}(t) = L_{yx}(t)$
i) If $L_{xz}(t) = 0$ and $L_{zy}(s) = 0$ then $L_{xy}(t + s) = 0$.
If in addition the triangle inequalities
j) $F_{xy}(t + s) \geq T(F_{xz}(t), F_{zy}(s))$
k) $L_{xy}(t + s) \leq S(L_{xz}(t), L_{zy}(s))$,

where T is a t-norm and S is a t-conorm, then $(X,\ F,\ L,\ T,\ S)$ is said to be an intuitionistic Menger space. The functions $F_{xy}(t)$ and $L_{xy}(t)$ denote the degree of nearness and the degree of non nearness between x and y with respect to t, respectively.

Example 1.8 ([10])(Induced intuitionistic probabilistic metric) Let (X, d) be a metric space. Then the metric d induces a distance distribution function F defined by $F_{xy}(t) = \mathcal{H}(t - d(x, y))$ and a non distribution function L defined by $L_{xy}(t) = \mathcal{G}(t - d(x, y))$ for all $x, y \in X$ and $t \geq 0$. Then $(X,\ F,\ L)$ is an intuitionistic probabilistic metric space. If t-norm T is $T(a, b) = \min\{a, b\}$ and t-conorm $S(a, b) = \min\{1, a+b\}$, for all $a, b \in [0, 1]$ then $(X,\ F,\ L,\ T_m,\ S_m)$ is an intuitionistic Menger space. Also the above example holds even with the t-norm $T(a, b) = \min\{a, b\}$ and $S(a, b) = \max\{a, b\}$ and hence $(X,\ F,\ L,\ T,\ S)$ is an intuitionistic Menger space with respect to any t-norm and t-conorm.

Lemma 1.9 *([9])* Consider the set L^* and operation $\leq_{L^*}$ defined by:

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$, for every $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice.

We denote its units by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$.

In the sequel we let that each t-norm T and its t-conorm S are satisfied in the conditions $\sup_{t<1} T(t, t) = 1$ and $\inf_{t<1} S(1 - t, 1 - t) = 0$, respectively.

Definition 1.10 ([10]) Let $(X,\ F,\ L,\ T,\ S)$ be an intuitionistic Menger space.
a) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for each $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists a positive integer $n_0 = n_0(\lambda, \varepsilon)$ such that $F_{x_n,x_m)}(\varepsilon) > (1 - \lambda)$ and $L_{x_n,x_m)}(\varepsilon) < \lambda$, for all $m, n \geq n_0$.
b) An intuitionistic Menger space in which every Cauchy sequence is convergent is said to be complete.

Theorem 1.11 ([10]) Let $(X,\ F,\ L,\ T,\ S)$ be an intuitionistic Menger space. Then for any sequence $\{x_n\}$ in X, $x_n \longrightarrow x$ if and only if $F_{x_n x}(t) \longrightarrow 1$ and $L_{x_n x}(t) \longrightarrow 0$, for each $t > 0$.

Definition 1.12 ([10]) A 5-tuple $(X,\ F,\ L,\ T,\ S)$ is said to be an intuitionistic Menger normed space if X is a real normed space, F is a distance distribution mapping from X into D, L is a non distance distribution mapping from X into E and T and S are continuous t-norm and continuous t-conorm, respectively, satisfying the following conditions for all $x, y, z \in X,\ s, t \geq 0$.

a) $F_x(t) + L_x(t) \leq 1$
b) $F_x(0) = 0$
c) $F_{x-y}(t) = \mathcal{H}(t)$ if and only if $x = y$
d) $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$, for each $\alpha \neq 0$
e) $F_{x+y}(t+s) \geq T(F_x(t), F_y(s))$
f) $L_x(0) = 1$
g) $L_{x-y}(t) = \mathcal{G}(t)$ if and only if $x = y$
h) $L_{\alpha x}(t) = L_x(\frac{t}{|\alpha|})$, for each $\alpha \neq 0$
i) $L_{x+y}(t+s) \leq S(L_x(t), L_y(s))$
In this case (F, L) is called an intuitionistic Menger norm.

Definition 1.13 ([10]) A sequence $\{x_n\}$ in an intuitionistic Menger normed space $(X,\ F,\ L,\ T,\ S)$ is said to be a Cauchy sequence if for each $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists a positive integer $n_0 = n_0(\varepsilon, \lambda)$ such that

$$F_{x_n - x_m}(\varepsilon) > 1 - \lambda \text{ and } L_{x_n - x_m}(\varepsilon) < \lambda,$$

for all $m, n \geq n_0$. The sequence $\{x_n\}$ is said to be convergent to x in X and denoted by $x_n \longrightarrow x$ if

$$F_{x_n - x}(t) \longrightarrow 1 \text{ and } L_{x_n - x}(t) \longrightarrow 0,$$

for each $t \geq 0$. An intuitionistic Menger normed space is said to be complete if and only if every Cauchy sequence is convergent.

2 Main results

Definition 2.1 An intuitionistic Menger space $(X,\ F,\ L,\ T,\ S)$ is said to be compact if and only if for any sequence $\{x_n\}$ in X there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \longrightarrow x$, for some $x \in X$.

Theorem 2.2 *Let g be a one-to-one self mapping on the intuitionistic Menger space $(X,\ F,\ L,\ T,\ S)$ and F^g and L^g defined by*

$$F^g_{xy}(t) = F_{g(x)g(y)}(t) \text{ and } L^g_{xy}(t) = L_{g(x)g(y)}(t),\ \forall t \in \mathbb{R}.$$

The following statements hold.
i) $(X_1,\ F^g,\ L^g,\ T,\ S)$ is an intuitionistic Menger space.
ii) If $X_1 = g(X)$ and $(X_1,\ F,\ L,\ T,\ S)$ is a complete intuitionistic Menger space, then $(X,\ F^g,\ L^g,\ T,\ S)$ is also a complete intuitionistic Menger space.
iii) If $(X_1,\ F,\ L,\ T,\ S)$ is compact then $(X,\ F^g,\ L^g,\ T,\ S)$ is also compact.

Proof : The proof of i) and ii) are straightforward. So we prove iii). Let $\{x_n\}$ be a sequence in X and $u_n = g(x_n)$, for each $n \in \mathbb{N}$. Since X_1 is compact we can find a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $v \in X_1$ such that $u_{n_k} \longrightarrow v$. But this means that

$$F_{u_{n_k}v}(t) \longrightarrow 1 \; and \; L_{u_{n_k}v}(t) \longrightarrow 0, \; \forall t > 0.$$

If we put $x_{n_k} = g^{-1}(u_{n_k})$ and $x = g^{-1}(v)$ then we have

$$F^g_{x_{n_k}x}(t) = F_{g(x_{n_k})g(x)}(t) \longrightarrow 1 \text{ and } L^g_{x_{n_k}x}(t) = F_{g(x_{n_k})g(x)}(t) = 0, \; \forall t > 0.$$

These show that $(X, \; F^g, \; L^g, \; T, \; S)$ is also compact. □

Definition 2.3 Let ϕ be the class of all mappings $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with the following properties
i) φ is right continuous.
ii) φ is non-decreasing.
iii) $\lim_{n\to\infty} \varphi^n(t) = 0, \; \forall t > 0$.

Definition 2.4 Let f be a self mapping defined on an intuitionistic Menger space $(X, \; F, \; L, \; T, \; S)$ and $\varphi \in \phi$. The mapping f is called an intuitionistic (g, φ)-contraction if there exists a bijection self mapping g on X such that for each $x, y \in X$ and for each $t > 0$

$$F_{g(x)g(y)}(t) > (1 - t) \text{ and } Ł_{g(x)g(y)}(t) \leq t$$

imply that (I)

$$F_{f(x)f(y)}(\varphi(t)) > (1 - \varphi(t)) \text{ and } L_{f(x)f(y)}(\varphi(t)) \leq \varphi(t).$$

Remark 1 : We see that the images of two points x and y under the function f are nearer than the images of the same points under the function g in the sense of lemma 2.9. So the generalization of intuitionistic (g, φ)-contraction is justified. Also we deduce the following results.
i) If for some $k \in (0, 1)$ we take $\varphi(t) = kt$ such that $t \in (0, \infty)$, $g(x) = x$ and take a generalized Menger space as an special intuitionistic menger space then we have the C- contraction in a generalized Menger space.
ii) In i), if we impose the condition $t \in (0, 1)$ then it gives the weak Hicks-contraction in a generalized Menger space.
iii) Also the other contractions such as $(\varepsilon - \lambda)$-probabilistic contraction and strict B-contraction can be as a special case of the (g, φ)-contraction, in a generalized Menger space.

Theorem 2.5 *Suppose that f and g be two self mappings on complete intuitionistic Menger space (X, F, L, T, S), g is bijective and f is an intuitionistic (g, φ)-contraction. Then there exists a unique point $p \in X$ such that $f(p) = g(p)$. Moreover if we have the sequence $\{x_n\}_{n\geq 1}$ given by $g(x_{n+1}) = f(x_n)$, for an arbitrary point $x_0 \in X$ then $p = \lim_{n\to\infty} x_n$.*

Proof : It is clear that for $t > 1$, $F^g_{xy}(t) > 1 - t$ and $L^g_{xy}(t) < t$. Also by definition of F^g and L^g we have $F_{g(x)g(y)}(t) > 1 - t$ and $L_{g(x)g(y)}(t) < t$. From conditions (I) we have

$$F_{f(x)f(y)}(\varphi(t)) > (1 - \varphi(t)) \text{ and } L_{f(x)f(y)}(\varphi(t)) < \varphi(t).$$

But this means that

$$F_{gg^{-1}f(x)gg^{-1}f(y)}(\varphi(t)) > (1 - \varphi(t)) \text{ and } L_{gg^{-1}f(x)gg^{-1}f(y)}(\varphi(t)) < \varphi(t),$$

and hence

$$F^g_{g^{-1}f(x)g^{-1}f(y)}(\varphi(t)) > (1 - \varphi(t)) \text{ and } L^g_{g^{-1}f(x)g^{-1}f(y)}(\varphi(t)) < \varphi(t).$$

By denoting $h = g^{-1}of$ and by iterations they imply that

$$F^g_{h^n(x)h^n(y)}(\varphi^n(t)) > (1 - \varphi^n(t)) \text{ and } L^g_{h^n(x)h^n(y)}(\varphi^n(t)) < \varphi^n(t).$$

Since $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists a natural number $n(\varepsilon, \lambda)$ such that $\varphi^n(t) \leq \min(\varepsilon, \lambda)$, for each $n \geq n(\varepsilon, \lambda)$. Since $F^g_{xy}(t)$ is non decreasing with respect to t it implies that

$$\begin{aligned} F^g_{h^n(x)h^n(y)}(\varepsilon) &\geq F^g_{h^n(x)h^n(y)}(\varphi^n(t)) \\ &> (1 - \varphi^n(t)) \\ &> (1 - \lambda). \end{aligned} \tag{1}$$

On the other hand $L^g_{xy}(t)$ is non-increasing with respect to t. So the similar argument shows that

$$L^g_{h^n(x)h^n(y)}(\varepsilon) < \lambda. \tag{2}$$

Now suppose that x_0 be an arbitrary point in X and $\{x_n\}$ be the sequence defined by $x_{n+1} = h(x_n)$ or equivalently $g(x_{n+1}) = f(x_n)$. If we take $x = x_m$ and $y = x_0$, from (1) and (2) we can obtain

$$F^g_{x_{m+n}x_n}(\varepsilon) = F^g_{h^n(x_m)h^n(x_0)}(\varepsilon) > (1 - \lambda),$$

and

$$L^g_{x_{m+n}x_n}(\varepsilon) = L^g_{h^n(x_m)h^n(x_0)}(\varepsilon) < \lambda,$$

for each $n \geq n(\lambda, \varepsilon)$ and $m \geq 1$. So $\{x_n\}$ is Cauchy sequence. But $(X,\ F,\ L,\ T,\ S)$ is complete and then by theorem 3.2, $(X,\ F^g,\ L^g,\ T,\ S)$ is complete. This implies that there exists a point $p \in X$ such that $x_n \longrightarrow p$, under the intuitionistic probabilistic metrics F^g and L^g. Now we will show that h is a continuous self map on $(X,\ F^g,\ L^g,\ T,\ S)$. First note that for a sequence $\{x_n\}$ in X such that $x_n \longrightarrow x$ in $(X,\ F^g,\ L^g,\ T,\ S)$ we deduce that

$$\lim_{n\to\infty} F^g_{x_n x}(t) = \lim_{n\to\infty} F_{g(x_n)g(x)}(t) = 1,\ \forall t > 0,$$

and

$$\lim_{n\to\infty} L^g_{x_n x}(t) = \lim_{n\to\infty} L_{g(x_n)g(x)}(t) = 0,\ \forall t > 0.$$

Also the condition (I) implies that

$$\lim_{n\to\infty} F_{f(x_n)f(x)}(t) \geq \lim_{n\to\infty} F_{f(x_n)f(x)}(\varphi(t)) = 1,\ \forall t > 0,$$

and

$$\lim_{n\to\infty} L_{f(x_n)f(x)}(t) \leq \lim_{n\to\infty} L_{f(x_n)f(x)}(\varphi(t)) = 0,\ \forall t > 0,$$

and hence f is a continuous mapping on $(X,\ F^g,\ L^g,\ T,\ S)$. Therefore if we take a sequence $\{x_n\}$ in X such that $x_n \longrightarrow x$ in $(X,\ F^g,\ L^g,\ T,\ S)$, we have

$$\begin{aligned}\lim_{n\to\infty} F^g_{h(x_n)h(x)}(t) &= \lim_{n\to\infty} F^g_{g^{-1}f(x_n)g^{-1}f(x)}(t)\\ &= \lim_{n\to\infty} F_{(gg^{-1}f(x_n)gg^{-1}f(x)}(t)\\ &= 1.\end{aligned}$$

Also the same argument shows that

$$\lim_{n\to\infty} L^g_{h(x_n)h(x)}(t) = 0.$$

Then $h(p) = p$, that is $g^{-1}of(p) = p$ or equivalently $f(p) = g(p)$. Now assume that $f(q) = g(q)$, for some $q \in X$. Then for any $t > 0$, by applying (I) repeatedly and by using the fact that $\lim_{n\to\infty} \varphi^n(t) = 0$, we can show that after n iterates we have

$$F_{g(p)g(q)}(t) > (1-t) \Rightarrow F_{f(p)f(q)}(\varphi^n(t)) > (1 - \varphi^n(t)).$$

Thus we have

$$\lim_{n\to\infty} F_{f(q)f(p)}(\varphi^n(t)) = 1,$$

and similarly

$$\lim_{n\to\infty} L_{f(q)f(p)}(\varphi^n(t)) = 0.$$

But this means that $F_{f(p)f(q)}(t) = \mathcal{H}(t)$, for all $t \in \mathbb{R}$ which implies that $f(p) = f(q)$, by definition 2.5(c). This shows that p is unique and hence the proof is complete. □

Note 1 : In the above theorem if we put $g(x) = x$, then p is a fixed point of f. So the above theorem is a generalized fixed point theorem.

If we consider the standard intuitionistic probabilistic metric space induced by a metric space $(X,\ d)$, proposed in example 2.8, their induced topologies are equivalent. So we can deduce the following corollary which will be a generalized condition for contractions on metric spaces.

Corollary 2.6 *Suppose that f,g be two self mappings on the complete metric space* $(X,\ d)$*, g is bijective and f is a* (φ, g)*-contraction, that is there exists a* $\varphi \in \phi$ *such that*

$$d(f(x), f(y)) \leq \varphi(d(g(x), g(y))),$$

for each $x, y \in X$*. Then there exists a unique point* $p \in X$ *such that* $f(p) = g(p)$*.*

3 Applications

In this section we try to show some applications of the fixed point theorem mentioned in section 2.
At first, in the following theorem, we shall utilize the results to study the existence and uniqueness of solution of the following kind of Uryson's integral equations

$$x(s) = y(s) + \int_G K(s, t, x(t))dt, \tag{3}$$

on the intuitionistic Menger normed space $(L^2(G),\ F,\ L,\ T,\ S)$, where G is a bounded closed set of $\mathbb{R}^n$, $L^2(G)$ is the real linear space of all Lebesgue square integrable functions, T and S are arbitrary t-norm and t-conorm respectively.

Theorem 3.1 *Suppose that K(s,t,u)* $(s, t \in G,\ -\infty < u < +\infty)$ *be separable and* $K(s, t, u) = k(s, t)u$*. Suppose further that* $\varphi(s) = \int_G k(s, x(t))dt$ *and for some* $0 < \alpha < 1$ *we have* $\varphi(s) = \alpha x(s)$*. If M is the self mapping on complete intuitionistic Menger normed space* $(L^2(G),\ F,\ L,\ T,\ S)$ *, defined by*

$$M(x(s)) = y(s) + \int_G K(s, t, x(t))dt, \tag{4}$$

then M has a unique fixed point which is the unique solution of equation (10), for given $y(s) \in L^2(G)$*. Moreover for any given* $x_0(s) \in L^2(G)$ *the iterative sequence*

$$x_{n+1}(s) = y(s) + \int_G K(s, t, x_n(t))dt \quad (n = 0, 1, 2, ...)$$

converges to $x^*(s)$.

Proposition 3.2 *Let S and T be a pair of continuous self mappings on the Banach space $(X, \| \,.\, \|_o)$ and*

$$\sum_{n=0}^{\infty} \|S^n x - T^n y\|_o < \infty, \ \forall x, y \in X. \tag{5}$$

Then there exists an element $x^* \in X$ such that
i) x^* is the unique common fixed point of S and T.
ii) For any given $x_0 \in X$, the iterative sequences $\{S^n x_0\}_{n=0}^{\infty}$ and $\{T^n x_0\}_{n=0}^{\infty}$ converge to the same point x^*.

Proof : For any $x_0 \in X$, put $\{x_n = S^n x_0\}$, for $n \geq 1$. Then we have

$$\begin{aligned} \|x_{k+m} - x_k\|_o &\leq \textstyle\sum_{i=k}^{k+m-1} \|x_{i+1} - x_i\|_o \\ &\leq \textstyle\sum_{i=k}^{k+m-1} (\|S^i x_1 - T^i x_1\|_o + \|T^i x_1 - S^i x_0\|_o). \end{aligned}$$

Hence, as $k \longrightarrow \infty$, we obtain

$$\lim_{k \to \infty} \|x_{k+m} - x_k\|_o = 0. \tag{6}$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. Suppose that $x_n \longrightarrow u^*$. By continuity of S, $Sx_n \longrightarrow Su^* = u^*$.i.e. u^* is a fixed point of S. Similarly, we can prove that for any given $x_0 \in X$, the sequence $T^n x_0$ converges to a point $y^* \in X$ and y^* is a fixed point of T. Now we prove that $x^* = y^* = u^*$. In fact let G(S) and G(T) be the sets of all fixed points of S and T, respectively. Since $u^* \in G(S)$ and $y^* \in G(T)$, $G(S)$ and $G(T)$ are not empty. For any $x \in G(S)$ and $y \in G(T)$ it follows from (12) that

$$\sum_{n=0}^{\infty} \|x - y\|_o = \sum_{n=0}^{\infty} \|S^n x - T^n y\|_o < \infty.$$

This implies that

$$\|x - y\|_o = 0, \ \forall x \in G(S), \ \forall y \in G(T).$$

Hence we have $G(S) = G(T) = \{u^*\} = \{y^*\}$ and this completes the proof. □

Corollary 3.3 *Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of continuous self mappings on the Banach space $(X, \|.\|_o)$. Suppose that for any positive integer i,j and $i \neq j$ the following holds,*

$$\sum_{n=0}^{\infty} \|T_i^n x - T_j^n y\|_o < \infty, \ \forall x, y \in X.$$

Then there exists an element $x^* \in X$ such that
i) x^* is the unique common fixed point of $\{T_i\}_{i=1}^{\infty}$.
ii) For any given $x_0 \in X$ and for any positive integer i, the iterative sequence $\{T_i^n x_0\}_{n=0}^{\infty}$ converges to the same point x^*.

Lemma 3.4 *Let $(X,\ F,\ L,\ T,\ S)$ be an intuitionistic Menger normed space such that*

$$\sup\{t \geq 0 : F_x(t) < 1\} < \infty \text{ and } \inf\{t \leq 0 : F_x(t) < 1\} < \infty.$$

For each $x \in X$, define

$$\|x\| = \sup\{t \geq 0 : F_x(t) < 1\},$$

0r

$$\|x\| = \inf\{t \leq 0 : L_x(t) < 1\}.$$

Then $(X,\ \|.\|)$ is a normed space. Moreover if $(X,\ F,\ L,\ T,\ S)$ is complete then $(X,\ \|.\|)$ is also a complete normed space.

Proof : Suppose that for $x \in X$,

$$\|x\| = \sup\{t \geq 0 : F_x(t) < 1\}.$$

By 2.12(d), we have $\| \alpha x \| = | \alpha | \| x \|$, for each $x \in X$ and $0 \neq \alpha \in \mathbb{R}$. We can easily show that $F_x(t)$ and $L_x(t)$ are non decreasing and non increasing in t, respectively. Now suppose that $\sup\{t \geq 0,\ F_x(t) < 1\} = 0$. Then for any $t > 0$, $F_x(t) = 1$. The nondecreasing property of $F_x(.)$, 2.12(b) and 2.12(c) implies that $x = 0$. Conversely suppose that $x = 0$. By 2.12(c)

$$\sup\{t \geq 0,\ F_0(t) < 1\} = \sup\{t \geq 0,\ H(t) < 1\} = 0.$$

Finally we prove that $\| x + y \| \leq \| x \| + \| y \|$, for each $x, y \in X$. By 2.12(e), for any $\varepsilon > 0$

$$\begin{aligned} F_{x+y}(\| x \| + \| y \| + \varepsilon) &\geq T(F_x(\| x \| + \tfrac{\varepsilon}{2}), F_y(\| y \| + \tfrac{\varepsilon}{2})) \\ &= T(1, 1) \\ &= 1. \end{aligned}$$

This means that

$$\sup\{t \geq 0,\ F_{x+y}(t) < 1\} \leq \| x \| + \| y \| + \varepsilon.$$

Since ε is arbitrary it implies that

$$\| x + y \| \leq \| x \| + \| y \| .$$

So $\|.\|$ is a norm on X. To show the second part we prove that the convergency in $(X,\ \|.\|)$ implies the convergency in $(X,\ F,\ L,\ T,\ S)$. Suppose that $x_n \stackrel{\|.\|}{\longrightarrow} x$. Then

$$\lim_{n \to \infty} \|x_n - x\| = 0.$$

If we put

$$t_n = \sup\{t \geq 0 : F_{x_n - x}(t) < 1\},$$

then for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $t_n < \varepsilon$. But this means that

$$F_{x_n - x}(\varepsilon) = 1, \ \forall n \geq n_0,$$

and hence

$$\lim_{n\to\infty} F_{x_n - x}(\varepsilon) = 1.$$

The same argument shows that

$$\lim_{n\to\infty} L_{x_n - x}(\varepsilon) = 0,$$

and hence $x_n \longrightarrow x$ in $(X,\ F,\ L,\ T,\ S)$. The similar argument shows that

$$\|x\| = \inf\{t \leq 0 : L_x(t) < 1\}, \ \forall x \in X,$$

is also a norm on X and the rest part of lemma holds. □

Lemma 3.5 *Let $(X,\ \|\ .\ \|_o)$ be a normed space. Suppose that $(X,\ F,\ L,\ T,\ S)$ be its induced intuitionistic Menger normed space, where*

$$F_x(t) = \mathcal{H}(t- \parallel x \parallel_o) \text{ and } L_x(t) = \mathcal{G}(t- \parallel x \parallel_o), \ \forall x, y \in H, \ \forall t \geq 0.$$

Then for any sequence $\{x_n\}$ in X, the followings are equivalent, for $x \in X$.
i) $\lim_{n\to\infty} \|x_n - x\|_o = 0$
ii) $\lim_{n\to\infty} \|x_n - x\| = 0$.

Proof : By definition of lemma 5.4, we have

$$\begin{aligned}\|x_n - x\| &= \sup\{t \geq 0,\ F_{x_n - x}(t) < 1\}\\ &= \sup\{t \geq 0,\ \mathcal{H}(t- \parallel x_n - x \parallel) < 1\}\\ &= \|x_n - x\|_o.\end{aligned}$$

Hence the proof is complete. □

Theorem 3.6 *Let $(X,\ F,\ L,\ T,\ S)$ be a complele intuitionistic Menger normed space such that*

$$\sup\{t \geq 0 : F_x(t) < 1\} < \infty \text{ and } \inf\{t \geq 0 : F_x(t) < 1\} < \infty,$$

S and T be self mappings on X and $\{c_n\}_{n=0}^{\infty}$ be a sequence of positive constants such that $\sum_{n=0}^{\infty} c_n < \infty$. Suppose further that for $n = 0, 1, 2, ...$

$$F_{S^n x - T^n y}(c_n t) \geq F_{x-y}(t), \tag{7}$$

and

$$L_{S^n x - T^n y}(c_n t) \leq L_{x-y}(t), \tag{8}$$

for all $x, y \in X$ and $t \geq 0$. Then the conclusions of proposition 5.2 hold.

Proof : Suppose there exists $c \in (0,1)$ such that

$$F_{Sx-Ty}(ct) \geq F_{x-y}(t), \tag{9}$$

and

$$L_{Sx-Ty}(ct) \leq L_{x-y}(t), \tag{10}$$

for all $x, y \in X$ and $t \geq 0$. Then we have

$$\begin{aligned} \sup\{t \geq 0 : F_{Sx-Ty}(t) < 1\} &\leq \sup\{t \geq 0 : F_{x-y}(\tfrac{t}{c}) < 1\} \\ &= c \sup\{t \geq 0 : F_{x-y}(t) < 1\}. \end{aligned} \tag{11}$$

Hence for any $n \in \mathbb{N}$, we have

$$\sup\{t \geq 0 : F_{S^n x - T^n y}(t) < 1\} \leq c^n \sup\{t \geq 0 : F_{x-y}(t) < 1\}.$$

Also the same argument shows that

$$\inf\{t \leq 0 : L_{S^n x - T^n y}(t) < 1\} \leq c^n \inf\{t \leq 0 : L_{x-y}(t) < 1\}.$$

Therefore according to notation used in lemma 5.4 we have

$$\begin{aligned} \textstyle\sum_{n=0}^{\infty} \|S^n x - T^n y\| &\leq \textstyle\sum_{n=0}^{\infty} c^n \|x-y\| \\ &< \infty. \end{aligned}$$

Moreover from lemma 5.4 we imply that

$$\begin{aligned} \|Tx - Ty\| &\leq \|Tx - Sx\| + \|Sx - Ty\| \\ &\leq (\sup\{t \geq 0 : F_{x-x}(\tfrac{t}{c}) < 1\}) \\ &\quad +(\sup\{t \geq 0 : F_{x-y}(\tfrac{t}{c}) < 1\}) \\ &= 0 + c\|x-y\|. \end{aligned}$$

This means that T is a continuous self mapping on $(X,\ F,\ L,\ T,\ S)$. The similar argument shows that S is also a continuous self mapping on $(X,\ F,\ L,\ T,\ S)$. If we apply the previous argument in (14) and (15) $(for\ n = 1)$ we can prove that S and T are continuous self mappings on $(X,\ F,\ L,\ T,\ S)$ and also for each $x, y \in X$ we have

$$\begin{aligned} \textstyle\sum_{n=0}^{\infty} \|S^n x - T^n y\| &\leq \textstyle\sum_{n=0}^{\infty} \sup\{t \geq 0 : F_{x-y}(\tfrac{t}{c_n}) < 1\} \\ &= \textstyle\sum_{n=0}^{\infty} c_n \|x-y\| \\ &< \infty, \end{aligned}$$

and by applying proposition 5.2, the proof is complete. □

Note 2 : In the above theorem we see that the condition of continuity on S and T is not imposed. We can easily show that S and T are continuous with the topology of $(X,\ F,\ L,\ T,\ S)$ which is generally weaker than the ordinary norm topology on X.

The proof of theorem 5.1 : Since $F_x(t)$ and $L_x(t)$ are increasing and non increasing in t, respectively, for all $t \geq 0$ and $x \in X$, a simple calculation shows that

$$\begin{aligned} F_{Mx(s)-My(s)}(t) &= F_{\int_G K(s,t,x(t))dt}(t) \\ &= F_{\alpha x(s))dt}(t) \\ &= F_{x(s)}(\tfrac{t}{|\alpha|}) \\ &\geq F_{x(s)}(\tfrac{t}{c}), \end{aligned}$$

for some $c \in (0,1)$. Hence we have

$$F_{Mx-My}(ct) \geq F_{x-y}(t). \tag{12}$$

Also the same argument shows that

$$L_{Mx-My}(ct) \leq L_{x-y}(t), \tag{13}$$

for all $x, y \in X$, $t \geq 0$. Now by applying theorem 5.6, the proof is complete. □

Note 3 : Suppose that $(X,\ \|\ .\ \|_o)$ be a Banach space and

$$F_x(t) = \mathcal{H}(t - \|x\|_o) \text{ and } L_x(t) = \mathcal{G}(t - \|x\|_o),$$

for all $t \geq 0$. If we consider the intuitionistic Menger normed space $(X,\ F,\ L,\ T,\ S)$, for any continuous t-norm T and any continuous t-conorm S, respectively, then by lemma 5.5 the topology of normed space is the same as its induced topology on $(X,\ F,\ L,\ T,\ S)$. On the other hand if $M : X \longrightarrow X$ be a self mapping and there exists a number $0 < c < 1$ such that

$$\|\ Mx - My\ \|_o \leq c\ \|\ x - y\ \|_o,\ \forall x, y \in X,$$

then we see that

$$F_{Mx-My}(ct) \geq F_{x-y}(t) \text{ and } L_{Mx-My}(ct) \leq L_{x-y}(t),$$

for all $x, y \in X$ and $t \geq 0$. If we apply the notation used in lemma 5.4 and see the first part of theorem 5.6, we imply that

$$\sum_{n=1}^{\infty} \|\ M^n x - M^n y\ \| < \infty,\ \forall x, y \in X.$$

Hence all conditions of theorem 5.6 hold and so M has a unique fixed point.

Conclusion : In this paper we generalized the definition of contractions in Menger spaces, intuitionistic Menger spaces and we proved a generalized fixed point theorem which can be helpful to extend the fixed point theory. We defined the intuitionistic 2-Menger spaces which probably apply as an effective instrument to interpret some notions in Physics. Also we could apply the probabilistic fixed point theory to gain some results in integral equations and Banach spaces.

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Received: August, 2010