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# Interpolation of n-Tuple Banach Spaces on $R^n$

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**Abstract**

Let $\bar{A} = (A_1, A_2, \cdots, A_n)$ be a compatible n-tuple of Banach spaces. We may define the interpolation method in $R^n$, and prove some related lemma and theorem.

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**Keywords:** n-tuple of Banach space, interpolation space

# 1 Introduction

Our main references to the theory of interpolation space are [1,5]. Let $\bar{A} = (A_1, A_2, \cdots, A_n)$ be a compatible n-tuple of Banach spaces. We define the space $F(\bar{A})$ of all function f with values in $\sum(\bar{A})$, which are bounded and continuous on $\overline{S} = \{x = (x_1, x_2, \cdots, x_n) \in R^n \ : \ 0 \leq x_1 \leq 1\}$ and analytic on $S = \{x = (x_1, x_2, \cdots, x_n) \in R^n \ : \ 0 < x_1 < 1\}$. And moreover, the functions $t \mapsto f(0, a_2, a_3, \cdots, a_{n-2}, t)(a_j = 0, 1 \ and \ j = 2, 3, \cdots, n-2)$ and $t \mapsto f(1, \cdots, 1, t)$ are continuous function from the real line into $\sum(\bar{A})$, which tend to zero as $|t| \longrightarrow \infty$. Clearly, $F(\bar{A})$ is a vector space. We provide $F$ with the norm

$$\begin{aligned} \|f\|_F &= max(sup\|f(0, \cdots, 0, t)\|_{A_1}, sup\|f(0, 1, 0, \cdots, 0, t)\|_{A_2} \\ &, \ sup\|f(0, 0, 1, 0, \cdots, 0, t)\|_{A_3}, \cdots, sup\|f(0, 0, 0, \cdots, 0, 1, t)\|_{A_{n-1}} \\ &, \ sup\|f(1, 1, \cdots, 1, t)\|_{A_n}). \end{aligned}$$

# 2 Main Results

**Theorem 2.1** *The space $\mathcal{F}$ is a Banach space.*

**Proof.**
Suppose that $\sum_n \|f_n\|_{\mathcal{F}} < \infty$. Since $f_n(x)$ is bounded in $\sum(\bar{A})$, we have

$$\begin{aligned}\|f_n(x)\|_{\sum(\bar{A})} &\leq max(sup\|f_n(0,0,\cdots,0,t)\|_{\sum(\bar{A})}, sup\|f(0,1,0,\cdots,0,t)\|_{\sum(\bar{A})} \\ &, \ sup\|f(0,1,0,\cdots,0,t)\|_{\sum(\bar{A})},\cdots, sup\|f(0,0,\cdots,0,1,t)\|_{A_{n-1}} \\ &, \ sup\|f_n(1,1,\cdots,t)\|_{\sum(\bar{A})})\end{aligned}$$

Since $A_j \subset \sum(\bar{A})$, we conclude that

$$\|f_n(x)\|_{\sum(\bar{A})} \leq \|f_n\|_{\mathcal{F}}.$$

Since $\sum(\bar{A})$ is a Banach space. it follows that $\sum_n f_n$ converges uniformly on S to a function f in $\sum(\bar{A})$. Thus f is bounded and continuous on S and analytic in $\bar{S}$. Furthermore, $\|f_n(0,\cdots,0,t)\|_{A_1} \leq \|f_n\|_{\mathcal{F}}$, $\|f_n(0,1,0,\cdots,0,t)\|_{A_2} \leq \|f_n\|_{\mathcal{F}}$, $\|f_n(0,0,1,0,\cdots,0,t)\|_{A_3} \leq \|f_n\|_{\mathcal{F}},\cdots,\|f_n(0,0,0,\cdots,0,1,t)\|_{A_{n-1}} \leq \|f_n\|_{\mathcal{F}}$ and $\|f_n(f(1,1,\cdots,1,t)\|_{A_n} \leq \|f_n\|_{\mathcal{F}}$. Thus $\sum_n f_n(0,\cdots,0,t)$, $\sum_n f_n(0,1,0,\cdots,0,t),\cdots,\sum_n f_n(0,\cdots,0,1,t)$ and $\sum_n f_n(1,\cdots,1,t)$ converges uniformly in $t$ to a limit in $A_1, A_2,\cdots,A_{n-1}$ and $A_n$, which must coincide with the limit in $\sum(\bar{A})$. Therefore, $f(0,0,\cdots,t) \in A_1$, $f(0,1,0,\cdots,0,t) \in A_2,\cdots,f(0,0,\cdots,0,1,t) \in A_{n-1}$, $f(1,1,\cdots,1,t) \in A_n$ and $\sum_n f_n(0,\cdots,0,t)$, $\sum_n f_n(0,1,0,\cdots,0,t),\cdots,\sum_n f_n(0,\cdots,0,1,t)$, $\sum_n f_n(1,\cdots,1,t)$ converges uniformly to $f(0,0,\cdots,0,t)$, $f(0,1,0,\cdots,0,t),\cdots,f(0,0,\cdots,0,1,t)$ in $A_1, A_2,\cdots,A_{n-1}$ and $f(1,1,\cdots,1,t)$ in $A_n$. But then it follows that $f = \mathcal{F}$, and that $\sum_n f_n$ converges to f in $\mathcal{F}$.

Let $\bar{A} = (A_0, A_1)$ be a quasi-Banach couple, let $0 < \theta < 1$ and $0 < q \leq \infty$. The real interpolation space $(A_0, A_1)_{\theta,q}$ consist of all elements $a \in A_0 + A_1$ having a finite quasi-norm

$$\|a\|_{\theta,q,} = \begin{cases} (\sum_{\nu\in Z}(2^{-\nu\theta}K(2^\nu,a))^q)^{1/q} & if\ 0 < q < \infty \\ sup_{\nu\in Z}\{2^{-\nu\theta}K(2^\nu,a)\} & if\ q = \infty \end{cases}.$$

Here, for $0 < t < \infty$, we put

$$K(t,a) = K(t,a;A_0,A_1) = inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1}\ :\ a = a_0 + a_1, a_j \in A_j\}.$$

**Theorem 2.2** *Let $(A_1, A_2)$ be a quasi-Banach couple that $A_1, A_2$ is $c_1, c_2$ normed with $c_2/c_1 \leq 1$.Then the following identities hold.*

$$(A_0 + A_1, A_1)_{\theta,q} \cap A_0 = (A_0, A_1)_{\theta,q} \cap A_0 = (A_0, A_0 \cap A_1)_{\theta,q}. \quad (1)$$

$$(A_0 \cap A_1, A_1)_{\theta,q} + A_0 = (A_0, A_1)_{\theta,q} + A_0 = (A_0, A_0 + A_1)_{\theta,q}. \quad (2)$$

**Proof.**
Let us prove the identity (1). The chain of inclusions " $\supset$ " is clear, whence we have to show $(A_0 + A_1, A_1)_{\theta,q} \cap A_0 \subset (A_0, A_0 \cap A_1)_{\theta,q}$. Take $a_0 \in (A_0 + A_1, A_1)_{\theta,q} \cap A_0$. Since $a_0 \in A_0$, only the behaviour of $K(t, a_0; A_0, A_0 \cap A_1)$ on $(0, 1)$ matters.

$$K(t, a_0; A_0, A_0 \cap A_1) \leq (c_0 + 1)K(t, a_0; A_0, A_1) + c_0 t\|a_0\|_{A_0}$$

$$= (c_0 + 1)tK(t^{-1}, a_0; A_1, A_0) + c_0 t\|a_0\|_{A_0}$$

$$= (c_0 + 1)tK(t^{-1}, a_0; A_1, A_0 + A_1) + c_0 t\|a_0\|_{A_0}$$

$$= (c_0 + 1)K(t, a_0; A_0 + A_1, A_1) + c_0 t\|a_0\|_{A_0}$$

also

$$\|a_0\|_{A_0, A_0 \cap A_1} \leq (\sum_{\nu \in Z} (2^{-\nu\theta} K(2^\nu, a))^q)^{1/q})$$

$$\leq (\sum_{\nu \leq 0} ((C_0 + 1)2^{-\nu\theta} K(2^\nu, a))^q)^{1/q} + (\sum_{\nu \leq 0} (c_0 2^{-\nu\theta} \|a_0\|_{A_0})^q)^{1/q}$$

$$\leq (c_0 + 1)[\|a_0\|_{A_0+A_1, A_1} + \|a_0\|_{A_0}]$$

Now, the identity (1)follows.

To prove the identity (2), we note as before that one chain of inclusions is trivial. Take $a \in (A_0, A_0 + A_1)_{\theta,q}$ and write $a = a_0 + a_1$ with $a_0 \in A_0, a_1 \in A_1$.We have

$$K(t, a_1; A_0, A_1) \leq c_0[K(c_1 t/c_0, a; A_0, A_1) + K(c_1 t/c_0, a_0; A_0, A_1)]$$

$$\leq c_0[K(c_1 t/c_0, a; A_0, A_1) + \|a_0\|_{A_0}]$$

for $t \geq 1, c_1/c_0 \leq 1$

$$\leq c_0[K(t, a; A_0, A_1) + \|a_0\|_{A_0}]$$

$$\leq c_0[K(t, a; A_0, A_0 + A_1) + \|a_0\|_{A_0}]$$

Then

$$\|a_1\|_{A_0, A_1} \leq c_0[\|a\|_{A_0, A_0 + A_1} + \|a_0\|_{A_0}]$$

And $K(t, a_1; A_0, A_1) \leq t\|a_1\|_{A_1}$ for $t \leq 1$. then $\|a_1\|A_0, A_1 \leq \|a_1\|_{A_1}$. Hence we have $a_1 \in (A_0, A_1)_{\theta,q}$.

In the proof of the next result we will have occasion to use the so called modular law for vector subspace A, B, C of a vector space Z:

$$B \subset C \Rightarrow (A + B) \cap C = (A \cap C) + B.$$

The proof of this fact is trivial.

We shall now define the interpolation functor $\bar{A}_{[\theta]}$. The space $\bar{A}_{[\theta]}$ consists of all $a \in \sum(\bar{A})$ such that $a = f(\theta)((\theta, 0, 0, \cdots, 0) = \theta)$ for some $f \in \mathcal{F}(\bar{A})$. The norm on $\bar{A}_{[\theta]}$ is

$$\|a\|_{[\theta]} = inf\{\|f\|_{\mathcal{F}} : \ f(\theta) = a, \ f \in \mathcal{F}\}.$$

**Theorem 2.3** *Let $\bar{A}$ be a compatible n-tuple Banach spaces and put*

$$X_j = \bar{A}_{[\theta_j]} \ \ (0 \leq \theta_j \leq 1; \ j = 1, 2, \cdots, n).$$

*Assume that $\triangle(\bar{A})$ is dense in the spaces $A_j$ and $\triangle(\bar{X})$. Then*

$$\bar{A}_{[\theta]} \subset \bar{X}_{[\theta]}.$$

*where $\theta = (1 - \sum_{i=2}^{n} c_i)\theta_1 + \sum_{i=2}^{n} c_i\theta_i$.*

**Proof.**
First we show that $\|a\|_{\bar{X}_{[\sum_{i=2}^{n} c_i]}} \leq \|a\|_{\bar{A}_{[\theta]}}$ if $a \in \bar{A}_{[\theta]}$. Take $a \in \bar{A}_{[\theta]}$ then there exists a function $f \in \mathcal{F}(\bar{A})$, such that $f(\theta) = a$ and $\|f\|_{\mathcal{F}} \leq \|a\|_{\bar{A}_{[\theta]}} + \varepsilon$, $\varepsilon > 0$ being arbitrary. Put $f_1(x) = f((1-x)\theta_1 + x\sum_{j=2}^{n}\theta_j - \sum_{i=2, i\neq j}^{n} c_i \sum_{j=2}^{n}\theta_j)$. Then $f_1(\sum_{i=2}^{n} c_i) = a$ and

$$\|f_1\|_{\mathcal{F}(\bar{X})} \leq \|a\|_{\bar{A}_{[\theta]}}$$

This gives $\|a\|_{\bar{X}_{[\theta]}} \leq \|a\|_{\bar{A}_{[\theta]}}$.

**Definition 2.4** *The interpolation space $A$ and $B$ are exact of exponent $\theta$ and $\eta$ with $\eta = \sum_{i=3}^{n} \eta_i$, $0 < \eta_i < 1$ and $0 \leq \theta + \eta \leq 1$ if*

$$\|T\|_{A,B} \leq \|T\|_{A_1,B_1}^{1-(\eta+\theta)} \|T\|_{A_2,B_2}^{\theta} \|T\|_{A_3,B_3}^{\eta_3} \cdots \|T\|_{A_i,B_i}^{\eta_i} \cdots \|T\|_{A_n,B_n}^{\eta_n}.$$

**Theorem 2.5** *The space $\bar{A}_{[\theta]}$ is a Banach space. The functor $\bar{A}_{[\theta]}$ is an exact interpolation functor of exponent $\theta$ and $\eta$.*

**Proof.**
The linear mapping $f \longrightarrow f(\theta)$ is a continuous mapping from $\mathcal{F}(\bar{A})$ to $\sum(\bar{A})$ since $\|f(\theta)\|_{\sum(\bar{A})} \leq \|f\|_{\mathcal{F}}$. The kernel of this mapping is $\mathcal{N}_\theta = \{f : \ f \in \mathcal{F}, \ f(\theta) = 0\}$. Clearly, $\bar{A}_{[\theta]}$ is isomorphic and isometric to the quotient space $\mathcal{F}(\bar{A})/\mathcal{N}_\theta$. Since $\mathcal{N}_\theta$ is closed, it follows that $\bar{A}_{[\theta]}$ is a Banach space. Assume that T maps $A_j$ to $B_j$ with norm $M_j(j = 1, 2, \cdots, n)$. Given $a \in \bar{A}_{[\theta]}$ and $\varepsilon > 0$,

there is a function $f \in \mathcal{F}$, such that $f(\theta) = a$ and $\|f\|_{\mathcal{F}} \leq \|a\|_{[\theta]}+\varepsilon$. Put $g(x) = M_1^{x-1}M_2^{-x+\eta}M_3^{-x+\theta+\sum_{i=4}^{n}\eta_i} \cdots M_j^{-x+\theta+\sum_{i=3,i\neq j}^{n}\eta_i} \cdots M_n^{-x+\theta+\sum_{i=3}^{n-1}\eta_i}T(f(x-\eta))$. g belongs to the class $\mathcal{F}(\bar{B})$. Moreover, $\|g\|_{\mathcal{F}} \leq \|f\|_{\mathcal{F}} \leq \|a\|_{[\theta]} + \varepsilon$. But now $g(\theta + \eta) = M_1^{\theta+\eta-1}M_2^{-\theta}M_3^{-\eta_3} \cdots M_j^{-\eta_j} \cdots M_n^{-\eta_n}T(a)$ and hence we conclude that $\|T(a)\|_{[\theta]} \leq M_1^{1-\theta+\eta}M_2^{\theta}M_3^{\eta_3} \cdots M_j^{\eta_j} \cdots M_n^{\eta_n}\|g\|_{\mathcal{F}} \leq M_1^{1-\theta+\eta}M_2^{\theta}M_3^{\eta_3} \cdots M_j^{\eta_j} \cdots M_n^{\eta_n}\|a\|_{[\theta]} + \acute{\varepsilon}$, where $\acute{\varepsilon} = M_1^{1-\theta+\eta}M_2^{\theta}M_3^{\eta_3} \cdots M_j^{\eta_j} \cdots M_n^{\eta_n}\varepsilon$. This gives the result.

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