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Numerical Comparison of Methods for Solving Systems of Conservation Laws of Mixed Type

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Abstract

In this paper, we apply Adomian decomposition method (shortly, ADM) to develop a fast and accurate algorithm for systems of conservation laws of mixed hyperbolic-elliptic type. The ADM does not require discretization and consequently of massive computations. A Sinc-Galerkin procedure is also developed for solving the same system. Sinc approximations to both derivatives and the indefinite integrals reduce the system to an explicit system of algebraic equations. It is shown that Sinc-Galerkin approximations produce an error of exponential order. Approximation by Sinc functions handles singularities in the problem, as well as changes in type of the system. A comparison between the two methods for the solution of Van der Waals system is analyzed for their solutions. The study outlines the significant features of the two methods. The results show that these methods are very efficient, convenient and can be applied to a large class of problems.

Mathematics Subject Classification: 65D05, 35M30

Keywords: Sinc-Galerkin method, Adomian decomposition method, conservation laws of mixed type

1 Introduction

Nonlinear coupled partial differential equations are very important in a variety of scientific fields, especially in fluid mechanics, solid state physics, plasma physics, plasma waves, capillary-gravity waves, and chemical physics [1]. The availability of exact solutions, for those nonlinear equations can greatly facilitate the verification of numerical solvers on the stability analysis of the solution

[7]. In this study, we consider the numerical approximations for a 2×2 systems of conservation laws with source terms (called balance laws) which can be written as

$$\frac{\partial}{\partial t}U + \frac{\partial}{\partial x}F(U) = G(U) \tag{1.0.1}$$

together with an initial data which is assumed to have bounded and small total variation. The functions F and G are called, respectively, the fluxes and the source terms. In equation (1.0.1) the solution vector is $U(x,t) = [u(x,t), v(x,t)]^T$, $F(U) = [f(u,v), g(u,v)]^T$ is the vector flux terms, and $G(U) = [h_1(x,t), h_2(x,t)]^T$ is the vector of the source terms. System (1.0.1) can be written equivalently in the following form

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} h_1(x,t) \\ h_2(x,t) \end{pmatrix} \tag{1.0.2}$$

The eigenvalues are formally $\lambda_{1,2} = \frac{1}{2}(f_u+g_v) \mp \frac{1}{2}\sqrt{(f_u+g_v)^2 - 4(f_v g_u - f_v g_u)}$. The system (1.0.2) is of mixed type, more precisely of elliptic type, for data lying in the phase $\varepsilon = \{(u,v) \in \mathbb{R}^2 : (f_u+g_v)^2 < (f_u+g_v)\}$ and of hyperbolic type for data lying in the phase $H = \{(u,v) \in \mathbb{R}^2 : (f_u+g_v)^2 > (f_u+g_v)\}$. The mathematical ill-posedness of the system in the elliptic region reveals the physical fact that the state in the elliptic region is not stable, and it typically evolves into phase transitions, i.e., jumps across the elliptic regions in the weak solution. The evolution of (1.0.1) will be governed by initial data. Here we pose some initial conditions

$$u(x,0) = u_0(x) \quad , \quad v(x,0) = v_0(x) \tag{1.0.3}$$

which makes (1.0.1), (1.0.3) into a mixed hyperbolic-elliptic initial value problem. The mathematical theory for this class of problems has been carried out in, e.g., [9]. Problems of this arise in computational fluid dynamics [13]. The study of liquid-gas phase transition and interface movement is important in science and engineering. In this paper, the Van der Waals equation will be used as a test example. Many numerical schemes have been proposed to solve Van der Waals equation [6] and [13]. There are a wide literature dedicated to the problem in (1.0.2) of construction numerical solutions; see [15]. The main objective of this contribution is to introduce a comparative study to examine the performance of the Adomian decomposition method (ADM for short)[4, 5], and the Sinc-Galerkin method [10, 12] in solving 2×2 systems of balance laws.

" The Sinc-Galerkin method, which builds an approximate solution valid on the entire spatial domain and on small interval in the time domain. The main idea is to replace derivatives and integrals by their discrete Sinc approximations. There are several reasons to approximate by Sinc functions:

1. They are easily implemented and give good accuracy for problems with singularities; approximation by Sinc are typified by errors of the form

$\mathcal{O}(\exp(-c/h))$ where $c > 0$ is a constant and h is a step size, which is an improvement over other methods with errors that are polynomials in h.

2. Approximation by Sinc functions handless singularities in the problem as will as changes in type of the system. The effect of any such singularities ,or changes in type will appear in some form in any scheme of numerical solution, and it is well known that polynomial method do not perform well near singularities, or in case of changes in type of the system " [2], for example, using ADM.

3. The obtained discrete system can be solved by iteration technique (fixed point iteration), which permits a significant reduction in both storage and computation compared with what would be required by traditional methods.

The paper is organized as follows: In Section 2, we review some basic facts about Sinc approximation. In Section 3 which contains Al-Khaled's previous work [2], where we apply the Sinc-Galerkin method for the problem presented in (1.0.1) to construct a numerical scheme and obtain a discrete system that can be solved by the fixed point iteration. In Section 4, we present the Adomian decomposition method. We numerically test the schemes on the Van der Waals equations in the last section.

2 Sinc Function: Notation and Background

The Sinc Galerkin method is developed by Stenger [12]. In this section we outline the main steps of our approximation using the Sinc function where notations and definitions are introduced and mentioned in [2]. Some known results of the Sinc function's theory that are important for this paper are also stated. In what follows, $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ denote respectively the set of all integers, the set of all real numbers and the set of complex numbers. The Sinc function is defined on the whole real line by

$$\text{sinc}\,(x) = \frac{\sin(\pi x)}{\pi x}, -\infty < x < \infty \tag{2.0.1}$$

For $h > 0$, the translated Sinc functions are given as

$$S(k;h)(x) = \text{sinc}\,\left((x-kh)/h\right), \quad k = 0, \pm 1, \pm 2, ... \tag{2.0.2}$$

If f is defined on the real line, then for $h > 0$ the series

$$C(f;h) = \sum_{k=-\infty}^{\infty} f(kh)\text{sinc}\,\left((x-kh)/h\right), \tag{2.0.3}$$

is called the Whittaker cardinal expansion of f whenever this series converges. The properties of (2.0.3) have been extensively studied and are thoroughly surveyed in [10, 12]. They are based in the infinite strip D_d in the complex plane $\mathbb{C}$

$$D_d = \{w = t + is : |s| < d \leq \pi/2, t \in \mathbb{R}\} \tag{2.0.4}$$

With regard to using Sinc as an approximation tool, an important class of functions must be identified.

Definition 2.1 *Let $h > 0$. The **Paly-Wiener** class of functions $\boldsymbol{W}(\frac{\pi}{h})$ is the family of all functions that are analytic in $\mathbb{C}$, such that $\int_{\mathbb{R}} |f(t)|^2 dt < \infty$, and for all $z \in \mathbb{C}$ we have $|f(z)| \leq C_1 \exp\left(\frac{\pi|z|}{h}\right)$, with C_1 a positive constant.*

The most important properties of Sinc function are summarized in the following theorem.

Theorem 2.1 *[12] If $f \in \boldsymbol{W}(\frac{\pi}{h})$, then $f' \in \boldsymbol{W}(\frac{\pi}{h})$. If $i, n \in \mathbb{Z}$ and if $\delta_{i-n}^{(k)}$ is defined by $\delta_{i-n}^{(k)} = (\frac{d}{dx})^k S(i,1)(x) \downarrow_{x=n}, k = 0, 1, 2, ...$ then*

$$f^{(k)}(x) = h^{-k} \sum_{n=-\infty}^{\infty} \left[\sum_{i=-\infty}^{\infty} f(ih) \delta_{i-n}^{(k)} \right] S(n,h)(x).$$

Let us now introduce the following definition which is fundamental to the development of Sinc methods on arc Γ in the complex plane $\mathbb{C}$. For more details see [12, 10]

Definition 2.2 *Let D be the simply-connected domain in the complex plane with boundary points $a \neq b$. Let ϕ be a conformal map of D onto the strip D_d with $\phi(a) = -\infty$ and $\phi(b) = \infty$. If the inverse map of ϕ is denoted by ψ, define*

$$\Gamma = \{z \in \mathbb{C} : z = \psi(u), u \in \mathbb{R}\}$$

Given ϕ, ψ and a positive number h, let us set $z_k = z_k(h) = \psi(kh), \quad k = 0, \mp 1, \mp 2, ...$ and define ρ by $\rho(z) = \exp(\phi(z))$. The following convenient notation will be useful in formulating the discrete system. The composite function $S(n,h) \circ \phi(x)$ define the basis functions for the space interval, which is our case $(-\infty, \infty)$, i.e., $\phi(x) = x$. The Sinc grid points $z_k \in \Gamma = \mathbb{R}$ will be denoted by x_i since they are real. They are the inverse images of the equispaced grid D_d, that is, $x_i = \phi^{-1}(ih) = ih$. Let

$$\delta_{i-n}^{(0)} = [S(n,h) \circ \phi(x)] \downarrow_{x=x_i} = \begin{cases} 1, & i = n \\ 0, & i \neq n \end{cases} \tag{2.0.5}$$

$$\delta_{i-n}^{(1)} = h \frac{d}{d\phi} [S(n,h) \circ \phi(x)] \downarrow_{x=x_i} = \begin{cases} 0 \quad , & i = n, \\ \frac{(-1)^{n-i}}{(n-i)}, & i \neq n, \end{cases} \tag{2.0.6}$$

denote the evaluation of the basis functions and their derivatives with respect to the map ϕ. Define the $m \times m$ matrix $I^{(1)}$ whose $in - th$ entry is given by $\delta^{(1)}_{i-n}$. The important class of functions for Sinc functions is denoted by $\mathbf{L}_\alpha(D)$ and defined next.

Definition 2.3 *Corresponding to numbers α and β, let $\boldsymbol{L}_{\alpha,\beta}(D)$ denote the family of all analytic functions F for which there exists a constant C_2 such that*

$$|F(z)| \leq C_2 \frac{|\rho(z)|^\alpha}{[1 + |\rho(z)|]^{\alpha+\beta}} \tag{2.0.7}$$

for all $z \in D$. Finally, we define $\boldsymbol{L}_\alpha(D) = \boldsymbol{L}_{\alpha,\alpha}(D)$.

Now we state the main theorem in this section, which shows how to approximate the derivative of functions from the class $\mathbf{L}_\alpha(D)$. The proof can be found in [12].

Theorem 2.2 *Let $\frac{F}{q} \in \boldsymbol{L}_\alpha(D)$, α is a positive constant, then taking $h = \sqrt{\frac{\pi d}{(\alpha N)}}$ gives*

$$\sup_{x \in \Gamma} |F^{(n)}(x) - (\frac{d}{dx})^n \sum_{k=-N}^{N} \frac{F(z_k)}{q(z_k)} q(x) S(k,h) \circ \phi(x)| \leq C_3 N^{\frac{n+1}{2}} exp(-\sqrt{\pi d \alpha N})$$

for $n = 0, 1, 2, ..., m$, with C_3 a constant depending only on m, ϕ, q, d, α and F.

The weight function q in the above theorem is chosen relative to the order of the derivative, to approximate the first derivative, the choice $q(x) = \frac{1}{\phi'(x)}$ is often suffices. An application of Theorem 2.2 to approximate the first derivative with respect to x of the function $u(x,t)$ in the domain $(-\infty, \infty)$ is as follows: Choose the map $\phi(x)$ which maps the infinite strip D_d defined in (2.0.3) onto D_d and the compositions $S(m, h_x) \circ \phi(x), \quad m = -N_x, ..., N_x$, define the basis elements for $-\infty, \infty$, where $S(m, h_x)$ as defined in Equation (2.0.2). The symbol h_x represent the mesh size of the infinite strip D_d for the infinite grid $\{ih_x\}, (-\infty < i < \infty)$. The Sinc grid points $x_i \in (-\infty, \infty)$ in D_d are the inverse images of the equispaced grid points; that is, $x_i = \phi^{-1}(ih_x) = ih_x$. Also in this case note that $\phi'(x) = 1$, and so $q(x) = 1$. This yield the approximation

$$u_x(t) \approx \frac{-1}{h_x} I^{(1)}_{m_x} u(t) \equiv Au(t) \tag{2.0.8}$$

where $u(t) = (u(x_{-N_x}, t), ..., u(x_{N_x}, t))^T$. With $m_x = 2N + 1$ and the skew-symmetric matrix $I^{(1)}_{m_x}$ is defined by (2.0.6). Now, we shall state a general formula for approximating the integral $\int_a^\nu F(t)dt, \nu \in \Gamma$, with Γ defined as in Definition 2.2 and with the indefinite integral taken over Γ. The proof of the following two facts can be found in [12].

Theorem 2.3 *If $F \in \boldsymbol{W}(\frac{\pi}{h})$, and if the integral $\int_{-\infty}^{x} F(t)dt$ exists for all $x \in \mathbb{R}$, then for all $k \in \mathbb{Z}$,*

$$\int_{a}^{kh} F(t)dt = h \sum_{n=-\infty}^{\infty} \delta_{k-n}^{(-1)} F(nh)$$

with

$$\delta_{k}^{(-1)} = \frac{1}{2} + \int_{o}^{k} \frac{\sin(\pi t)}{\pi t} dt.$$

To facilitate the description of the discrete system, we define the matrix $I^{(-1)}$. For a positive integer N define the $(2N+1) \times (2N+1)$ matrix $I^{(-1)}$ by

$$I^{(-1)} = \left[\delta_{k-n}^{(-1)}\right], \quad k, n = -N, ..., N \tag{2.0.9}$$

In certain cases the indefinite sum in Theorem 2.3 can be evaluated directly, but in general it must be truncated. To do that, we give more assumption of the function F as in the following Theorem (see, [12]). For the conformal map χ replaced by ϕ in Definition 2.2 we have

Theorem 2.4 *Let $\frac{F}{\chi'} \in \boldsymbol{L}_{\alpha}(D)$, with $\alpha > 0, d > 0$, let $\delta_{k}^{(-1)}$ be defined as above, and let $h = \sqrt{\frac{\pi d}{\alpha N}}$. Then there exists a constant C_4, which independent of N, such that*

$$\left| \int_{a}^{z_k} F(t)dt - h \sum_{j=-N}^{N} \delta_{k-j}^{(-1)} \frac{F(z_j)}{\chi'(z_j)} \right| \leq C_4 \exp(-\sqrt{\pi d \alpha N}). \tag{2.0.10}$$

3 The Sinc-Galerkin Method: Balance Laws

To derive an approximate solution for the system in Equation(1.0.2), we start with one-dimensional balance law

$$u_t(x,t) + F(u(x,t))_x = H(x,t), \quad (x,t) \in \mathbb{R} \times (0, T_0) \tag{3.0.1}$$

subject to the initial condition

$$u(x,0) = k(x), \quad x \in \mathbb{R}. \tag{3.0.2}$$

We should mention that Al-Khaled [2] derived in more details the solution for one-dimensional conservation laws by using Sinc method. Suppose $k(x) \in \mathbf{L}_{\alpha}(D)$, and taking $h = \sqrt{\frac{\pi d}{\alpha N}}$, Sinc collocation with respect to x results into a system of Volterra integral equations

$$u(t) = \int_{0}^{t} [H - F(u(\tau))Au(\tau)]d\tau + k \tag{3.0.3}$$

with $u(t) = (u_{-N_x}(t), ..., u_{N_x}(t))^T$, where in general $u_i(t) = u_i(x_i, t)$, $k = (k(z_{-N_x}), ..., k(z_{N_x}))^T$ and $H = (H(z_{-N_x}), ..., H(z_{N_x}))^T$, and with the square matrix $A = \frac{-1}{h_x} I^{(1)}_{m_x}$. We next collocate with respect to the variable t via the use of the indefinite integration formula (2.0.10). As follows: since our domain in the time direction is $(0, T_0)$, the eye-shaped domain $D_E = \{t = x + iy : |arg(t/(T_0 - t))| < d \leq \frac{\pi}{2}\}$ is mapped conformably onto D_d via $\chi(t) = \ln(t/(T_0 - t))$. Define $S(k, h_t) \circ \chi(t), \;\; k = -N_t, ..., N_t$, to be the basis function for the interval $(0, T_0)$. Thus defining a matrix E by $E = h_t I^{(-1)}_{m_t} D\left(\frac{1}{\chi'(t_j)}\right)$ with the nodes $t_j = \chi^{-1}(jh_t), \;\; j = -N_t, ..., N_t$, where

$$D\left(\frac{1}{\chi'(t_j)}\right) = diag\left[D\left(\frac{1}{\chi'(t_{-N_t})}\right), ..., D\left(\frac{1}{\chi'(t_{N_t})}\right)\right]$$

$h_t = \sqrt{\frac{\pi d}{\alpha N_t}}$, and $I^{(-1)}_{m_t}$ as defined in (2.0.9), with $m_t = 2N_t + 1$. Then the solution of equation (3.0.1) is in matrix form is given by the rectangular $m_x \times m_t$ matrix $U = [u_{ij}]$:

$$U = \mathcal{H} - (F'(U) \circ AU)\, E^T + K \tag{3.0.4}$$

where the notation $\circ$ denote the Hadamard matrix multiplication. The vector K has the same dimension as the vector U, and every column of K consists of the same vector k. For the convergence of our approximate solution obtained in (3.0.4), we state the following two Theorems where the proof can be found in [2].

Theorem 3.1 *For the function $u(x, t)$ in equation (3.0.1) and the initial condition (3.0.2), let the matrix U be as defined in (3.0.4). Then for $N_x, N_t > \frac{4}{\pi d \alpha}$ there exists a constant C independent of N_x, N_t such that*

$$\sup_{(x_i, t_j)} \|u(x_i, t_j) - U\| \leq CN \exp(-\sqrt{\pi d \alpha N})$$

where $N = \min\{N_x, N_t\}$

The existence of the solution for the discrete system (3.0.4) is guarantee as in the following theorem.

Theorem 3.2 *Given a constant $R > 0$, there is a constant $T_0 > 0$ such that if $\|U_1 - U_0\| < \frac{R}{2}$, then the solution of (3.0.4) has a unique solution. Moreover, the iteration scheme $U_{n+1} = \mathcal{H} - (F'(U_n) \circ AU_n)\, E^T + K$ converges to this unique solution.*

4 The Adomian Decomposition Method (ADM)

To solve the 2×2 system of balance laws by ADM [4, 5], we rewrite the system in (1.0.2) as

$$\begin{aligned} u_t + f_u u_x + f_v v_x &= h_1(x,t), \quad u(x,0) = u_0(x) \\ v_t + g_u u_x + g_v v_x &= h_2(x,t), \quad v(x,0) = v_0(x) \end{aligned} \tag{4.0.1}$$

which can be written in an operator form

$$\begin{aligned} L_t u + f_u L_x u + f_v L_x v &= h_1(x,t), \quad u(x,0) = u_0(x) \\ L_t v + g_u L_x u + g_v L_x v &= h_2(x,t), \quad v(x,0) = v_0(x) \end{aligned} \tag{4.0.2}$$

where the notations $L_t = \frac{\partial}{\partial t}$ and $L_x = \frac{\partial}{\partial x}$ symbolize the linear differential operator. Applying the inverse operator $L_t^{-1} = \int_0^t .dt$ to the system in (4.0.2) yields

$$\begin{aligned} u(x,t) &= u_0(x) - L_t^{-1}\left[h_1(x,t) - (\phi_1(u,v) + \phi_2(u,v))\right] \\ v(x,t) &= v_0(x) - L_t^{-1}\left[h_2(x,t) - (\phi_3(u,v) + \phi_4(u,v))\right] \end{aligned} \tag{4.0.3}$$

where $\phi_1(u,v) = f_u u_x$, $\phi_2(u,v) = f_v v_x$, $\phi_3(u,v) = g_u u_x$ and $\phi_4(u,v) = g_v v_x$. The ADM [4, 5] assumes an infinite series solution for unknown functions $u(x,t)$ and $v(x,t)$ in the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \quad , \quad v(x,t) = \sum_{n=0}^{\infty} v_n(x,t)$$

and the nonlinear operators $\phi_i(u,v), \;\; i = 1,2,3,4$ by the infinite series of Adomian polynomials given by $\phi_(u,v) = \sum_{n=0}^{\infty} A_{ni}$, where A_{ni} are the appropriate Adomian's polynomials which are constructed according to algorithm determined in [14]. In our case, the general form of formulas for $A_{ni}, i = 1,2,3,4$, Adomian's polynomials as

$$A_{ni}(u_0, u_1, ..., u_n, v_0, v_1, ..., v_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\phi_i\left(\sum_{k=0}^{\infty}\lambda^k u_k, \sum_{k=0}^{\infty}\lambda^k v_k\right)\right]_{\downarrow \lambda=0} \quad , \quad n \geq 0. \tag{4.0.4}$$

These formulas are easy to set computer code to get as many as we need in the calculations of the numerical solutions. Following the decomposition method, the nonlinear system (4.0.3) is constructed in a form of the recursive relation given by

$$\begin{aligned} u_0(x,t) &= u_0(x), \quad u_{n+1}(x,t) = -L_t^{-1}\left[h_1(x,t) - (A_{n1} + A_{n2})\right], n \geq 1 \\ v_0(x,t) &= v_0(x), \quad v_{n+1}(x,t) = -L_t^{-1}\left[h_2(x,t) - (A_{n3} + A_{n4})\right], n \geq 1 \end{aligned} \tag{4.0.5}$$

We construct the solution $u(x,t)$ and $v(x,t)$ as

$$u(x,t) = \lim_{n\to\infty} \sum_{k=0}^{n} u_k(x,t) \quad , \quad v(x,t) = \lim_{n\to\infty} \sum_{k=0}^{n} v_k(x,t) \tag{4.0.6}$$

4.1 Convergence of the ADM Approximation

The decomposition series solutions are generally converge very rapidly in real physical problems. The convergence of the decomposition series have investigated by several authors [8, 3]. In [11] the authors have given a new condition for obtaining convergence of the decomposition series to the classical presentation of the ADM. Here we study the convergence analysis presented in [11] applied to the Van der Waals system. For more applications on different models for the method in [11], one can look at [3]. Let us consider the Hilbert space H, defined by $H = L^2((\alpha,\beta)\times[0,T])$ the set of applications: $u : (\alpha,\beta)\times[0,T] \longrightarrow \mathbb{R}$; with $\int_{(\alpha,\beta)\times[0,T]} u^2(x,s)dsd\tau < \infty$, and the following scalar products:

- $(u,v)_H = \int_{(\alpha,\beta)\times[0,T]} u(x,s)v(x,s)dsd\tau$
- $\| u \|_H^2 = \int_{(\alpha,\beta)\times[0,T]} u^2(x,s)dsd\tau$

Let us denote $L(u) = \frac{\partial}{\partial t}u$, and $L(v) = \frac{\partial}{\partial t}v$ so, Van der Waals can be written as

$$L(u) = -p(v)_x, \quad L(v) = u_x$$

According to [11], the ADM converges if the following two hypothesis are satisfied.

1. $H_1 :\ (L(u)-L(w), u-w) \geq K(u,w) \| u-w \|^2,\ K(u,w) > 0,\ \forall\ u,w \in H$
2. H_2: For any $M > 0$, there exists a constant $C(M) > 0$ such that for $u,w \in H$ with $\| u \| \leq M, \| w \| \leq M$, we have $(L(u)-L(w), r) \leq C(M) \| u-w \| \ \| r \|$ for every $r \in H$.

The convergence analysis, based on the above hypothesis of the ADM applied to various other nonlinear equations that has been conducted by several authors [8, 3]. We can prove the following Theorem by resembles the convergence proof in [8, 11, 3].

Theorem 4.1 *(***Sufficient condition of convergence***). The Adomian method applied to the Van der Waals system converges towards a particular solution.*

5 Applications: Fluid Dynamics

The ADM provides an analytical solution in the form of an infinite power series. However, there is a practical need to evaluate this solution, and to obtain numerical values from the infinite power series. In order to investigate the accuracy of the ADM with a finite number of terms. The Van der Waals equation in fluid dynamics is solved numerically by both the Sinc method and ADM, and the corresponding results are compared with the Adomian solution. The numerical method adopted in this paper was the Sinc method, and the parameter N is chosen to be 16.

Example 5.1 " *The one-dimensional isothermal motion of a compressible elastic fluid solid can be described in Lagrangian coordinates by the coupled system*

$$u_t + p(v)_x = 0, \quad v_t - u_x = 0, \quad x \in \mathbb{R}, \quad t > 0 \tag{5.0.1}$$

which describes the one-dimensional longitudinal motion in elastic bars or fluids, where u is the velocity, v is the specific volume, and p is the pressure. The pressure $p(v)$ varies for different materials. Typically, for an ideal gas, $p(v)$ is strictly increasing, i.e., $p'(v) > 0$, so that the system (5.0.1) is hyperbolic. However, for some matrical models $p'(v)$ may not monotone. A typical example is the Van der Waals fluid, whose constitute function $p(v)$ is given by

$$p(v) = \frac{RT}{v-b} - \frac{a}{v^2} \tag{5.0.2}$$

where R is the gas constant, T is the temperature, a, b are positive constants. For more details, see [13]. During the co-existence of gas and liquid, $p'(v)$ may become positive within an interval, as in the case (5.0.2) with suitable parameters, making the system (5.0.1) elliptic in this region " [2], Al-Khaled [2] discussed the same problem (5.0.1) but subject to Riemann initial conditions, in this example, following [13], we take $RT = 1, a = 0.9$ and $b = 0.25$. The system is elliptic for $\alpha \leq v \leq \beta$, where $\alpha = 0.574912$ and $\beta = 1.036251$. We discuss problem (5.0.1) with smooth initial conditions defined in the interval $(-\infty, \infty)$ as

$$u(x,0) = 1 - 0.5\cos x, \;\; v(x,0) = 1 + 0.5\sin x \tag{5.0.3}$$

Figures, $1-4$, shows that the initial condition crosses the elliptic region, in other words, a solution may start with an initial condition outside the elliptic region, then enter the elliptic region after some finite x and some time t. Our calculations shows that ADM is unstable inside the elliptic region, see Figure 2 and 4. While the solution using the Sinc method is stable inside the elliptic region, see Figure 1 and 3.

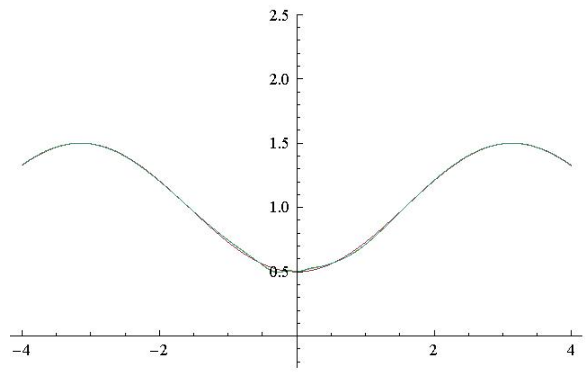

Figure 1: The approximate solution of velocity $u(x,t)$ using Sinc Method for $t = 0, 0.2, 0.4, 1.0$

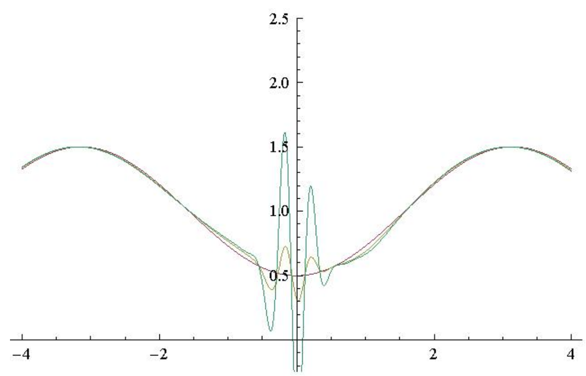

Figure 2: The approximate solution of velocity $u(x,t)$ using ADM for $t = 0, 0.2, 0.4, 0.5$

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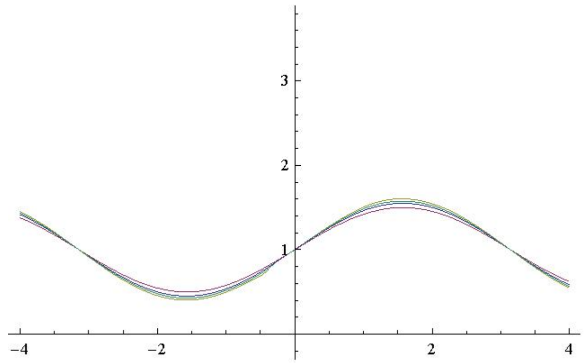

Figure 3: The approximate solution of volume $v(x,t)$ using Sinc method for $t = 0, 0.2, 0.4, 1.0$

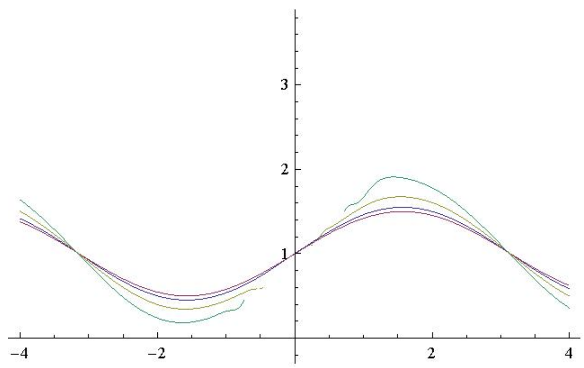

Figure 4: The approximate solution of volume $v(x,t)$ using ADM for $t = 0, 0.2, 0.4, 0.5$

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