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Control Lyapunov Function for Feedback Stabilization of Affine in the Control Stochastic Time-Varying Systems

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Abstract

The aim of this paper is to study the problem of nonuniform in time global asymptotic stability in probability of affine in the control stochastic time-varying systems. We extend Artstein-Sontag theorem to the concept of control Lyapunov function to derive the necessary and sufficient conditions for feedback stabilization for affine in the control stochastic time-varying systems.

Keywords: Stochastic time-varying system; Control Lyapunov function; Global asymptotic stability in probability.
AMS Subjects Classifications: 60H10; 93C10; 93D05; 93D15; 93D21; 93E15.

1 Introduction

Lyapunov functions play an important role to synthesis and design in control theory. The stochastic version of the Lyapunov theorem has been used to derive the necessary and sufficient conditions for stabilization of stochastic differential system at their equilibrium state. The stabilizability of various types of nonlinear stochastic differential system has been studied for different notations of stochastic stability in the recent years (see, for instance, Handel

[5], Florchinger [4],[3], Krstic and Deng [10], Deng et.al [2]).

Tsinias [17], Karafyllis [8] and Tsinias and Karafyllis [16] have shown that for a class of triangular systems whose dynamics contain time-varying unknown parameters, it is possible to find, by applying a backstepping design procedure, a smooth time-varying feedback controller in such a way that the equilibrium of the resulting closed-loop system is globally asymptotically stable, in general nonuniform with respect to initial values of time. Karafyllis and Tsinias [6] established converse Lyapunov theorems for the concepts of nonuniform in time robust globally asymptotically stable and nonuniform in time input-to-state stability and gave explicit formula of a feedback law exhibiting input-to-state stability for time-varying system.

The aim of this paper is to study the problem of nonuniform in time global asymptotic stability in probability (GASP) of stochastic time-varying systems. We extend Artstein-Sontag theorem (see Artstein [1] and Sontag [14]) by introducing the concept of time-varying control Lyapunov function. An explicit formula for a time-varying feedback stabilizer is proposed in Theorem 3.2. We show that, even for a class of autonomous systems, it is possible to achieve nonuniform in time globally asymptotic stabilization in probability by smooth time-varying feedback, although a smooth time-independent feedback exhibiting uniform in time stabilization does not exist. The main tools used in this paper are the stochastic Lyapunov theorem proved by Khasminiskii [9] and the converse stability theorem of Kushner [11].

The paper is organized as follows. Section 2 introduces the class of affine in the control stochastic time-varying systems and some basic definitions and results that we are dealing with in this paper. In section 3, we state and prove the main results of the paper on the feedback stabilization of the class of stochastic time-varying systems. Finally in section 4 we provide a numerical example illustrating our results.

2 Stochastic Stability

The purpose of this section is to introduce the class of control stochastic time-varying systems with which we are concerned in this paper.

A detailed exposition on the subject can be found in the books of Speyer and Chung [15] and Khasminiskii [9].

Let (Ω, F, P) be a complete probability space, and denote by $(w_t)_{t\geq 0}$ a standard

R^m-valued Wiener process defined on this space.
Consider the multi-input stochastic time-varying system in R^n

$$dx = f(t, x, d)dt + \sum_{k=1}^{m} h_k(t, x)dw \tag{1}$$

$$x \in R^n,\ d \in D,\ t \geq 0.$$

We assume that $D \subset R^q$ is a nonempty compact set and $f : R^+ \times R^n \times D \to R^n, h_k : R^+ \times R^n \to R^{n\times m}, 1 \leq k \leq m$ are mappings with $f(t, 0, d) = 0, h_k(t, 0) = 0$ for all $(t, d) \in R^+ \times D$ that satisfies the following hypotheses:
(i) The functions $f(t, x, d), h_k(t, x)$ are Borel measurable in t for all $(x, d) \in R^n \times D$.
(ii) The functions $f(t, x, d)$ is continuous in d for all $(t, x) \in R^+ \times R^n$.
(iii) The functions $f(t, x, d)$ and $h_k(t, x)$ are locally bounded and locally Lipschitz continuous in $x \in R^n$, uniformly in $d \in D$, in the sense that for every bounded interval $I \subset R^+$ and for every compact subset S of R^n, there exists a constant $C \geq 0$ such that

$$|f(t, x, d) - f(t, y, d)| + \sum_{k=1}^{m} |h_k(t, x) - h_k(t, y)| \leq C|x - y|.$$

$$\forall t \in I,\ (x, y) \in S \times S,\ d \in D,$$

where $|.|$ denotes the usual Euclidian norm.

Notations. Throughout this paper we adopt the following notations:
• M_D denotes the set of all measurable functions from $R^+ = [0, +\infty)$ to D and functions $d \in M_D$ are time-varying parameters, where D is any given compact subset of R^q. For each $d \in M_D$, we denote by $x(t) = x(t_0, x_0, d)$ the solution of (1) at time t that corresponds to some input $d \in M_D$ initiated from x_0 at time t_0.
• K^+ denotes the class of positive nondecreasing C^∞ functions $\phi : R^+ \to (0, +\infty)$.

Definition 2.1 *A function $\gamma : R^+ \to R^+$ is*
∗ *A K-function if it is continuous, strictly increasing and $\gamma(0) = 0$,*

* *A K_∞-function if it is a K-function and also $\gamma(r) \to \infty$ as $r \to \infty$ and*
* *A positive definite function if $\gamma(r) > 0$ for all $r > 0$, and $\gamma(0) = 0$.*

Definition 2.2 *The equilibrium $x = 0$ of the system (1) is*
• *globally stable in probability, if for every $t_0 \geq 0, d \in M_D$ and $\varepsilon > 0$ there exists a class K-function $\gamma(.)$ such that*

$$P\left\{|x(t)| < \gamma(|x_0|)\right\} \geq 1 - \varepsilon, \tag{2}$$

$$\forall x_0 \in R^n \backslash \{0\}.$$

• *GASP, if it is globally stable in probability and*

$$P\{\lim_{t\to\infty} |x(t)| = 0\} = 1, \tag{3}$$

$$\forall x_0 \in R^n.$$

Next, consider the multi-input stochastic time-varying system in R^n

$$dx = f(t, x, v)dt + \sum_{z=1}^{p} g_z(t, x)u^z dt + \sum_{k=1}^{m} h_k(t, x)dw \tag{4}$$

$$x \in R^n,\ v \in R^l,\ u \in R^p,\ t \geq 0,$$

where the dynamics $f(.), h_k(.)$ and $g_z : R^+ \times R^n \to R^{n\times p}$, $1 \leq z \leq p$ are C^0 and locally Lipschitz with respect to (x, v) with $f(., 0, 0) = 0$ and $h_k(., 0) = 0$. The main results of this paper (Theorem 3.1 and Theorem 3.2) constitute extensions of the well-known Artstein-Sontag theorem and guarantees existence of a C^∞ mapping $u = k(t, x)$ in such a way that the resulting closed-loop system

$$dx = f(t, x, v)dt + \sum_{z=1}^{p} g_z(t, x)k(t, x)^z dt + \sum_{k=1}^{m} h_k(t, x)dw \tag{5}$$

satisfies the nonuniform in time GASP with v as input.
Denoting by **D** the infinitesimal generator of the stochastic process solution of the stochastic differential system (4), that is, **D** is the second-order differential operator defined for any function Φ in $C^2(R^+ \times R^n, R)$ by

$$\mathbf{D}\Phi(t, x) = \sum_{i=1}^{n} f^i(t, x, v)\frac{\partial \Phi(x)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m} h_k^i h_k^j \frac{\partial^2 \Phi(t, x)}{\partial x_i \partial x_j}, \tag{6}$$

where $1 \leq i, j \leq n$. We also denote by $\mathbf{D}_z$ the first order differential operator defined for any function Φ in $C^1(R^+ \times R^n, R)$ by

$$\mathbf{D}_z\Phi(t,x) = \sum_{i=1}^{n} g_z^i(t,x)\frac{\partial \Phi(t,x)}{\partial x_i}, \tag{7}$$

where $z \in (1, ..., p)$, one can introduce the notion of control Lyapunov function as follows.

Definition 2.3 *Let $\gamma(t,x) : R^+ \times R^+ \to R^+$ be a positive definite function, which is C^0, locally Lipschitz in r, the stochastic time-varying system (4) admits a "* control Lyapunov function,*" if there exists a C^2 function $\Phi : R^+ \times R^n \to R^+$, a class K_∞ functions a_1, a_2, and a positive definite function $\rho : R^+ \to R^+$ such that for all $(t,x,d) \in R^+ \times R^n \times D$ the following conditions hold:*

$$a_1(t,|x|) \leq \Phi(t,x) \leq a_2(t,|x|), \tag{8}$$

$$\boldsymbol{D}_z\Phi(t,x) = 0, \quad |v| \leq \gamma(t,|x|) \quad \Rightarrow \quad \boldsymbol{D}\Phi(t,x) \leq -\rho(\Phi(t,x)) < 0. \tag{9}$$

Remark 2.4 *Alternatively, the condition (9) obtained in Definition 2.3 can be replace with the following condition*

$$|v| \leq \gamma(t,|x|) \quad \Rightarrow \quad \boldsymbol{D}\Phi(t,x) + \sum_{z=1}^{p} \boldsymbol{D}_z\Phi(t,x)u^z \leq -\rho(\Phi(t,x)) < 0.$$

3 Main Results

We extend the well-known Artstein-sontag theorem by introducing the concept of time-varying control Lyapunov function (Theorem 3.1). Among other things we establish that, even for a class of autonomous systems, it is possible to achieve nonuniform in time globally asymptotic stabilization in probability by smooth time-varying feedback, although a smooth time-independent feedback exhibiting uniform in time stabilization does not exist (Corollary 3.4).

Theorem 3.1 *Consider the system (4) and let $\gamma(t,s) : R^+ \times R^+ \to R^+$ be a positive definite function, which is C^0, locally Lipschitz in s. Then the following statements are equivalent;*
(i) There exists a C^∞ function $k : R^+ \times R^n \to R^p$ with $k(t,0) = 0$ for all $t \geq 0$

such that the resulting closed-loop system (5) satisfies the nonuniform in time GASP with v as input.
(ii) There exists a C^0 function $k : R^+ \times R^n \to R^p$ with $k(t, 0) = 0$ for all $t \geq 0$ such that the resulting closed-loop system (5) satisfies the same property as in statement (i).
(iii) The system (4) admits a control Lyapunov function.

Proof : $(i \to ii)$ Is obvious.
$(ii \to iii)$ Suppose that there exists a map $k(.)$, as in statement (ii) of the theorem, such that the closed-loop system (5) satisfies the GASP with v as input. Then by Kushner's converse Lyapunov theorem [11], there exists a C^2 function $\Phi : R^+ \times R^n \to R^+$ in such a way that (8) holds and

$$|v| \leq \gamma(t, |x|) \;\Rightarrow\; \mathbf{D}_0\Phi(t, x) = \mathbf{D}\Phi(t, x) + \sum_{z=1}^{p} \mathbf{D}_z\Phi(t, x)k(t, x)^z \leq -\Phi(t, x).$$

The latter implies (9) with $\rho(s) = s$ and condition $\mathbf{D}_z\Phi(t, x) = 0$. Therefore, Definition 2.3 holds and $\Phi(t, x)$ is a control Lyapunov function for the system (4).
$(iii \to i)$ Consider the functions a_1, a_2 and Φ as defined in (8) and (9). Notice, by virtue of (8), that

$$\frac{\partial \Phi}{\partial x}(t, 0) = 0. \tag{10}$$

For those t, x condition (9) in conjunction with (10) enables us to build by standard partition of unity arguments a C^∞ map $K : R^+ \times R^n \to R^p$ with $K(., 0) = 0$ such that

$$\mathbf{D}_0\Phi(t, x) \leq -\rho(\Phi(t, x)). \tag{11}$$

From (11) and $\Phi(t, x) \geq 0, \Phi_t = \Phi(t, x)$ is a supermartingale. By a supermartingale inequality (see Rogers and Williams [13]), for any class K_∞ function $\beta(.)$, we get

$$P\{\sup_{0 \leq r \leq t} \Phi_r \geq \beta(\Phi_0)\} \leq \frac{L\Phi_0}{\beta(\Phi_0)}, \tag{12}$$

for any $t \geq 0, \Phi_0 \neq 0$ and constant $L > 0$. Therefore,

$$P\{\sup_{0 \leq r \leq t} \Phi_r < \beta(\Phi_0)\} \geq 1 - \frac{L\Phi_0}{\beta(\Phi_0)}. \tag{13}$$

Denote $\lambda = a_1^{-1} \circ \beta \circ a_2$. Then $\sup_{0 \le r \le t} \Phi_r < \beta(\Phi_0)$ implies that

$$\sup_{0 \le r \le t} |x(r)| < \lambda(|x_0|),$$

and thus

$$P\{\sup_{0 \le r \le t} |x(r)| < \lambda(|x_0|)\} \ge 1 - \frac{L\Phi_0}{\beta(\Phi_0)}. \tag{14}$$

For a given $\varepsilon > 0$, choose $\beta(.)$ such that

$$\beta(\Phi_0) \ge \frac{L\Phi_0}{\varepsilon}. \tag{15}$$

With (14) and (15) we get

$$P\{\sup_{0 \le r \le t} |x(r)| < \lambda(|x_0|)\} \ge 1 - \varepsilon, \tag{16}$$

for any $t \ge 0, x_0 \in R^n \backslash \{0\}$. This implies that

$$P\{|x(t)| < \lambda(|x_0|)\} \ge 1 - \varepsilon,$$

for any $t \ge 0, x_0 \in R^n \backslash \{0\}$. From (16) and Definition 2.2 the equilibrium is globally stable in probability with respect to (5). Finally, from (11) and the stochastic version of La Salle's invariance theorem (see Kushner [12]), we get the stochastic process solution $x(t)$ of the closed-loop system (5) which tends to 0 with probability 1, that is

$$P\{\lim_{t \to \infty} |x(t)| = 0\} = 1, \tag{17}$$

for all $x_0 \in R^n$. Therefore, from (17) and the above globally stability in probability and Definition 2.2 we get the equilibrium is GASP with respect to (5). This completes the proof of Theorem 3.1. □

In the following theorem we use of an explicit formula of a feedback law established by Florchinger [4] and exhibiting GASP property for the system (5).

Theorem 3.2 *Let Φ be a control Lyapunov function associated to the stochastic time varying system (4), and for any $(t, x) \in R^+ \times R^n$, denote by $b(t, x)$ and $\Psi(t, x)$ the functions defined by*

$$b(t, x) = \boldsymbol{D}_z \Phi(t, x), \tag{18}$$

and

$$\Psi(t,x) = \max_{|v|\leq\gamma(t,|x|)} \boldsymbol{D}\Phi(t,x) + \rho(\Phi(t,x)), \tag{19}$$

then the feedback law

$$k(t,x) = \xi(\Psi(t,x),(b(t,x))^2)(b(t,x)), \tag{20}$$

where $b(t,x)$ *and* $\Psi(t,x)$ *are given by (18) and (19), respectively, and*

$$\xi(\Psi,b) = \begin{cases} -\frac{\Psi+\sqrt{\Psi^2+b^2}}{b(1+\sqrt{1+b})} & if\ b>0, \\ 0 & if\ b=0, \end{cases} \tag{21}$$

renders the stochastic system (5) GASP with v *as input.*

Proof : Suppose Φ satisfies Definition 2.3, from (9) and definition (19) of $\Psi(t,x)$ it follows that

$$\mathbf{D}_z\Phi(t,x) = 0 \ \Rightarrow \Psi(t,x) \leq 0. \tag{22}$$

Notice that from (20) and (21) the feedback law $k(t,x)$ is well defined for all (t,x), since the denominator in (20) is strictly positive for all $(t,x)\in R^+\times R^n$, and is of class $C^0(R^+\times R^n)$. Indeed, $\mathbf{D}_z\Phi(t,x)\geq 0$ for all $(t,x)\in R^+\times R^n$, and suppose that $\mathbf{D}_z\Phi(t,x)=0$ for certain $(t,x)\in R^+\times R^n$. It then follows from (9) and definition (19) of $\Psi(.)$ that $\Psi(t,x)\leq 0$. Furthermore, according to regularity assumptions made for $\Phi(t,x), f(t,x,v), g_z(t,x), h_k(t,x)$, $\gamma(t,x)$ and $\rho(.)$, the map $k(t,x)$ as defined by (20) is C^0 on $R^+\times R^n$ and locally Lipschitz with respect to $x\in R^n$, with $k(t,0)=0$ for all $t\geq 0$.
Denoting by $\mathbf{D}_0$ the infinitesimal generator of the stochastic process solution of the resulting closed-loop system (5) we get

$$\mathbf{D}_0\Phi(t,x) = \max_{|v|\leq\gamma(t,|x|)} \mathbf{D}\Phi(t,x) + \sum_{z=1}^{p} \mathbf{D}_z\Phi(t,x)k(t,x)^z.$$

Then, we have by taking into account (20) and (22) that

$$\mathbf{D}_0\Phi(t,x) \leq -\rho(\Phi(t,x)). \tag{23}$$

The rest of the proof is straightforward consequence of (23) and Theorem 3.1 (implication $iii\to i$). This completes the proof of Theorem 3.2. □

We next specialize the results of Theorem 3.1 to the following case of time-varying systems:

$$dx = f(t,x)dt + \sum_{z=1}^{p} g_z(t,x)u^z dt + \sum_{k=1}^{m} h_k(t,x)dw \tag{24}$$
$$x \in R^n,\ u \in R^p,\ t \geq 0,$$

where $f(.), h_k(.)$ and $g_z(.)$ are C^0 and locally Lipschitz with respect to x with $f(.,0) = 0$ and $h_k(.,0) = 0$.

Corollary 3.3 *Consider the system (24) then the following statements are equivalent:*
(i) There exist a C^2 function $\Phi : R^+ \times R^n \to R^+$, a class K_∞ functions a_1, a_2, and a C^0 positive definite function $\rho : R^+ \to R^+$ such that for all $(t,x) \in R^+ \times R^n$

$$a_1(t,|x|) \leq \Phi(t,x) \leq a_2(t,|x|), \tag{25}$$

$$\boldsymbol{D}_z\Phi(.) = 0,\ \Rightarrow\ \boldsymbol{D}\Phi(t,x) \leq -\rho(\Phi(t,x)), \tag{26}$$

where

$$\boldsymbol{D}_z\Phi(t,x) = \sum_{i=1}^{n} g_z^i(t,x)\frac{\partial\Phi(t,x)}{\partial x_i},$$

and

$$\boldsymbol{D}\Phi(t,x) = \sum_{i=1}^{n} f^i(t,x)\frac{\partial\Phi(t,x)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m} h_k^i(t,x)h_k^j(t,x)\frac{\partial^2\Phi(t,x)}{\partial x_i\partial x_j}.$$

(ii) There exists a C^∞ function $k : R^+ \times R^n \to R^p$ with $k(t,0) = 0$ for all $t \geq 0$ such that $0 \in R^n$ is GASP for the resulting closed-loop system

$$dx = (f(t,x) + \sum_{z=1}^{p} g_z(t,x)k(t,x)^z)dt + \sum_{k=1}^{m} h_k(t,x)dw.$$

(iii) For every C^0 function $\gamma(t,s) : R^+ \times R^+ \to R^+$, being locally Lipschitz in s, and in such that, for each $t \geq 0$ the mapping $\gamma(t,.)$ is positive definite, there exists a C^∞ function $\widetilde{k} : R^+ \times R^n \to R^p$ with $\widetilde{k}(t,0) = 0$ for all $t \geq 0$, in such a way that the system

$$dx = (f(t,x) + \sum_{z=1}^{p} g_z(t,x)(\widetilde{k}(t,x)^z + v))dt + \sum_{k=1}^{m} h_k(t,x)dw, \tag{27}$$

satisfies GASP with v as input.

Proof : $(i \leftrightarrow ii)$ Is an immediate consequence of Theorem 3.1.
In order to establish $(i) \leftrightarrow (iii)$ consider the system (27) in the following form

$$dx = (f(t,x) + \sum_{z=1}^{p} g_z(t,x)v + \sum_{z=1}^{p} g_z(t,x)u^z)dt + \sum_{k=1}^{m} h_k(t,x)dw,$$

where $u = \widetilde{k}(t,x)$. The latter has the form time-varying system (4) with $\widetilde{f}(t,x,v) = f(t,x) + \sum_{z=1}^{p} g_z(t,x)v$, that is

$$dx = (\widetilde{f}(t,x,v) + \sum_{z=1}^{p} g_z(t,x)u^z)dt + \sum_{k=1}^{m} h_k(t,x)dw. \qquad (28)$$

The equivalence between (i) and (iii) follows directly from theorem 3.1 and the obvious consequence of (28):

$$\mathbf{D}_z\Phi(.) = 0, \quad |v| \leq \gamma(t,|x|) \quad \Rightarrow \quad \mathbf{D}\Phi(t,x) \leq -\rho(\Phi(t,x)),$$

where

$$\mathbf{D}\Phi(t,x) = \sum_{i=1}^{n} \widetilde{f}^i(t,x,v)\frac{\partial\Phi(t,x)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m} h_k^i(t,x)h_k^j(t,x)\frac{\partial^2\Phi(t,x)}{\partial x_i \partial x_j}.$$

The rest part of proof is straightforward and is left to the reader. This completes the proof of Corollary 3.3. □

Corollary 3.4 *Consider the system*

$$dx = (f(x) + \sum_{z=1}^{p} g_z(x)u^z)dt + \sum_{k=1}^{m} h_k(x)dw \qquad (29)$$
$$x \in R^n, \; u \in R,$$

where $f(.), h_k(.)$ and $g_z(.)$ are locally Lipschitz with $f(0) = 0$ and $h_k(0) = 0$, and suppose that (29) is globally uniformly asymptotically stabilized at the origin by means of a C^0 static feedback $u = k(x)$ with $k(0) = 0$. Then for every C^0 function $\gamma(t,s) : R^+ \times R^+ \to R^+$, being locally Lipschitz in s, and in such that, for each $t \geq 0$ the mapping $\gamma(t,.)$ is positive definite, there exists a C^∞ time-varying feedback law $u = k(t,x)$ with $k(.,0) = 0$ such that the system

$$dx = (f(x) + \sum_{z=1}^{p} g_z(x)(k(t,x)^z + u))dt + \sum_{k=1}^{m} h_k(x)dw$$

satisfies the GASP property with u as input.

Proof : Using Kushner's converse Lyapunov theorem [11] we may find a C^2 radially unbounded, positive definite function $\Phi : R^n \to R^+$ that satisfies

$$\mathbf{D}_0\Phi(x) = \mathbf{D}\Phi(x) + \sum_{z=1}^{p} \mathbf{D}_z\Phi(x)k_0(x)^z < 0,$$

where $\mathbf{D}_0$ is the infinitesimal generator of the stochastic process solution of the closed-loop system

$$dx = (f(x) + \sum_{z=1}^{p} g_z(x)k_0(x)^z)dt + \sum_{k=1}^{m} h_k(x)dw.$$

It then follows that

$$\mathbf{D}_z\Phi(x) = 0, \ \Rightarrow \mathbf{D}\Phi(x) \leq -\rho(\Phi(t,x)), \tag{30}$$

for a certain C^0 positive definite function $\rho : R^+ \to R^+$. The rest of the proof is straightforward consequence of (30) and Corollary 3.3. This completes the proof of Corollary 3.4. □

4 Application

In this section we illustrate our results by a numerical example.

Example 4.1 *Denote by $x(t) \in R^2$ the solution of the stochastic time-varying system*

$$dx = \begin{pmatrix} x_2 - x_1 \\ -x_1 - x_2 \end{pmatrix} dt + \begin{pmatrix} x_2 \\ 0 \end{pmatrix} udt + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dw_t, \tag{31}$$

where $(w_t)_{t\geq 0}$ is a standard real-valued Wiener process and u is a real-valued measurable control law.
We apply Corollary 3.3 to establish existence of a locally Lipschitz time-varying feedback $k(t,x)$ that guarantees nonuniform in time GASP for the resulting system:

$$dx = \begin{pmatrix} x_2 - x_1 \\ -x_1 - x_2 \end{pmatrix} dt + \begin{pmatrix} x_2 \\ 0 \end{pmatrix} (k(t,x) + u)dt + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dw_t, \tag{32}$$

with u as input.
Consider the Lyapunov function candidate

$$\Phi(t,x) = \frac{1}{2}(x_1^2 + x_2^2).$$

A simple calculation shows that

$$\boldsymbol{D}_z\Phi(t,x) = x_1x_2, \quad \boldsymbol{D}\Phi(t,x) = -\frac{1}{2}(x_1^2 + x_2^2), \quad \Psi(t,x) = 0. \tag{33}$$

From (33) for those (t,x) we get

$$\begin{aligned}\boldsymbol{D}_0\Phi(t,x) &= -\frac{1}{2}(x_1^2+x_2^2) - \frac{\sqrt{(x_1x_2)^4}}{1+\sqrt{1+(x_1x_2)^2}} \\ &\leq -\Phi(t,x).\end{aligned}$$

Therefore, both (8) and (9) are satisfied with $\rho(s) = s, a_1(s) = s$ and $a_2(s) = 2s$, and thus, according to Corollary 3.3, there exists a C^∞-feedback law $k(t,x)$ with $k(.,0) = 0$ such that the GASP property is fulfilled for the resulting system (32).
Finally, by application of the results of Theorem 3.2 we fined an explicit formula for a locally Lipschitz feedback law. Indeed, by (33) and (20) we find feedback law

$$k(t,x) = -\frac{\sqrt{(x_1x_2)^4}}{x_1x_2(1+\sqrt{1+(x_1x_2)^2})},$$

that renders the equilibrium solution of the resulting system (32) satisfies GASP property.

5 Conclusions

In this paper, we have studied the problem of nonuniform in time globally asymptotic stability in probability of affine in the control stochastic time-varying systems. We have extended Artstein-Sontag theorem to the concept of control Lyapunov function to derive the necessary and sufficient conditions for feedback stabilization for affine in the control stochastic time-varying systems. We also establish that, even for a class of autonomous systems, it is possible to achieve nonuniform in time globally asymptotic stabilization in probability by smooth time-varying feedback, although a smooth time-independent feedback exhibiting uniform in time stabilization does not exist.

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