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Matrix Transformations on BS-Spaces in p-Adic Analysis

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Abstract

In this work we give a generalization of the Toeplitz-Silverman and the Kojima-Schur theorems for sequence spaces over non archimedean BS-spaces. We define others matrix transformations and establish some results which characterize these matrices.

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1 Introduction

The classical study concerning the characterization of matrix transformations between different sequence spaces, in the complex case, is based over the Kojima-Schur theorem ([3], theorem 2.7, p. 10) for the conservative matrices, and over the Toeplitz-Selverman theorem ([3], theorem 2.9, p. 10) for the regular matrices. These two theorems are generalized, for the sequence spaces over a Banach spaces, by Robinson [10] and Melvin-Melvin [4], afterwords for the sequence spaces over a Frechet spaces by Ramanujan [6]. The Kojima-Schur theorem has been generalized by Wu Junde, Kim Dohan and Cho Minhyung for the sequence spaces over a barrelled spaces ([13] theorem1, p. 286). The version of these two theorems in case of sequence spaces over a non archimedean valuated field K is gave by Rangachari and Srinivasan [7].

In this work we give a generalization of these two theorems for the sequence spaces over a non archimedean locally $K-$ convex BS-spaces thanks to a family of quasi-seminorms that we define in this paper. We are also define at the same other matrix transformations and establish others results characterizing them.

Throughout this paper, K is a non-archimedean $(n.a)$ non trivially valued complete field with valuation $|.|$, (X, τ_X) and (Y, τ_Y) are two n.a locally $K-$ convex spaces, $(\mathcal{P})$ and $(\mathcal{Q})$ are two family of n.a seminorms which define the topologies τ_X and τ_Y respectively.

A Hausdorff topological vector space (TVS) X has the Banach-Steinhaus Property or is a $BS-$ space (respectively is a $wBS-$ space) if for all locally $K-$ convex space Y and all sequence $(f_k)_k \subset B(X,Y)$ we have $(f_k)_k$ is equicontinuous if and only if it is pointwise bounded (respectively $\lim_k f_k(x) = f(x)$ for each $x \in X$ imply that $f \in B(X,Y)$). For example, all $F-$ space or barrelled space or $\mathcal{N}_0-$ barrelled space or $l^\infty-$ barrelled space is a $BS-$ space. We have also each $BS-$ space is an $wBS-$ space.

$(\omega(X), \tau_\omega(X)) :=$ the linear space of all sequences in X endowed with the product topology $\tau_\omega(X)$ which is generated by the family of n.a semi-norms $(p_n)_{n\in I\!N, p\in(\mathcal{P})}$ which is defined by $p_n(\overline{x}) = p(x_n)$ for all $\overline{x} = (x_n)_n \in \omega(X)$ and all $p \in (\mathcal{P})$. This space is noted $\omega(K)$ or ω in case when $X = K$.

A sequence space on X is a subspace of $\omega(X)$, we define the following sequence spaces over X

$c_0(X) := \{(x_k)_k \in \omega(X) : (x_k)_k \textit{ converges to } 0 \textit{ in } X\}$,

$\varphi(X) := \{(x_k)_k \in \omega(X) : \textit{ there exists } k_0 \in I\!N \ : \ x_k = 0 \textit{ for all } k \geq k_0\}$,

$c(X) := \{(x_k)_k \in \omega(X) : (x_k)_k \textit{ converges in } X\}$,

$m(X) := \{(x_k)_k \in \omega(X) : (x_k)_k \textit{ is bounded in } X\}$.

Over $m(X)$ we define the sequence of n.a semi-norms $(\overline{p})_{p\in(\mathcal{P})}$ by :

$\overline{p}(\overline{x}) = \sup_k \ p(x_k)$ for all $\overline{x} = (x_k)_k \in m(X)$ and all $p \in (\mathcal{P})$.

Let τ_∞ the topology on $m(X)$ which is defined by the sequence of $n.a$ semi-norms $(\overline{p})_{p\in(\mathcal{P})}$. Throughout $c_0(X)$ is equipped with the τ_∞ topology.

Proposition 1 *lim* :$(c(X), \tau_\infty) \longrightarrow (X, \tau)$ *is continuous.*

Proof. Let $\overline{x} = (x_j)_j \in c(X)$, then for all $p \in (\mathcal{P})$ we have:

$$p(\lim \overline{x}) = \lim_j p(x_j)$$

$$\leq \sup_j p(x_j)$$

$$\leq \overline{p}(\overline{x}).$$

$\square$

A sequence space (E,τ_E) on X, is called $X-$ space if the inclusion map $:(E,\tau_E) \longrightarrow (\omega(X),\tau_\omega(X))$ is continuous (it is equivalent to the continuity of the canonical projections $\pi_n : (E,\tau_E) \longrightarrow X,\ \overline{x}=(x_j)_j \longmapsto x_n$ for all $n \in I\!N$).

If (E,τ_E) contains $\varphi(X)$, we put $\delta_n^X : X \longrightarrow E,\ x \longmapsto (0,0,...,x,0,0,...)$, where x is in the nth position and $\delta^X : X \longrightarrow \omega(X),\ x \longmapsto (x,x,...)$; if (E,τ_E) is a $X-$ space, then δ^X and δ_n^X are a continuous linear mapping for every $n \in I\!N$.

A sequence space (E,τ_E) which contains $\varphi(X)$ is called an $AX-$ space if it is an $X-$ space and for all $\overline{x}=(x_n)_n$ in E the sequence $\left(x^{[n]}\right)_n$ converges to $\overline{x}$ in (E,τ_E) $\left(i.e.\ \overline{x}=\sum_{n=1}^{\infty}\delta_n^X(x_n)\right)$, where $x^{[n]}=\sum_{k=1}^{n}\delta_k^X(x)$ for all $n\in I\!N$. It is obvious that $(c_0(X),\tau_\infty)$ is an $AX-$ space.

A pairs (X,Y) of locally $K-$ convex spaces has the $I-$ property if for each linear map $f:(c_0(X),\tau_\infty)\longrightarrow Y$, we have: f is continuous if and only if for all $i\in I\!N$, $f\circ\delta_i^X$ is continuous.

A sequence $(f_n)_n$ converges to f in $B(X,Y)$ if, for all x in X $\lim_n f_n(x)=f(x)$ in (Y,τ_Y).

Proposition 2 *Let X and Y be two locally $K-$ convex spaces, then for all f in $B(c_0(X),Y)$ there exists an unique sequence $(f_n)_n$ in $B(X,Y)$ such that $f(\overline{x})=\sum_{n=1}^{\infty}f_n(x_n)$ for all $\overline{x}=(x_n)_n\in c_0(X)$. Conversely, if (X,Y) has the $I-$ property then any such sequence for which the series is convergent for all $\overline{x}$ in $c_0(X)$ defines an element of $B(c_0(X),Y)$.*

Proof. Let $f\in B(c_0(X),Y)$ and for each $i\in I\!N$ let $f_i=f\circ\delta_i^X$, then for any $\overline{x}=(x_n)_n\in c_0(X)$ we have:

$$\begin{aligned} f(\overline{x}) &= f\left(\lim_k x^{[k]}\right)\\ &= \lim_k f\left(x^{[k]}\right)\\ &= \lim_k \sum_{n=1}^{k} f_n(x_n)\\ &= \sum_{n=1}^{\infty} f_n(x_n). \end{aligned}$$

If also $f(\overline{x})=\sum_{n=1}^{\infty}g_n(x_n)$, $g_n\in B(X,Y)$ for all $n\in I\!N$, then for any x in X and any $k\in I\!N$ we have:

$$\begin{aligned} f_k(x) &= f\circ\delta_k^X(x)\\ &= f\left(\delta_k^X(x)\right)\\ &= g_k(x). \end{aligned}$$

Thus the representation is unique.

Conversely, suppose that $f_n\in B(X,Y)$ for all $n\in I\!N$ such that $\sum_{n\geq1}f_n(x_n)$ is convergent to $f(\overline{x})$ in Y for all $\overline{x}=(x_n)_n\in c_0(X)$.

So f is a linear map and for all $k\in I\!N$ and all $x\in X$ $f\circ\delta_k^X(x)=f\left(\delta_k^X(x)\right)=f_k(x)$, then for all $k\in I\!N$ $f\circ\delta_k^X$ is continuous and so f is continuous because (X,Y) has the $I-$ property. $\square$

Proposition 3 *Let X and Y be two locally $K-$ convex spaces such that (X,Y) has the $I-$ property then:*

if X is a wBS space then $c_0(X)$ is a $wBS-$ space.

Proof. Let $(f_n)_n$ be a sequence in $B(c_0(X),Y)$ such that for all $\overline{x} \in c_0(X)$ $\lim_n f_n(\overline{x}) = f(\overline{x})$ in Y.

For each $n \in I\!N$, each $i \in I\!N$ and each x in X we have:

$f_n \circ \delta_i^X(x) - f \circ \delta_i^X(x) = f_n\left(\delta_i^X(x)\right) - f\left(\delta_i^X(x)\right)$ and so $\lim_n f_n \circ \delta_i^X(x) = f \circ \delta_i^X(x)$ in Y. Then $f \circ \delta_i^X \in B(X,Y)$ for all $i \in I\!N$. Therefore $f \in B(X,Y)$ since (X,Y) has the $I-$ property. □

Let E and F be two locally $K-$ convex sequence spaces over X and Y respectively and $T : E \longrightarrow F$ be a linear map, T is called a matrix transformation (or matricial operator) if there exists an infinite matrix $A = (A_{ij})_{i,j}$ such that $A_{ij} \in B(X,Y)$ for every $i,j \geq 1$, $E \subset \omega_T$ and for every $\overline{x} = (x_j)_j \in E$ $T\overline{x} = A\overline{x} = \left(\sum_{j\geq 1} A_{ij}x_j\right)_i$, where $\omega_T = \left\{(x_j)_j \in \omega(X) : \sum_{j\geq 1} A_{ij}x_j \textit{ converges in } Y \textit{ for all } i\right\}$, which is called domain of application of T.

We define the following sequence spaces:

$c_T := \{\overline{x} \in \omega_T : T\overline{x} \in c(Y)\}$,

$c_{0T} := \{\overline{x} \in \omega_T : T\overline{x} \in c_0(Y)\}$,

$m_T := \{\overline{x} \in \omega_T : T\overline{x} \in m(Y)\}$.

The topology of ω_T is given by the semi-norms:

$\{p_n : n \in I\!N,\ p \in (\mathcal{P})\} \cup \{r_{qi} : i \in I\!N,\ q \in (\mathcal{Q})\}$,

where $r_{qi}(\overline{x}) = \sup_m q\left(\sum_{j=1}^m A_{ij}x_j\right)$ for all $\overline{x} = (x_j)_j \in \omega_T$ and the topology of c_T is given by:

$\{p_n : n \in I\!N,\ p \in (\mathcal{P})\} \cup \{r_{qi} : i \in I\!N,\ q \in (\mathcal{Q})\} \cup \{\overline{q}_T : q \in (\mathcal{Q})\}$,

where $\overline{q}_T(\overline{x}) = \sup_i q\left(\sum_{j=1}^\infty A_{ij}x_j\right)$ for all $\overline{x} = (x_j)_j \in c_T$.

It is easily to see that if $\lambda(X)$ is any one of $\omega(X), m(X), c(X), c_0(X), \omega_T, c_T$, then every $\pi_n : \lambda(X) \longrightarrow X$ is continuous.

The next result generalizes the proposition established by L. W. Baric in the classical case ([1], proposition 2.7, p. 167).

Proposition 4 *Let T be a matricial operator which takes $E(X)$ an BS and $X-$ space into $\left(E(Y), \tau_\omega(Y)_{/E(Y)}\right)$.*

Then T is continuous from $E(X)$ into $E(Y)$.

Proof. For each $i \in I\!N$ and each $\overline{x} = (x_j)_j \in E(X)$, let $T_i(\overline{x}) = \lim_m \sum_{j=1}^m A_{ij}x_j = \lim_m \sum_{j=1}^m A_{ij} \circ \pi_j(\overline{x})$.

Each T_i is thus continuous because $E(X)$ is an $BS-$ space and $X-$ space.

Now for all $q \in (\mathcal{Q})$, all $i \in I\!N$ and all $\overline{x} = (x_j)_j \in E\ (X)$ we have:
$$q_i(T\overline{x}) = q_i\left(\left(\sum_{j=1}^{\infty} A_{ij}x_j\right)_i\right)$$
$$= q\left(\sum_{j=1}^{\infty} A_{ij}x_j\right)$$
$$= q(T_i\overline{x})$$
T_i is continuous, then the proposition holds. □

δ_n^X is continuous into ω_T, even if T is not continuous.

Proposition 2 is also true if we replace $c_0(X)$ by ω_T, since $\left(x^{[n]}\right)_n$ converges to $\overline{x}$ in ω_T.

1. T is called:
(a). conservative, if $c(X) \subset c_T$,
(b). conservative for the null sequences, if $c_0(X) \subset c_T$,
(c). null conservative, if $c(X) \subset c_{0T}$,
(d). regular for the null sequences, if $c_0(X) \subset c_{0T}$,
(e). coercive, if $m(X) \subset c_T$,
(f). null coercive, if $m(X) \subset c_{0T}$,
(g). $c_0(X)-$ permanent, if $c_0(X) \subset m_T$,
(h). $c(X)-$ permanent, if $c(X) \subset m_T$,
(i). $m(X)-$ permanent, if $m(X) \subset m_T$,
2. Let $f \in B(X,Y)$, T is called $f-$ regular if T is conservative and for all $\overline{x} \in c(X)$ $\lim T\ \overline{x} = f\ (\lim \overline{x})$.
3. If $X = Y$, T is said to be :
j. $S-$ regular if T is conservative for the null sequences and for all $\overline{x} = (x_j)_j \in c_0(X)$ $\lim T\ \overline{x} = \sum_{j=1}^{\infty} x_j$,
k. regular if it is conservative and for all $\overline{x} \in c(X)$ $\lim\ T\ \overline{x} = \lim \overline{x}$.

Let X be a K topological vector space; a map p: $X \longrightarrow [0\ ,+\infty]$ is called a quasi-seminorm if
i. $p(\lambda x) = |\lambda|\ p(x)$ for every λ in K and x in X,
ii. $p(x+y) \leq \max(p(x), p(y))$ for every x and y in X.

Over $B(X,Y)$ and $\omega(B(X,Y))$, we define respectively the family of quasi-seminorms n.a $\left(\|.\|_{p,q}\right)_{(p,q)\in(\mathcal{P})\times(\mathcal{Q})}$ by :
$\|f\|_{p,q} = \sup\{q(f(x)) : p(x) \leq 1\}$ for all $f \in B(X,Y)$ and all $(p,q) \in (\mathcal{P}) \times (\mathcal{Q})$,
$\|(f_k)_k\|_{p,q} = \sup\{q\left(\sum_{k=1}^{n} f_k x_k\right) : p(x_k) \leq 1\ for\ all\ k,\ 1 \leq k \leq n\}$ for all $(f_k)_k \in \omega(B(X,Y))$ and all $(p,q) \in (\mathcal{P}) \times (\mathcal{Q})$.

We have the following propositions:

Proposition 5 *Let* $f \in B(X,Y)$ *and* $(p,q) \in (\mathcal{P}) \times (\mathcal{Q})$ *such that* $N_p \subset N_K$ *and* $\|f\|_{p,q} \prec \infty$ *then*

(i). $\|f\|_{p,q} = \sup\left\{\frac{q(f(x))}{p(x)} : p(x) \neq 0\right\}$

(ii). $q(f(x)) \leq \|f\|_{p,q}\, p(x)$ *for every* x *in* X.

Where $N_K = \{|\lambda| : \lambda \in K\}$ *and* $N_p = \{p(x) : x \in E\}$.

Proof. **(i).** Put $\rho_{p,q}(f) = \sup\left\{\frac{q(f(x))}{p(x)} : p(x) \neq 0\right\}$ and let $x \in E$ such that $p(x) \neq 0$, then there exists $\lambda \in K$ such that $p(x) = |\lambda|$, so $p(\frac{x}{|\lambda|}) \leq 1$ and then $q\left(f\left(\frac{x}{|\lambda|}\right)\right) \leq \|f\|_{p,q}$, thus $\frac{q(f(x))}{p(x)} \leq \|f\|_{p,q}$ and therefore $\rho_{p,q}(f) \leq \|f\|_{p,q}$.

Let $x \in E$ such that $p(x) \leq 1$, then:

If $p(x) = 0$, then if $\|f\|_{p,q} = 0$, $\|f\|_{p,q} \leq \rho_{p,q}(f)$, and if not, let $\rho \succ 1$ such that there exists $(\lambda_r)_r \in \omega(K)$ that verifies $|\lambda_r| = \rho^r$ for all $r \in I\!N$ ([5], p. 30). For all $r \in I\!N$ $p(\lambda_r x) = 0$ then $q(f(\lambda_r x)) \leq \|f\|_{p,q}$ therefore for all $r \in I\!N$ $q(f(x)) \leq \frac{\|f\|_{p,q}}{\rho^r}$ so $q(f(x)) = 0$. If $p(x) \neq 0$, then $q(f(x)) \leq \frac{q(f(x))}{p(x)}$ so $\|f\|_{p,q} \leq \rho_{p,q}(f)$.

(ii). Let $x \in E$, if $p(x) = 0$, then by **(i)**, if we suppose that $\|f\|_{p,q} \neq 0$ we have $q(f(x)) = 0$ and the inequality is satisfied, if $p(x) \neq 0$, we have $\frac{q(f(x))}{p(x)} \leq \|f\|_{p,q}$ then $q(f(x)) \leq \|f\|_{p,q}\, p(x)$. □

Proposition 6 *Let* $(f_k)_k \in \omega(B(X,Y))$, *then for every* $p \in (\mathcal{P})$ *and* $q \in (\mathcal{Q})$ *we have* $\|(f_k)_k\|_{p,q} = \sup_k \|f_k\|_{p,q}$.

Proof. Let $n \in I\!N^*$ and $x_1, x_2, ..., x_n \in B_p(0,1)$, then we have: $q\left(\sum_{k=1}^n f_k x_k\right) \leq \max_{1\leq k\leq n} q(f_k x_k) \leq \max_{1\leq k\leq n} \|f_k\|_{p,q}$, and so $\|(f_k)_k\|_{p,q} \leq \sup_k \|f_k\|_{p,q}$. On the other hand, for all $k \geq 1$ $\|f_k\|_{p,q} \leq \|(f_k)_k\|_{p,q}$ and then $\sup_k \|f_k\|_{p,q} \leq \|(f_k)_k\|_{p,q}$. □

Let E be a non empty subset of $\omega(X)$, the generalize $\beta-$ dual of E is the set noted by E^β and defined as follow:

$$E^\beta = \left\{(f_j)_j \in \omega(B(X,Y)) : \sum_{j\geq 1} f_j x_j \textit{ converges in } Y \textit{ for all } (x_j)_j \in E\right\}.$$

Proposition 7 *Let* $(f_k)_k \in \omega(B(X,Y))$ *such that* Y *be sequentially complete and* $c_0(X)$ *is a* $BS-$ *space, then* $(f_k)_k \in c_0(X)^\beta$ *if and only if for all* $q \in (\mathcal{Q})$ *there exists* $p \in (\mathcal{P})$ *such that* $\|(f_k)_k\|_{p,q} \prec \infty$.

Proof. If $(f_k)_k \in c_0(X)^\beta$ then the linear map: $c_0(X) \longrightarrow Y$, $(x_k)_k \longmapsto \sum_{k=1}^\infty f_k x_k$ is continuous, and so for all $q \in (\mathcal{Q})$ there exists $p \in (\mathcal{P})$ and there exists $R \succ 0$ such that $q\left(\sum_{k=1}^\infty f_k x_k\right) \leq R\overline{p}(\overline{x})$ for all $\overline{x} = (x_k)_k \in c_0(X)$; then

$\|(f_k)_k\|_{p,q} \prec \infty$.
Conversely, let $\overline{x} = (x_k)_k \in c_0(X)$ and let $q \in (\mathcal{Q})$, then there exists $p \in (\mathcal{P})$ such that $\|(f_k)_k\|_{p,q} \prec \infty$.
But we have $q(f_k x_k) \leq \|(f_k)_k\|_{p,q} p(x_k) \overset{k\to\infty}{\longrightarrow} 0$; then $\sum_k f_k x_k$ converges in Y. □

corollary 1 *Under the conditions of proposition before, if $(f_k)_k \in \omega(B(X,Y))$, then $(f_k)_k \in c(X)^\beta$ if, and only if the two conditions below are met*
(i). *For all $q \in (\mathcal{Q})$ there exists $p \in (\mathcal{P})$ such that $\|(f_k)_k\|_{p,q} \prec \infty$,*
(ii). $\lim_k f_k = 0$.

Proof. $(f_k)_k \in c(X)^\beta$ then $(f_k)_k \in c_0(X)^\beta$ and so we have **(i)**, on the other hand for all $x \in X$ $\sum_k f_k x$ converges in Y, then $\lim_k A_k x =, 0$ and so $\lim_k f_k = 0$.
Conversely, let $\overline{x} = (x_k)_k \in c(X)$ and let us put $x = \lim_k x_k$; $(x_k - x)_k \in c_0(X)$ then $\sum_k f_k(x_k - x)$ converges in Y (proposition before), and by **(ii)** $\sum_k f_k x$ converge in Y, therefore $\sum_k f_k x_k$ converges in Y. □

Proposition 8 *Let X be an $F-$ space, $(p_u)_u$ is an increasing sequence of n.a seminorms which define the topology of X and Y be a sequentially complete locally $K-$ convex space. If $(f_k)_k \in m(X)^\beta$ then for all $q \in (\mathcal{Q})$ and all $\varepsilon \succ 0$ there exists $u \in I\!N$ such that $\lim_n \|R_n\|_{p_u,q} \leq \varepsilon$, where $R_n = (f_n, f_{n+1}, ...)$ for every $n \in I\!N$.*

Proof. Let $(f_k)_k \in m(X)^\beta$, then $(f_k)_k \in c_0(X)^\beta$, and so for all $q \in (\mathcal{Q})$ there exists $u \in I\!N$ such that $\|(f_k)_k\|_{p_u,q} \prec \infty$ and then for all $n \in I\!N$ $\|R_n\|_{p_u,q} \prec \infty$. Suppose that there exists $q \in (\mathcal{Q})$ and there exists $\varepsilon \succ 0$ such that for all $u \in I\!N$ $\lim_n \sup \|R_n\|_{p_u,q} \succ \varepsilon$. Then For $u = 1$, $\lim_n \sup \|R_n\|_{p_1,q} \succ \varepsilon$, so there exists $n_0 \succ 1$ such that $\|R_{n_0}\|_{p_1,q} \succ \frac{\varepsilon}{2}$ then there exist $n_1 \succ n_0$ and $z_{n_1} \in B_{p_1}(0,1)$ such that $q(f_{n_1} z_{n_1}) \succ \frac{\varepsilon}{4}$. For $u = 2$, $\lim_n \sup \|R_n\|_{p_2,q} \succ \varepsilon$, then there exists $n_2 \succ n_1$ and $z_{n_2} \in B_{p_2}(0,1)$ such that $q(f_{n_2} z_{n_2}) \succ \frac{\varepsilon}{4}$. And by induction over $u = r \succ 2$, there exists $n_r \succ n_{r-1}$ and $z_{n_r} \in B_{p_r}(0,1)$ such that $q(f_{n_r} z_{n_r}) \succ \frac{\varepsilon}{4}$. Let us put $\overline{x} = (x_k)_k$, with $x_k = z_k$ if $k = n_r$ $(r \geq 1)$ and $x_k = 0$ elsewhere. Then for every $u \in I\!N$, we have $\sup_k p_u(x_k) = \sup_r (p_u(z_{n_r})) = \max(\max_{r \prec u} p_u(z_{n_r}), \max_{r \geq u} p_u(z_{n_r}))$ $\leq \max(\max_{r \prec u} p_u(z_{n_r}), \max_{r \geq u} p_r(z_{n_r})) \leq \max(\max_{r \prec u} p_u(z_{n_r}), 1) \prec \infty$. Then $\overline{x} \in m(X)$; but $\sum_{k \geq 1} f_k x_k$ is not converging in Y, which is absurd. □

If X is a Banach space then we have the following characterization of the space $m(X)^\beta$.

corollary 2 *If $(X, \|.\|)$ is a Banach space and Y is a sequentially complete, then for every $(f_k)_k \in \omega(B(X,Y))$, $(f_k)_k \in m(X)^\beta$ if and only if for all $q \in (\mathcal{Q})$ $\lim_n$ $\|R_n\|_{\|.\|,q} = 0$.*

2 Characterization of matrix transformations

In this paragraph we generalize the classical theorems of summation theory in case of n.a $BS-$ spaces thanks to technics of p-adic functional analysis, notably the Kojima-Schur theorem ([3], theorem 2.7, p. 10) and the Toeplitz-Silverman theorem ([3], theorem 2.9, p. 10). We already know the version of these two theorems in case of sequence spaces over a Banach spaces [3], [4] and [10], in case of sequence spaces over a Frechet spaces [6], in case of sequence spaces over a Barrelled spaces [13] and in case of sequence spaces over a n.a valued field K [7].

Throughout this paragraph $T = (A_{ij})_{i,j}$ is a matrix transformation.

Lemma 1 *If T is conservative for the null sequence, then for every $\overline{x} = (x_j)_j \in c_0(X)$, the series $\sum_{j\geq 1} A_{ij}x_j$ converges uniformly with respect to $i \in I\!N$.*

Proof. $([13], \ lemma 1, \ p. \ 284)$. □

Lemma 2 *If X is a $BS-$ space and T is conservative for the null sequence, then for every bounded subset M of (X, τ_X) and $q \in (\mathcal{Q})$, there exists $K_M \succ 0$ such that for any $i, m \in I\!N$ and $x_j \in M$, $\max_{1\leq j\leq m} q(A_{ij}x_j) \leq K_M$.*

Proof. If not, we can choose a bounded subset M of (X, τ_X) and $q \in (\mathcal{Q})$ such that for every $K \succ 0$ there exist $i_0, m_0 \in I\!N$ and $x_j^{(0)} \in M$ satisfying that:

(1) $$\max_{1\leq j\leq m_0} q\left(A_{i_0 j}x_j^{(0)}\right) \succ K.$$

Note that the matrix transformation T transforms $c_0(X)$ in to $c(Y)$ and X is a $BS-$ space, it is easily to see that for every $j \in I\!N$, $\{A_{ij}(x) : x \in M, \ i \in I\!N\}$ is a bounded subset of (Y, τ_Y).

At first, we show that for every $i \in I\!N$, there exists $K_i \succ 0$ such that for any $m \in I\!N$ and $x_j \in M$,

(2) $$\max_{1\leq j\leq m} q(A_{ij}x_j) \leq K_i.$$

If not, there exists $i \in I\!N$ such that for every $K \succ 0$ there exist $m_0 \in I\!N$ and $x_j^{(0)} \in M$, $j = 1, 2, ..., m_0$, such that $\max_{1\leq j\leq m_0} q\left(A_{ij}x_j^{(0)}\right) \succ K$.

Let $K = 1$, there exist $m_1 \in I\!N$ and $x_j^{(1)} \in M$, $j = 1, 2, ..., m_1$, such that:
$\max_{1\leq j\leq m_1} q\left(A_{ij}x_j^{(1)}\right) \succ 1$.

For $K = \varrho^2 + \max_{1\leq j\leq m_1} r_j$ where $r_j = \sup_{i\in I\!N, x\in M} q(A_{ij}x)$ there exists $m_2 \in I\!N$ and $x_j^{(2)} \in M$, $j = 1, 2, ..., m_2$, such that $\max_{1\leq j\leq m_2} q\left(A_{ij}x_j^{(2)}\right) \succ \varrho^2 + \max_{1\leq j\leq m_1} r_j$.

Thus we have $m_2 \succ m_1$ and, $\max_{m_1+1\leq j\leq m_2} q\left(A_{ij}x_j^{(2)}\right) \succ \varrho^2$.

Inductively, we can obtain a sequence $(m_n)_n$ of $I\!N$ such that:
$\max_{1\leq j\leq m_1} q\left(A_{ij}x_j^{(1)}\right) \succ 1$ and $\max_{m_n+1\leq j\leq m_{n+1}} q\left(A_{ij}x_j^{(n+1)}\right) \succ \varrho^{n+1}$, $n = 1, 2, ...$

Let $z_j = x_j^{(1)}, 1 \leq j \leq m_1$ and $z_j = \frac{x_j^{(n+1)}}{\lambda_{n+1}}$, $m_n + 1 \leq j \leq m_{n+1}$, $n = 1, 2, ...,$ where $\lambda_n \in K$ such that $|\lambda_n| = \varrho^n$ for all $n \in I\!N$.
From that M is a bounded subset of (X, τ_X) it follows that $(z_j)_j \in c_0(X)$.
On the other hand, from $\max_{m_n+1 \leq j \leq m_{n+1}} q\left(A_{ij} x_j^{(n+1)}\right) \succ 1$, the series $\sum_{j \geq 1} A_{ij} z_j$ is not convergent. This is a contradiction and so the conclusion holds.
Now, we show that the inequality (1) is not true. In fact, let $K = 1$, there exist $i_1, m_1 \in I\!N$ and $x_j^{(1)} \in M$, $j = 1, 2, ..., m_1$, such that $\max_{1 \leq j \leq m_1} q\left(A_{i_1 j} x_j^{(1)}\right) \succ 1$.
For $K = \max\left(\max_{1 \leq i \leq i_1} K_i, \varrho^2 + \max_{1 \leq j \leq m_1} r_j\right)$ there exist $i_2, m_2 \in I\!N$ and $x_j^{(2)} \in M$, $j = 1, 2, ..., m_2$, such that:

(3) $$\max_{1 \leq j \leq m_2} q\left(A_{i_2 j} x_j^{(2)}\right) \succ K.$$

It is obvious that $m_2 \succ m_1$, $i_2 \succ i_1$ and from (3) it follows that $max_{m_1+1 \leq j \leq m_2} q\left(A_{i_2 j} x_j^{(2)}\right) \succ \varrho^2$.
Inductively, we can obtain two strictly increasing sequences $(m_n)_n$ and $(i_n)_n$ such that $\max_{1 \leq j \leq m_1} q\left(A_{i_1 j} x_j^{(1)}\right) \succ 1$ and $\max_{m_n+1 \leq j \leq m_{n+1}} q\left(A_{i_{n+1} j} x_j^{(n+1)}\right) \succ \varrho^{n+1}$, $n = 1, 2, ...$
Let $z_j = x_j^{(1)}, 1 \leq j \leq m_1$ and $z_j = \frac{x_j^{(n+1)}}{\lambda_{n+1}}$, $m_n + 1 \leq j \leq m_{n+1}$, $n = 1, 2, ...;$ where $\lambda_n \in K$ and $|\lambda_n| = \varrho^n$ for all $n \in I\!N$. Then $\max_{m_n+1 \leq j \leq m_{n+1}} q\left(A_{i_{n+1} j} z_j\right) \succ 1$, $n = 1, 2, ...$ This shows that the series $\sum_{j \geq 1} A_{ij} z_j$ does note convergent uniformly with respect to $i \in I\!N$. This contradicts lemma 1 and hence lemma 2 holds. □

Remark 1 *Under the conditions of lemma 2 we have for every bounded subset M of (X, τ_X) and $q \in (\mathcal{Q})$, there exists $K_M \succ 0$ such that for any $i, m \in I\!N$ and $x_j \in M$, $q\left(\sum_{j=1}^m A_{ij} x_j\right) \leq K_M$.*

Theorem 1 *If X and $c_0(X)$ are $BS-$ spaces and (Y, τ_Y) is sequentially complete, then T is conservative for the null sequence if, and only if the two conditions below are met*
(i). *For every bounded subset M of (X, τ_X) and $q \in (\mathcal{Q})$, there exists $K_M \succ 0$ such that for any $i, m \in I\!N$ and $x_j \in M$, $q\left(\sum_{j=1}^m A_{ij} x_j\right) \leq K_M$.*
(ii). *For every $j \in I\!N$ there exists $A_j \in B(X, Y)$ such that $\lim_i A_{ij} = A_j$. In this case we have $\lim_i \sum_{j=1}^\infty A_{ij} x_j = \sum_{j=1}^\infty A_j x_j$ for all $\overline{x} = (x_j)_j \in c_0(X)$.*

Proof. The necessity follows from remark before and the fact that for any $x \in X$ and $j \in I\!N$ $\delta_j(x) \in c_0(X)$.
Sufficiency, let $p \in (\mathcal{P})$ and $q \in (\mathcal{Q})$, then there exists $K \succ 0$ such that for any $i, m \in I\!N$ and $x_j \in B_p(o, 1)$, $q\left(\sum_{j=1}^m A_{ij} x_j\right) \leq K$, therefore $\sup_i \left\|(A_{ij})_j\right\|_{p,q} \prec \infty$ and so, by proposition 7, for all $i \in I\!N$ $(A_{ij})_j \in c_0(X)^\beta$, then for all $i \in I\!N$ $\sum_j A_{ij} x_j$

converges in Y.
On the other hand for all $j \in I\!N$ $q(A_j x_j) = q(\lim_i A_{ij} x_j) \leq \left(\sup_i \| (A_{ij})_j \|_{p,q}\right) p(x_j) \overset{j\to\infty}{\longrightarrow} 0$, then $A_j x_j \overset{j\to\infty}{\longrightarrow} 0$, and so $\sum_j A_j x_j$ converges in Y.
Let us prove that $\lim_i \sum_{j=1}^{\infty} A_{ij} x_j = \sum_{j=1}^{\infty} A_j x_j$ in Y.
Let $q \in (\mathcal{Q})$, then there exist $p \in (\mathcal{P})$ and $R \succ 0$ such that $\sup_i \left\| (A_{ij})_j \right\|_{p,q} \leq R$.
Let $\varepsilon \succ 0$, then there exists $r \in I\!N$ such that $p(x_j) \leq \frac{\varepsilon}{R}$ for all $j \geq r$ and $q\left(\sum_{j\geq r} A_j x_j\right) \leq \varepsilon$.
Let $i_0 \in I\!N$ such that for all $i \geq i_0$ $q\left(\sum_{j\prec r} (A_{ij} - A_j) x_j\right) \leq \varepsilon$, then for all $i \geq i_0$
$q\left(\sum_{j=1}^{\infty} (A_{ij} - A_j) x_j\right) \leq \max\left\{ q\left(\sum_{j\prec r} (A_{ij} - A_j) x_j\right), q\left(\sum_{j\geq r} A_{ij} x_j\right), q\left(\sum_{j\geq r} A_j x_j\right) \right\}$
$\leq \max\left\{ q\left(\sum_{j\prec r} (A_{ij} - A_j) x_j\right), \sup_i \left\| (A_{ij})_j \right\|_{p,q} \sup_{j\geq r} p(x_j), q\left(\sum_{j\geq r} A_j x_j\right) \right\} \leq \varepsilon$. □

We have the following proposition which is a generalization of Kojima-Schur theorem in case of sequence spaces over a n.a $BS-$ spaces.

Proposition 9 ***(Kojima-Schur theorem)*** *If X and $c_0(X)$ are $BS-$ space and (Y, τ_Y) is sequentially complete, then T is conservative if, and only if the three conditions below are met*
(i). *For every bounded subset M of (X, τ_X) and $q \in (\mathcal{Q})$, there exists $K_M \succ 0$ such that for any $i, m \in I\!N$ and $x_j \in M$, $q\left(\sum_{j=1}^{m} A_{ij} x_j\right) \leq K_M$,*
(ii). *For every $i \in I\!N$ $\lim_j A_{ij} = 0$ and for all $x \in X$ $\lim_i \sum_{j=1}^{\infty} A_{ij} x$ exists,*
(iii). *For all $j \in I\!N$ there exists $A_j \in B(X, Y)$ such that $\lim_i A_{ij} = A_j$. In this case for all $\overline{x} = (x_j)_j \in c(X)$ we have $\lim_i T\overline{x} = \lim \sum_{j=1}^{\infty} A_{ij} (\lim \overline{x}) + \sum_{j=1}^{\infty} A_j (x_j - \lim \overline{x})$.*

Proof. Suppose that T is conservative, then T is conservative for the null sequence so **(i)** and **(iii)** holds by theorem before.
Let $x \in X$, $\delta(x) = (x, x, ...) \in c(X)$, then $T(\delta(x)) = \left(\sum_{j=1}^{\infty} A_{ij} x\right)_i \in c(Y)$ so for all $i \in I\!N$ and all $x \in X$, $\lim_j A_{ij} x = 0$ and $\lim_i \sum_{j=1}^{\infty} A_{ij} x$ exists.
Conversely, **(i)** and **(iii)** implies that T is conservative for the null sequence. On the other hand, let $\overline{x} = (x_j)_j \in c(X)$ and $x = \lim \overline{x}$, then $\overline{x} = (x_j - x)_j + \delta(x)$ so $\sum_{j\geq 1} A_{ij} x_j$ converges in Y and by **(ii)** and **(iii)** we have $\lim_i \sum_{j=1}^{\infty} A_{ij} (x - x_j) = \sum_{j=1}^{\infty} A_j (x - x_j)$, then $\lim T\overline{x} = \sum_{j=1}^{\infty} A_j (x - x_j) + \lim_i \sum_{j=1}^{\infty} A_{ij} (x)$. □

Under the conditions of theorem before we have the following corollaries and propositions 10 and 11:

corollary 3 *$T = (A_{ij})_{i,j}$ is conservative for the null sequences if, and only if the two conditions below are met*

(i). *For all* $q \in (\mathcal{Q})$ *there exists* $p \in (\mathcal{P})$ *such that* $\sup_i \left\| (A_{ij})_j \right\|_{p,q} \prec \infty$.
(ii). *For all* $j \in I\!N$ *there exists* $A_j \in B(X,Y)$ *such that* $\lim_i A_{ij} = A_j$, *and for all* $\overline{x} = (x_j)_j \in c_0(X)$ *we have* $\lim T\overline{x} = \lim_i \sum_{j=1}^{\infty} A_{ij}x_j = \sum_{j=1}^{\infty} A_j x_j$.

corollary 4 *T is regular for the null sequences if, and only if the two conditions below are met*
(i). *For all* $q \in (\mathcal{Q})$ *there exists* $p \in (\mathcal{P})$ *such that* $\sup_i \left\| (A_{ij})_j \right\|_{p,q} \prec \infty$,
(ii). *For all* $j \in I\!N$ $\lim_i A_{ij} = 0$.

corollary 5 ***(Other version of Kojima-Schur theorem)*** *T is conservative if, and only if the three conditions below are met*
(i). *For all* $q \in (\mathcal{Q})$ *there exists* $p \in (\mathcal{P})$ *such that* $\sup_i \left\| (A_{ij})_j \right\|_{p,q} \prec \infty$,
(ii). *For all* $i \in I\!N$ $\lim_j A_{ij} = 0$ *and for all* $x \in X$ $\lim T(\delta(x)) = \lim_i \sum_j A_{ij}x$ *exists,*
(iii). *For all* $j \in I\!N$ *there exists* $A_j \in B(X,Y)$ *such that* $\lim_i A_{ij} = A_j$. *And for all* $\overline{x} = (x_j)_j \in c(X)$ *we have* $\lim T\overline{x} = \lim_i \sum_j A_{ij}(\lim \overline{x}) + \sum_j A_j(x_j - \lim \overline{x})$.

corollary 6 *T is null conservative if, and only if the three conditions below are met*
(i). *For all* $q \in (\mathcal{Q})$ *there exists* $p \in (\mathcal{P})$ *such that* $\sup_i \left\| (A_{ij})_j \right\|_{p,q} \prec \infty$,
(ii). *For all* $i \in I\!N$ $\lim_j A_{ij} = 0$ *and for all* $x \in X$ $\lim T(\delta(x)) = \lim_i \sum_j A_{ij}x$ $= 0$,
(iii). *For all* $j \in I\!N$ $\lim_i A_{ij} = 0$. *And for all* $\overline{x} = (x_j)_j \in c(X)$ *we have* $\lim T\overline{x} = \lim_i \sum_j A_{ij}(\lim \overline{x}) + \sum_j A_j(x_j - \lim \overline{x})$.

Proposition 10 *Let* $f \in B(X,Y)$, *then T is* $f-$ *regular if, and only if the three conditions below are met*
(i). *For all* $q \in (\mathcal{Q})$ *there exists* $p \in (\mathcal{P})$ *such that* $\sup_i \left\| (A_{ij})_j \right\|_{p,q} \prec \infty$,
(ii). *For all* $i \in I\!N$ $\lim_j A_{ij} = 0$ *and for all* $x \in X$ $\lim T(\delta(x)) = \lim_i \sum_j A_{ij}x$ $= f(x)$,
(iii). *For all* $j \in I\!N$ $\lim_i A_{ij} = 0$.

Proof. Suppose that T be $f-$ regular, then T is in particular regular for the null sequences, and so we have **(i)** and **(iii)** (corollary 4). Let $x \in X$, $\overline{x} = \delta(x) \in c(X)$ and so $\sum_j A_{ij}x = f(x)$, on the other hand we have for all $i \in I\!N$ $lim_j\ A_{ij}x = 0$.
Conversely, by corollary 5 T is conservative; let $\overline{x} = (x_j)_j \in c(X)$ and let us put $x = \lim \overline{x}$, $(x_j - x)_j \in c_0(X)$, then by corollary 3 T is conservative for the null sequences, thus $\lim_i \sum_{j=1}^{\infty} A_{ij}(x_j - x) = \sum_{j=1}^{\infty} \lim_i A_{ij}(x_j - x) = 0$, and so $\lim_i \sum_{j=1}^{\infty} A_{ij}x_j = \lim_i \sum_{j=1}^{\infty} A_{ij}x$. Therefore $\lim T\,\overline{x} = f\,(\lim \overline{x})$. □

The following corollary is a generalization of Toeplitz-Silverman theorem in case of sequence spaces over a n.a $BS-$ space.

corollary 7 ***(Theorem of Toeplitz-Silverman)*** *T is regular if, and only if the three conditions below are met*

(i). *For all $q \in (\mathcal{Q})$ there exists $p \in (\mathcal{P})$ such that $sup_i \left\| (A_{ij})_j \right\|_{p,q} \prec \infty$,*

(ii). *For all $i \in I\!N$ $\lim_j A_{ij} = 0$ and for all $x \in X$ $\lim T(\delta(x)) = x$,*

(iii). *For all $j \in I\!N$ $\lim_i A_{ij} = 0$.*

Proof. T is regular if, and only if T is $f-$ regular, where $f = id_X$ $(X = Y)$. □

Proposition 11 *T is S-regular if, and only if the two conditions below are met*

(i). *For all $q \in (\mathcal{Q})$ there exists $p \in (\mathcal{P})$ such that $\sup_i \left\| (A_{ij})_j \right\|_{p,q} \prec \infty$,*

(ii). *For all $j \in I\!N$ $\lim_i A_{ij} = id_X$.*

Proof. Suppose that T be S-regular and let $k \in I\!N$, then for all $x \in X$, $\overline{x} = \delta_k(x) \in c_0(X)$ and so $\lim_i A_{ij}x = x$. T is in particular conservative for the null sequences then **(i)** holds.

Conversely, let $\overline{x} = (x_j)_j \in c_0(X)$, by corollary 3 T is conservative for the null sequences then $\lim_i \sum_{j=1}^{\infty} A_{ij}x_j = \sum_{j=1}^{\infty} \lim_i A_{ij}x_j$, this means that $\lim T\,\overline{x} = \sum_{j=1}^{\infty} x_j$. □

If $(X, \|.\|)$ be a $n.a$ Banach space, thanks corollary 2 we have the following characterization of coercive matrix transformations

Theorem 2 *T is coercive if, and only if the three conditions below are met*

(i). *For all $i \in I\!N$ and all $q \in (\mathcal{Q})$ $\lim_j \|R_{ij}\|_{\|.\|,q} = 0$ where $R_{ij} = (A_{ij},\ A_{ij+1},\ ...)$, $i, j = 1, 2, ...$*

(ii). *For all $j \in I\!N$ there exists $A_j \in B(X,Y)$ such that lim_i $A_{ij} = A_j$.*

And for all $\overline{x} = (x_j)_j \in m(X)$ we have $\lim T\overline{x} = \lim_i \sum_j A_{ij}x_j = \sum_j A_jx_j$.

(iii). *For all $q \in (\mathcal{Q})$, $\lim_j \left\{ \sup_i \|R_{ij} - R_j\|_{\|.\|,q} \right\} = 0$ where $R_j = (A_j,\ A_{j+1},\ ...)$, $j = 1, 2, ...$*

Proof. Suppose that T be coercive, then T is conservative for the null sequences so **(ii)** is given by theorem 1, and **(i)** holds by corollary 2.

Let us prove **(iii)**. Let $B = \left\{ \overline{x} = (x_j)_j \in m(X) : \|\overline{x}\|_\infty = \sup_j \|x_j\| \le 1 \right\}$, over B we define the n.a metric $d(x,y) = \sup_j 2^{-j} \|x_j - y_j\|$. For all m, n and all $q \in (\mathcal{Q})$ we consider $f_{mn} : (B,\ d) \longrightarrow (Y, q)$, $(x_j)_j \longmapsto \sum_{j=1}^{\infty} (A_{nj} - A_{mj})\, x_j$,

$q(f_{mn}(x) - f_{mn}(y)) \le \max \left\{ d(x,y)\,.2^p \max_{1 \le j \le p} \|A_{nj} - A_{mj}\|_{\|.\|,q}, \|R_{mp}\|_{\|.\|,q}, \|R_{np}\|_{\|.\|,q} \right\}$; then f_{mn} is continuous.

Let $\varepsilon \succ 0$, let us put $F_{mn} = \{x \in B : q(f_{mn}(x)) \leq \varepsilon\}$ and $E_p = \cap\{F_{mn} \ : \ m, n \geq p\}$. For all $p \geq 1$ E_p is closed in B. Let $\overline{x} = (x_j)_j \in B$, then $T\overline{x} = \left(\sum_j A_{ij} x_j\right)_i \in c(Y)$, and so there exists $r \in I\!N$ such that for all $m,\ n \geq r$ $q\left(\sum_{j=1}^{\infty} (A_{nj} - A_{mj}) x_j\right) \leq \varepsilon$, then $x \in E_r$ and so $B = \cup_p\ E_p$. (B, d) being complete, then by the Baire-Hausdorff theorem ([14], p. 11) there exists $r \geq 1$ such that $U \subset E_r$ where U is an open in (B, d) and then there exist $s \succ 0$ and $a \in B$ such that $B\ (a, s) = \{x \in B : d(x, a) \leq s\} \subset E_r$.

Let $i \geq 1$ such that $\sup_{k \geq i}\ 2^{-k} \leq s$ and let $\overline{x} = (x_k)_k \in B$ and $j \geq 1$; let us put $y_k = a_k$ if $k \prec i$, $y_k = x_k$ if $i \leq k \leq i + j$ and $y_k = 0$ if $k \succ i + j$ and $y = (y_k)_k$; then $d(y, a) = \left(\max_{i \leq k \leq i+j} \max 2^{-k} \|x_k - a_k\|, \sup_{k \succ i+j} 2^{-k} \|a_k\|\right)$ $\leq \max\left(\max_{i \leq k \leq i+j} 2^{-k} \|x_k\|, \sup_{k \geq i} 2^{-k} \|a_k\|\right) \leq \sup_{k \geq i}\ 2^{-k} \leq s$. Then $y \in B(a, s)$ and so $y \in E_r$ thus for all $m,\ n \geq r$ $q\left(\sum_{k=1}^{\infty} (A_{nk} - A_{mk}) .y_k\right) \leq \varepsilon$, therefore for all $m,\ n \geq r$ $q\left(\sum_{k=1}^{i-1} (A_{nk} - A_{mk}) a_k + \sum_{k=i}^{i+j} (A_{nk} - A_{mk}) x_k\right) \leq \varepsilon$.

Let $M_0 \in I\!N$ such that for all $m,\ n \geq M_0$ $q\left(\sum_{k=1}^{i-1} (A_{nk} - A_{mk}) a_k\right) \leq \varepsilon$.

Put $N = \max(M_0, r)$, then for all $m, n \geq N$ we have $q\left(\sum_{k=i}^{i+j} (A_{nk} - A_{mk}) x_k\right) \leq \varepsilon$.

Then if we taking $m \longrightarrow +\infty$ and taking the supremum over $x \in B$ and over $j \geq i$, we have for all $n \geq N$ $\|R_{ni} - R_i\|_{\|.\|,q} \leq \varepsilon$ and so for all $n \geq N$ and $p \geq i$ $\|R_{np} - R_p\|_{\|.\|,q} \leq \varepsilon$.

For all $n = 1,\ \ldots,\ N$ $\lim_p \|R_{np}\|_{\|.\|,q} = 0$, then there exists $p_0 \geq i$ such that for all $p \geq p_0$ and all $n = 1,\ \ldots,\ N$ $\|R_{np}\|_{\|.\|,q} \leq \varepsilon$. For all $n \leq N$ and $p \geq p_0$, $\|R_{np} - R_p\|_{\|.\|,q} \leq \max\left(\|R_{np}\|_{\|.\|,q}, \|R_{Np}\|_{\|.\|,q}, \|R_{Np} - R_p\|_{\|.\|,q}\right) \leq \varepsilon$, and so we have for all $p \geq p_0$ $\sup_n \|R_{np} - R_p\|_{\|.\|,q} \leq\ \varepsilon$, and therefore $\lim_p \left\{\sup_n \|R_{np} - R_p\|_{\|.\|,q}\right\} = 0$.

Conversely, let $\overline{x} = (x_j)_j \in m(X)$, let us prove that $\left(\sum_{j=1}^{\infty} A_{ij} x_j\right)_i \in c(Y)$. By **(i)**, for all $n \in I\!N$ $\sum_j A_{ij} x_j$ converge in Y (Corollary 2). For all $q \in (\mathcal{Q})$ $\|R_p\|_{\|.\|,q} \leq \max\left(\|R_{1p}\|_{\|.\|,q}, \|R_{1p} - R_p\|_{\|.\|,q}\right) \overset{p \to \infty}{\longrightarrow} 0$, then $\|R_p\|_{\|.\|,q} \overset{p \to \infty}{\longrightarrow} 0$, and then $\sum_j A_j x_j$ converges in Y.

Let us show that $\lim_i \sum_{j=1}^{\infty} A_{ij} x_j = \sum_{j=1}^{\infty} A_j x_j$. Let $\lambda \in K^*$ such that $\|\overline{x}\|_\infty \leq |\lambda|$. Then for all $q \in (\mathcal{Q})$ and all $\varepsilon \succ 0$ there exists $p \in I\!N$ such that $sup_n \|R_{np} - R_p\|_{\|.\|,q} \leq \frac{\varepsilon}{|\lambda|}$.

Let $N \in I\!N$ such that for all $n \geq N$ $q\left(\sum_{j \prec p} (A_{nj} - A_j) x_j\right) \leq \varepsilon$, then for all $n \geq N$ we have $q\left(\sum_{j=1}^{\infty} (A_{nj} - A_j) x_j\right) \leq \max\begin{pmatrix} q\left(\sum_{j \prec p} (A_{nj} - A_j) x_j\right), \\ q\left(\sum_{j \geq p} (A_{nj} - A_j) x_j\right) \end{pmatrix} \leq \max\left(\varepsilon, |\lambda|, \|R_{np} - R_p\|_{\|.\|,q}\right) \leq \varepsilon$.

Then $\lim_n \sum_{j=1}^{\infty} A_{nj}x_j = \sum_{j=1}^{\infty} A_j x_j$ and so T is coercive. $\square$

Remark 2 *This theorem generalize, in n.a case, the theorem proved by Madox ([3], theorem 4.6, p. 46) in case of sequence spaces in Banach spaces, however it remains to find a generalization for a n.a BS− spaces .*

corollary 8 *T is null-coercive if, and only if **(i)** and **(ii)** hold*

(i). *For all* $q \in (\mathcal{Q})$, $\lim_j \left\{\sup_i \|R_{ij}\|_{\|.\|,q}\right\} = 0$,

(ii). *For all* $j \in I\!N$ $\lim_i A_{ij} = 0$.

Theorem 3 *Suppose that* $c_0(X)$ *be a BS− space and* Y *be sequentially complete, then* T *is* $c_0(X)-$ *permanent if, and only if for all* $q \in (\mathcal{Q})$ *there exists* $p \in (\mathcal{P})$ *such that* $sup_i \left\|(A_{ij})_j\right\|_{p,q} \prec \infty$.

Proof. Suppose that T be $c_0(X)-$ permanent; then for all $i \in I\!N$ let us put $T_i : c_0(X) \longrightarrow Y$, $\overline{x} = (x_j)_j \longmapsto \sum_{j=1}^{\infty} A_{ij}x_j$, then T_i is continuous and it is pointwise bounded, and so $(T_i)_i$ is equicontinuous over $c_0(X)$, then we have for all $q \in (\mathcal{Q})$ there exist $p \in (\mathcal{P})$ and $R \succ 0$ such that for all $\overline{x} \in c_0(X)$ and all $i \in I\!N$ $q(T_i\overline{x}) \leq R\overline{p}(\overline{x})$ and then $sup_i \left\|(A_{ij})_j\right\|_{p,q} \leq R \prec \infty$.

Conversely, let $\overline{x} = (x_j)_j \in c_0(X)$ and let $q \in (\mathcal{Q})$, then there exists $p \in (\mathcal{P})$ such that $sup_i \left\|(A_{ij})_j\right\|_{p,q} \prec \infty$, and so for all $i \in I\!N$ $(A_{ij})_j \in c_0(X)^{\beta}$ (proposition 7), then for all $i \in I\!N$ $\sum_j A_{ij}x_j$ converges in Y. Let $\rho \succ 1$ such that there exists $(\lambda_r)_r \in \omega(K)$ verifying $|\lambda_r| = \rho^r$ for all $r \in I\!N$, and let $r_0 \geq 1$ such that $\overline{p}(\overline{x}) \leq \rho^{r_0}$, then $sup_i\ q\left(\sum_{j=1}^{\infty} A_{ij}x_j\right) \leq \rho^{r_0} \sup_i \left\|(A_{ij})_j\right\|_{p,q} \prec +\infty$. $\square$

Proposition 12 *Under the conditions of theorem before we have* T *is* $c(X)-$ *permanent if, and only if **(i)** and **(ii)** hold*

(i). *For all* $q \in (\mathcal{Q})$ *there exists* $p \in (\mathcal{P})$ *such that* $sup_i \left\|(A_{ij})_j\right\|_{p,q} \prec \infty$,

(ii). *For all* $i \in I\!N$ $\lim_j A_{ij} = 0$.

Proof. Suppose that T be $c(X)-$ permanent, then T is $c_0(X)-$ permanent, and so we have **(i).** Moreover for all $x \in X$ and all $i \in I\!N$ $\sum_j A_{ij}x$ converge in Y, then for all $i \in I\!N$ $\lim_j A_{ij}x = 0$.

Conversely, let $\overline{x} = (x_j)_j \in c(X)$, then for all $i \in I\!N$ $\sum_j A_{ij}x_j$ converges in Y (corollary 1), then for all $q \in (\mathcal{Q})$ there exists $p \in (\mathcal{P})$ such that $\overline{p}(\overline{x}) \leq \rho^{r_0}$ where ρ and r_0 are these defined in proof of theorem before, finally $sup_i\ q\left(\sum_{j=1}^{\infty} A_{ij}x_j\right) \leq \rho^{r_0} \sup_i \left\|(A_{ij})_j\right\|_{p,q} \prec +\infty$. $\square$

If X is a n.a Banach space, then we have the following proposition which characterize the $m(X)-$ permanent matrix transformations

Proposition 13 *T is $m(X)-$ permanent if, and only if **(i)** and **(ii)** hold*

(i). *For all $q \in (\mathcal{Q})$ $\sup_i \left\|(A_{ij})_j\right\|_{\|.\|,q} \prec \infty$,*

(ii). *For all $i \in I\!N$ there exists $q \in (\mathcal{Q})$ such that $\lim_j \|R_{ij}\|_{\|.\|,q} = 0$.*

Proof. Suppose that T be $m(X)-$ permanent, then for all $i \in I\!N$ $(A_{ij})_j \in m(X)^{\beta}$ then we have **(ii)** (Corollary 2). On the other hand T is in particular $c_0(X)-$ permanent, and so we have **(i)** (theorem 3).

Conversely, let $\overline{x} = (x_j)_j \in m(X)$, then for all $i \in I\!N$ $\sum_j A_{ij}x_j$ converges in Y (Corollary 2), then for all $q \in (\mathcal{Q})$ we have, $sup_i\, q\left(\sum_{j=1}^{\infty} A_{ij}x_j\right) \le \rho^{r_0} \sup_i \left\|(A_{ij})_j\right\|_{\|.\|,q}$ $\prec +\infty$, where ρ and r_0 are these defined in proof of theorem before. □

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