rgα-Closed and rgα-Open Maps

in Topological Spaces

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Abstract

A set $A$ in a topological space $(X, \tau)$ is said to be a regular generalized $\alpha$-closed if $\alpha cl(A) \subset U$ whenever $A \subset U$ and $U$ is regular $\alpha$-open in $X$. In this paper, we introduce $rg\alpha$-closed map from a topological space $X$ to a topological space $Y$ as the image of every closed set is $rg\alpha$-closed, and also we prove that the composition of two $rg\alpha$-closed maps need not be $rg\alpha$-closed map. We also obtain some properties of $rg\alpha$-closed maps.

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1 Introduction

Generalized closed mappings were introduced and studied by Malghan[18]. $wg$-closed maps and $rwg$-closed maps were introduced and studied by Nagavani[19]. Regular closed maps, $gpr$-closed maps and $rg$-closed maps have been introduced and studied by Long[12], Gnanambal[9] and Arockiarani[1] respectively. In this paper, a new class of maps called regular generalized $\alpha$-closed (briefly, $rg\alpha$-closed) maps, $rg\alpha^*$-closed maps have been introduced and studied their relations with various generalized closed maps. We prove that the composition of two $rg\alpha$-closed maps need not be $rg\alpha$-closed map. We also obtain some properties of $rg\alpha$-closed maps.

Let us recall the following definition which we shall require later.
Definition 1.1. A subset $A$ of a space $(X, \tau)$ is called
1) a preopen set\,[17] if $A \subseteq \text{intcl}(A)$ and a preclosed set if $\text{clint}(A) \subseteq A$.
2) a semiopen set\,[10] if $A \subseteq \text{clint}(A)$ and a semiclosed set if $\text{intcl}(A) \subseteq A$.
3) a $\alpha$-open set\,[20] if $A \subseteq \text{intclint}(A)$ and a $\alpha$-closed set if $\text{clintcl}(A) \subseteq A$.
4) a semi-preopen set\,[1] if $A \subseteq \text{clintcl}(A)$ and a semi-preclosed set if $\text{intclint}(A) \subseteq A$.
5) a regular open set\,[26] if $A = \text{intcl}(A)$ and a regular closed set if $A = \text{clint}(A)$.

The intersection of all semiclosed (resp. semiopen) subsets of $(X, \tau)$ containing $A$ is called the semi-closure (resp. semi-kernal) of $A$ and is denoted by $\text{scl}(A)$ (resp. $\text{sker}(A)$).

Definition 1.2. A subset $A$ of a space $(X, \tau)$ is called
1) a generalized closed set (briefly, $g$-closed)\,[11] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.
2) a semi-generalized closed set (briefly, $sg$-closed)\,[5] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semiopen in $X$.
3) a generalized semi-closed set (briefly, $gs$-closed)\,[2] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.
4) a generalized $\alpha$-closed set (briefly, $g\alpha$-closed)\,[15] if $\alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $X$.
5) an $\alpha$-generalized closed set (briefly, $\alpha g$-closed)\,[14] if $\alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.
6) a generalized semi-preclosed set (briefly, $gsp$-closed)\,[7] if $\text{spcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.
7) a regular generalized closed set (briefly, $rg$-closed)\,[21] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $X$.
8) a generalized preclosed set (briefly, $gp$-closed)\,[16] if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.
9) a generalized preregular closed set (briefly, $gpr$-closed)\,[9] if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $X$.
10) a weakely generalized closed set (briefly, $wg$-closed)\,[19] if $\text{clint}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $X$.
11) a strongly generalized semi-closed set\,[23] (briefly, $g^*$-closed) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$-open in $X$.
12) a $\pi$-generalized closed set (briefly, $\pi g$-closed)\,[8] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\pi$-open in $X$.
13) a weakely closed set (briefly, $w$-closed)\,[25] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semiopen in $X$. 
14) mildly generalized closed set (briefly, mildly $g$-closed)\cite{22} if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$-open in $X$.
15) semi weakly generalized closed set (briefly, $\text{swg}$-closed)\cite{19} if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $X$.
16) regular weakly generalized closed set (briefly, $\text{rwg}$-closed)\cite{19} if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $X$.
17) regular semiopen set\cite{6} if there is a regular open set $U$ such that $U \subset A \subset \text{cl}(U)$.
18) regular $\alpha$-open set (briefly, $\text{r}\alpha$-open)\cite{28} if there is a regular open set $U$ such that $U \subset A \subset \alpha\text{cl}(U)$.
19) regular $w$-closed set (briefly, $\text{rw}$-closed)\cite{4} if $\text{cl}(A) \subset U$ whenever $A \subset U$ and $U$ is regular semiopen in $X$.
20) regular generalized $\alpha$-closed set (briefly, $\text{rg}\alpha$-closed)\cite{28} if $\alpha\text{cl}(A) \subset U$ whenever $A \subset U$ and $U$ is regular $\alpha$-open in $X$.

The complements of the above mentioned closed sets are their respective open sets.

**Definition 1.3.** A map $f : (X, \tau) \to (Y, \sigma)$ is called
(i) $g$-continuous\cite{3} if $f^{-1}(V)$ is $g$-closed set of $(X, \tau)$ for every closed set $V$ of $(Y, \sigma)$,
(ii) $\text{rga}$-continuous\cite{30} if the inverse image of every closed set in $(Y, \sigma)$ is $\text{rga}$-closed set in $(X, \tau)$.
(iii) $\text{rga}$-irresolute map\cite{30} if the inverse image of every $\text{rga}$-closed set in $(Y, \sigma)$ is $\text{rga}$-closed in $(X, \tau)$.
(iv) strongly $\text{rga}$-continuous\cite{30} if the inverse image of every $\text{rga}$-open set in $(Y, \sigma)$ is open in $(X, \tau)$.

**Definition 1.4.** A map $f : (X, \tau) \to (Y, \sigma)$ is said to be
(i) $g$-closed\cite{18} if $f(F)$ is $g$-closed in $(Y, \sigma)$ for every closed set $F$ of $(X, \tau)$,
(ii) $w$-closed\cite{24} if $f(F)$ is $w$-closed in $(Y, \sigma)$ for every closed set $F$ of $(X, \tau)$,
(iii) $\text{wg}$-closed\cite{19} if $f(F)$ is $\text{wg}$-closed in $(Y, \sigma)$ for every closed set $F$ of $(X, \tau)$,
(iv) $\text{rwg}$-closed\cite{19} if $f(F)$ is $\text{rwg}$-closed in $(Y, \sigma)$ for every closed set $F$ of $(X, \tau)$,
(v) $\text{rg}$-closed\cite{1} if $f(F)$ is $\text{rg}$-closed in $(Y, \sigma)$ for every closed set $F$ of $(X, \tau)$,
(vi) $\text{gpr}$-closed\cite{9} if $f(F)$ is $\text{gpr}$-closed in $(Y, \sigma)$ for every closed set $F$ of $(X, \tau)$,
(vii) regular closed\cite{13} if $f(F)$ is closed in $(Y, \sigma)$ for every regular closed set $F$ of $(X, \tau)$. 
Definition 1.5. A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be
(i) \( g \)-open[27] if \( f(U) \) is \( g \)-open in \( (Y, \sigma) \) for every open set \( U \) of \( (X, \tau) \),
(ii) \( w \)-open[24] if \( f(U) \) is \( w \)-open in \( (Y, \sigma) \) for every open set \( U \) of \( (X, \tau) \),
(iii) \( wg \)-open[19] if \( f(U) \) is \( wg \)-open in \( (Y, \sigma) \) for every open set \( U \) of \( (X, \tau) \),
(iv) \( rwg \)-open[19] if \( f(U) \) is \( rwg \)-open in \( (Y, \sigma) \) for every open set \( U \) of \( (X, \tau) \),
(v) \( rg \)-open[1] if \( f(U) \) is \( rg \)-open in \( (Y, \sigma) \) for every open set \( U \) of \( (X, \tau) \),
(vi) \( gpr \)-open[9] if \( f(U) \) is \( gpr \)-open in \( (Y, \sigma) \) for every open set \( U \) of \( (X, \tau) \),
(vii) regular open[13] if \( f(U) \) is open in \( (Y, \sigma) \) for every regular open set \( U \) of \( (X, \tau) \).

2 \( rg\alpha \)-closed Maps and \( rg\alpha \)-open Maps

We introduce the following definition
Definition 2.1. A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be \( regular \) \( generalized \) \( \alpha \)-\( closed \) (briefly, \( rg\alpha \)-\( closed \)) if the image of every closed set in \( (X, \tau) \) is \( rg\alpha \)-closed in \( (Y, \sigma) \).

Theorem 2.1. Every closed map is \( rg\alpha \)-\( closed \) map, but not conversely.

Proof. The proof follows from the definitions and fact that every closed set is \( rg\alpha \)-\( closed \).

The converse of the above Theorem need not be true, as seen from the following example.

Example 2.1 Consider \( X = Y = \{a, b, c\} \) with topologies \( \tau = \{X, \phi, \{a\}, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{a\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity map. Then this function is \( rg\alpha \)-\( closed \) but not closed, as the image of closed set \( \{c\} \) in \( X \) is \( \{c\} \) which is not closed set in \( Y \).

Theorem 2.2. Every \( w \)-\( closed \) map is \( rg\alpha \)-\( closed \) map but not conversely.

Proof. The proof follows from the definitions and fact that every \( w \)-\( closed \) set is \( rg\alpha \)-\( closed \).

The converse of the above Theorem need not be true, as seen from the following example.

Example 2.2 Consider \( X = Y = \{a, b, c\} \) with topologies \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{a\}\} \). Let the map \( f : (X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a) = f(b) = b \) and \( f(c) = c \). Then this function is \( rg\alpha \)-\( closed \) but not \( w \)-\( closed \), as the image of closed set \( \{c\} \) in \( X \) is \( \{c\} \) which is not \( w \)-\( closed \) set in \( Y \).

Theorem 2.3. Every \( rw \)-\( closed \) map is \( rg\alpha \)-\( closed \) map but not conversely.
Proof. The proof follows from the definitions and fact that every rw-closed set is rga-closed.

The converse of the above Theorem need not be true, as seen from the following example.

Example 2.3 Consider $X = Y = \{a, b, c, d\}$ with topologies
\[\tau = \sigma = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}.\]
Let the map $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = f(b) = b$, $f(c) = d$ and $f(d) = c$. Then this function is rga-closed but not rw-closed, as the image of closed set $\{d\}$ in $X$ is $\{c\}$ which is not rw-closed set in $Y$.

Theorem 2.4. Every rga-closed map is rg-closed map but not conversely.

Proof. The proof follows from the definitions and fact that every rga-closed set is rg-closed.

The converse of the above Theorem need not be true, as seen from the following example.

Example 2.4 Consider $X = \{a, b, c\}$, $Y = \{a, b, c, d\}$ with topologies
\[\tau = \sigma = \{X, \phi, \{a\}, \{c\}, \{a, c\}\} and \sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}.\]
Let the map $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = a$, $f(b) = c$ and $f(c) = d$. Then this function is rg-closed but not rga-closed, as the image of the closed set $\{a, b\}$ in $X$ is $\{a, c\}$ which is not rga-closed set in $Y$.

Theorem 2.5. Every rga-closed map is rwg-closed map but not conversely.

Proof. The proof follows from the definitions and fact that every rga-closed set is rwg-closed.

The converse of the above Theorem need not be true, as seen from the following example.

Example 2.5 Consider $X = \{a, b, c\}$, $Y = \{a, b, c, d\}$ with topologies
\[\tau = \sigma = \{X, \phi, \{b, c\}\} and \sigma = \{Y, \phi, \{d\}, \{a, c\}, \{a, c, d\}\}.\]
Let the map $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = a$, $f(b) = b$ and $f(c) = d$. Then this function is rwg-closed but not rga-closed, as the image of the closed set $\{a\}$ in $X$ is $\{a\}$ which is not rga-closed set in $Y$.

Theorem 2.6. Every rga-closed map is gpr-closed map but not conversely.

Proof. The proof follows from the definitions and fact that every rga-closed set is gpr-closed.

The converse of the above Theorem need not be true, as seen from the following example.
Example 2.6 Consider $X = Y = \{a, b, c, d\}$ with topologies 
$
\tau = \sigma = \{Y, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}.
$
Let the map $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = d$, $f(b) = b$, $f(c) = a$ and $f(d) = c$. Then this function is gpr-closed but not rgα-closed, as the image of the closed set $\{c, d\}$ in $X$ is $\{a, c\}$ which is not rgα-closed set in $Y$.

Remark 2.1. The following examples show that the regular closed maps and rgα-closed maps are independent.

Example 2.7 Let $X = Y = \{a, b, c\}$, and a map $f : (X, \tau) \to (Y, \sigma)$ be the identity map with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}\}$. Then $f$ is rgα-closed but not regular closed, as the image of the regular closed set $\{a, c\}$ in $X$ is $\{a, c\}$ which is not closed set in $Y$.

Example 2.8 Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}\}$. Let a map $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = c$, $f(b) = b$ and $f(c) = a$. Then $f$ is regular closed but not rgα-closed, as the image of the closed set $\{c\}$ in $X$ is $\{a\}$ which is not rgα-closed in $Y$.

Remark 2.2. The following examples show that the g-closed maps and rgα-closed maps are independent.

Example 2.9 Let $X = Y = \{a, b, c\}$, and a map $f : (X, \tau) \to (Y, \sigma)$ be the identity map with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Then $f$ is rgα-closed but not g-closed, as the image of the closed set $\{a\}$ in $X$ is $\{a\}$ which is not g-closed set in $Y$.

Example 2.10 Consider $X = \{a, b, c\}$, $Y = \{a, b, c, d\}$, $\tau = \{X, \phi, \{c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Let a map $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = a$, $f(b) = d$ and $f(c) = c$. Then this function is g-closed but not rgα-closed, as the image of the closed set $\{a, b\}$ in $X$ is $\{a, d\}$ which is not rgα-closed in $Y$.

Remark 2.3. The following examples show that the wg-closed maps and rgα-closed maps are independent.

Example 2.11 Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Let a map $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then this function is rgα-closed but not wg-closed, as the image of the closed set $\{a\}$ in $X$ is $\{a\}$ which is not wg-closed set in $Y$. 
Example 2.12 Consider $X = \{a, b, c\}$, $Y = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Let a map $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = c$, $f(b) = b$ and $f(c) = d$. Then this function is $wg$-closed but not $rg\alpha$-closed, as the image of the closed set $\{b, c\}$ in $X$ is $\{b, d\}$ which is not $rg\alpha$-closed in $Y$.

Remark 2.4. From the above discussions and known results we have the following implications

In the following diagram, by $A \to B$ we mean $A$ implies $B$ but not conversely and $A \leftrightarrow B$ means $A$ and $B$ are independent of each other.

![Diagram]

Theorem 2.7. If a mapping $f : (X, \tau) \to (Y, \sigma)$ is $rg\alpha$-closed, then $rg\alpha$-cl$(f(A)) \subset f(cl(A))$ for every subset $A$ of $(X, \tau)$.

Proof. Suppose that $f$ is $rg\alpha$-closed and $A \subset X$. Then $cl(A)$ is closed in $X$ and so $f(cl(A))$ is $rg\alpha$-closed in $(Y, \sigma)$. We have $f(A) \subset f(cl(A))$, by Theorem 2.9 (iv) in [29], $rg\alpha$-cl$(f(A)) \subset rg\alpha$-cl$(f(cl(A))) \to (i)$. Since $f(cl(A))$ is $rg\alpha$-closed in $(Y, \sigma)$, $rg\alpha$-cl$(f(cl(A))) = f(cl(A)) \to (ii)$, by the Theorem 2.10 in [29]. From (i) and (ii), we have $rg\alpha$-cl$(f(A)) \subset f(cl(A))$ for every subset $A$ of $(X, \tau)$.

Remark 2.5. The converse of the above Theorem 2.7. is not true in general as seen from the following example

Example 2.13 Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$, $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$ Define $f : (X, \tau) \to (Y, \sigma)$ by $f(x) = x$ for every $x \in X$. Then $rg\alpha$-cl$(f(A)) \subset f(cl(A))$ for every subset $A$ of $(X, \tau)$. But $f$ is not $rg\alpha$-closed, since $f(\{b\}) = \{b\}$ is not $rg\alpha$-closed in $(Y, \sigma)$.

Corollary 2.1. If a mapping $f : (X, \tau) \to (Y, \sigma)$ is $rg\alpha$-closed, then the image $f(A)$ of closed set $A$ in $(X, \tau)$ is $\tau_{rg\alpha}$-closed in $(Y, \sigma)$.
Proof. Let $A$ be a closed set in $(X, \tau)$. Since $f$ is $\mathrm{rga}$-closed, by above Theorem 2.7, $\mathrm{rga-cl}(f(A)) \subset f(\mathrm{cl}(A)) \rightarrow (i)$. Also $\mathrm{cl}(A) = A$, as $A$ is a closed set and so $f(\mathrm{cl}(A)) = f(A) \rightarrow (ii)$. From (i) and (ii), we have $\mathrm{rga-cl}(f(A)) \subset f(A)$. We know that $f(A) \subset \mathrm{rga-cl}(f(A))$ and so $\mathrm{rga-cl}(f(A)) = f(A)$. Therefore $f(A)$ is $\tau_{\mathrm{rga}}$-closed in $(Y, \sigma)$. 

Theorem 2.8. Let $(X, \tau)$ be any topological spaces and $(Y, \sigma)$ be a topological space where "$\mathrm{rga-cl}(A) = w\mathrm{-cl}(A)$ for every subset $A$ of $Y$" and $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map, then the following are equivalent. 

(i) $f$ is $\mathrm{rga}$-closed map. 
(ii) $\mathrm{rga-cl}(f(A)) \subset f(\mathrm{cl}(A))$ for every subset $A$ of $(X, \tau)$. 

Proof. $(i) \Rightarrow (ii)$ Follows from the Theorem 2.7. 

$(ii) \Rightarrow (i)$ Let $A$ be any closed set of $(X, \tau)$. Then $A = \mathrm{cl}(A)$ and so $f(A) = f(\mathrm{cl}(A)) \supset \mathrm{rga-cl}(f(A))$ by hypothesis. We have $f(A) \subset \mathrm{rga-cl}(f(A))$, by Theorem 2.9(ii) in [29]. Therefore $f(A) = \mathrm{rga-cl}(f(A))$. Also $f(A) = \mathrm{rga-cl}(f(A)) = w\mathrm{-cl}(f(A))$, by hypothesis. That is $f(A) = w\mathrm{-cl}(f(A))$ and so $f(A)$ is $w\mathrm{-closed}$ in $(Y, \sigma)$. Thus $f(A)$ is $\mathrm{rga}$-closed set in $(Y, \sigma)$ and hence $f$ is $\mathrm{rga}$-closed map. 

Theorem 2.9. A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\mathrm{rga}$-closed if and only if for each subset $S$ of $(Y, \sigma)$ and each open set $U$ containing $f^{-1}(S) \subset U$, there is a $\mathrm{rga}$-open set $V$ of $(Y, \sigma)$ such that $S \subset V$ and $f^{-1}(V) \subset U$. 

Proof. Suppose $f$ is $\mathrm{rga}$-closed. Let $S \subset Y$ and $U$ be an open set of $(X, \tau)$ such that $f^{-1}(S) \subset U$. Now $X - U$ is closed set in $(X, \tau)$. Since $f$ is $\mathrm{rga}$-closed, $f(X - U)$ is $\mathrm{rga}$-closed set in $(Y, \sigma)$. Then $V = Y - f(X - U)$ is a $\mathrm{rga}$-open set in $(Y, \sigma)$. Note that $f^{-1}(S) \subset U$ implies $S \subset V$ and $f^{-1}(V) = X - f^{-1}(f(X - U)) \subset X - (X - U) = U$. That is $f^{-1}(V) \subset U$. 

For the converse, let $F$ be a closed set of $(X, \tau)$. Then $f^{-1}(f(F)^c) \subset F^c$ and $F^c$ is an open in $(X, \tau)$. By hypothesis, there exists a $\mathrm{rga}$-open set $V$ in $(Y, \sigma)$ such that $f(F)^c \subset V$ and $f^{-1}(V) \subset F^c$ and so $F \subset f^{-1}(V)^c$. Hence $V^c \subset f(F) \subset f((f^{-1}(V))^c) \subset V^c$ which implies $f(V) \subset V^c$. Since $V^c$ is $\mathrm{rga}$-closed, $f(F)$ is $\mathrm{rga}$-closed. That is $f(F)$ is $\mathrm{rga}$-closed in $(Y, \sigma)$ and therefore $f$ is $\mathrm{rga}$-closed. 

Remark 2.6. The composition of two $\mathrm{rga}$-closed maps need not be $\mathrm{rga}$-closed map in general and this is shown by the following example.

Example 2.14 Let $X = Y = Z = \{a, b, c\}$, $\tau = P(X)$, $\sigma = \{Y, \phi, \{e\}, \{a, b\}\}$ and $\eta = \{Z, \phi, \{a\}, \{b\}, \{a, b\}\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = a$, $f(b) = b$ and $f(c) = c$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be the identity map. Then
f and g are rga-closed maps, but their composition \( g \circ f : (X, \tau) \to (Z, \eta) \) is not rga-closed map, because \( F = \{ a \} \) is closed in \((X, \tau)\), but \( g \circ f(F) = g \circ f(\{ a \}) = g(f(\{ a \})) = g(\{ a \}) = \{ a \} \) which is not rga-closed in \((Z, \eta)\).

**Theorem 2.10.** If \( f : (X, \tau) \to (Y, \sigma) \) is closed map and \( g : (Y, \sigma) \to (Z, \eta) \) is rga-closed map, then the composition \( g \circ f : (X, \tau) \to (Z, \eta) \) is rga-closed map.

**Proof.** Let \( F \) be any closed set in \((X, \tau)\). Since \( f \) is closed map, \( f(F) \) is closed set in \((Y, \sigma)\). Since \( g \) is rga-closed map, \( g(f(F)) \) is rga-closed set in \((Z, \eta)\). That is \( g \circ f(F) = g(f(F)) \) is rga-closed and hence \( g \circ f \) is rga-closed map.

**Remark 2.7.** If \( f : (X, \tau) \to (Y, \sigma) \) is rga-closed map and \( g : (Y, \sigma) \to (Z, \eta) \) is closed map, then the composition need not be rga-closed map as seen from the following example.

**Example 2.15** Consider \( X = Y = Z = \{ a, b, c \}, \ \tau = \{ X, \phi, \{ a \}, \{ b \}, \{ a, b \} \}, \ \sigma = \{ Y, \phi, \{ a \}, \{ b, c \} \} \) and \( \eta = \{ Z, \phi, \{ b \}, \{ c \}, \{ b, c \} \} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be the identity map and \( g : (Y, \sigma) \to (Z, \eta) \) is defined by \( g(a) = g(b) = a \) and \( g(c) = b \). Then \( f \) is rga-closed map and \( g \) is a closed map. But their composition \( g \circ f : (X, \tau) \to (Z, \eta) \) is not rga-closed map, since for the closed set \( \{ c \} \) in \((X, \tau)\), but \( g \circ f(\{ c \}) = g(f(\{ c \})) = g(\{ c \}) = \{ b \} \) which is not rga-closed in \((Z, \eta)\).

**Theorem 2.11.** Let \((X, \tau), (Z, \eta)\) be topological spaces, and \((Y, \sigma)\) be topological spaces where ”every rga-closed subset is closed”. Then the composition \( g \circ f : (X, \tau) \to (Z, \eta) \) of the rga-closed maps \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) is rga-closed.

**Proof.** Let \( A \) be a closed set of \((X, \tau)\). Since \( f \) is rga-closed, \( f(A) \) is rga-closed in \((Y, \sigma)\). Then by hypothesis, \( f(A) \) is closed. Since \( g \) is rga-closed, \( g(f(A)) \) is rga-closed in \((Z, \eta)\) and \( g(f(A)) = g \circ f(A) \). Therefore \( g \circ f \) is rga-closed.

**Theorem 2.12.** If \( f : (X, \tau) \to (Y, \sigma) \) is g-closed, \( g : (Y, \sigma) \to (Z, \eta) \) be rga-closed and \((Y, \sigma)\) is \( T_{1/2} \)-space then their composition \( g \circ f : (X, \tau) \to (Z, \eta) \) is rga-closed map.

**Proof.** Let \( A \) be a closed set of \((X, \tau)\). Since \( f \) is g-closed, \( f(A) \) is g-closed in \((Y, \sigma)\). Since \((Y, \sigma)\) is \( T_{1/2} \)-space, \( f(A) \) is closed in \((Y, \sigma)\). Since \( g \) is rga-closed, \( g(f(A)) \) is rga-closed in \((Z, \eta)\) and \( g(f(A)) = g \circ f(A) \). Therefore \( g \circ f \) is rga-closed.
Theorem 2.13. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two mappings such that their composition $g \circ f : (X, \tau) \to (Z, \eta)$ be $rg\alpha$-closed mapping. Then the following statements are true.

(i) If $f$ is continuous and surjective, then $g$ is $rg\alpha$-closed.

(ii) If $g$ is $rg\alpha$-irresolute and injective, then $f$ is $rg\alpha$-closed.

(iii) If $f$ is $g$-continuous, surjective and $(X, \tau)$ is a $T_{1/2}$-space, then $g$ is $rg\alpha$-closed.

(iv) If $g$ is strongly $rg\alpha$-continuous and injective, then $f$ is $rg\alpha$-closed.

Proof. (i) Let $A$ be a closed set of $(Y, \sigma)$. Since $f$ is continuous, $f^{-1}(A)$ is closed in $(X, \tau)$ and since $g \circ f$ is $rg\alpha$-closed, $(g \circ f)(f^{-1}(A))$ is $rg\alpha$-closed in $(Z, \eta)$. That is $g(A)$ is $rg\alpha$-closed in $(Z, \eta)$, since $f$ is surjective. Therefore $g$ is $rg\alpha$-closed.

(ii) Let $B$ be a closed set of $(X, \tau)$. Since $g \circ f$ is $rg\alpha$-closed, $g \circ f(B)$ is $rg\alpha$-closed in $(Z, \eta)$. Since $g$ is $rg\alpha$-irresolute, $g^{-1}(g \circ f(B))$ is $rg\alpha$-closed set in $(Y, \sigma)$. That is $f(B)$ is $rg\alpha$-closed in $(Y, \sigma)$, since $f$ is injective. Therefore $f$ is $rg\alpha$-closed.

(iii) Let $C$ be a closed set of $(Y, \sigma)$. Since $f$ is $g$-continuous, $f^{-1}(C)$ is $g$-closed set in $(X, \tau)$. Since $(X, \tau)$ is a $T_{1/2}$-space, $f^{-1}(C)$ is closed set in $(X, \tau)$. Since $g \circ f$ is $rg\alpha$-closed, $(g \circ f)(f^{-1}(C))$ is $rg\alpha$-closed in $(Z, \eta)$. That is $g(C)$ is $rg\alpha$-closed in $(Z, \eta)$, since $f$ is surjective. Therefore $g$ is $rg\alpha$-closed.

(iv) Let $D$ be a closed set of $(X, \tau)$. Since $g \circ f$ is $rg\alpha$-closed, $(g \circ f)(D)$ is $rg\alpha$-closed in $(Z, \eta)$. Since $g$ is strongly $rg\alpha$-continuous, $g^{-1}((g \circ f)(D))$ is closed set in $(Y, \sigma)$. That is $f(D)$ is closed set in $(Y, \sigma)$, since $g$ is injective, Therefore $f$ is closed. □

Theorem 2.14. If $f : (X, \tau) \to (Y, \sigma)$ is an open, continuous, $rg\alpha$-closed surjection and $cl(F) = F$ for every $rg\alpha$-closed set in $(Y, \sigma)$, where $X$ is regular, then $Y$ is regular.

Proof. Let $U$ be an open set in $Y$ and $p \in U$. Since $f$ is surjection, there exists a point $x \in X$ such that $f(x) = p$. Since $X$ is regular and $f$ is continuous, there is an open set $V$ in $X$ such that $x \in V \subset cl(V) \subset f^{-1}(U)$. Here $p \in f(V) \subset f(cl(V)) \subset U \to (i)$. Since $f$ is $rg\alpha$-closed, $f(cl(V))$ is $rg\alpha$-closed set contained in the open set $U$. By hypothesis, $cl(f(cl(V))) = f(cl(V))$ and $cl(f(V)) = cl(f(cl(V))) \to (ii)$. From (i) and (ii), we have $p \in f(V) \subset cl(f(V)) \subset U$ and $f(V)$ is open, since $f$ is open. Hence $Y$ is regular. □

Theorem 2.15. If a map $f : (X, \tau) \to (Y, \sigma)$ is $rg\alpha$-closed and $A$ is closed set of $X$, then $f_A : (A, \tau_A) \to (Y, \sigma)$ is $rg\alpha$-closed.
Proof. Let $F$ be a closed set of $A$. Then $F = A \cap E$ for some closed set $E$ of $(X, \tau)$ and so $F$ is closed set of $(X, \tau)$. Since $f$ is $rg\alpha$-closed, $f(F)$ is $rg\alpha$-closed set in $(Y, \sigma)$. But $f(F) = f_A(F)$ and therefore $f_A : (A, \tau_A) \rightarrow (Y, \sigma)$ is $rg\alpha$-closed.

Analogous to $rg\alpha$-closed maps, we define $rg\alpha$-open map as follows.

Definition 2.2. A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is called a $rg\alpha$-open map if the image $f(A)$ is $rg\alpha$-open in $(Y, \sigma)$ for each open set $A$ in $(X, \tau)$.

From the definitions we have the following results.

Theorem 2.16. (i) Every open map is $rg\alpha$-open but not conversely. 
(ii) Every $w$-open map is $rg\alpha$-open but not conversely. 
(iii) Every $rg\alpha$-open map is $rg$-open but not conversely. 
(iv) Every $rg\alpha$-open map is $rwg$-open but not conversely. 
(v) Every $rg\alpha$-open map is $gpr$-open but not conversely.

Theorem 2.17. For any bijection map $f : (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent:
(i) $f^{-1} : (Y, \sigma) \rightarrow (X, \tau)$ is $rg\alpha$-continuous.
(ii) $f$ is $rg\alpha$-open map and (iii) $f$ is $rg\alpha$-closed map.

Proof. (i) $\Rightarrow$ (ii) Let $U$ be an open set of $(X, \tau)$. By assumption, $(f^{-1})^{-1}(U) = f(U)$ is $rg\alpha$-open in $(Y, \sigma)$ and so $f$ is $rg\alpha$-open.
(ii) $\Rightarrow$ (iii) Let $F$ be a closed set of $(X, \tau)$. Then $F^c$ is open set in $(X, \tau)$.

By assumption, $f(F^c)$ is $rg\alpha$-open in $(Y, \sigma)$. That is $f(F^c) = f(F)^c$ is $rg\alpha$-open in $(Y, \sigma)$ and therefore $f(F)$ is $rg\alpha$-closed in $(Y, \sigma)$. Hence $f$ is $rg\alpha$-closed.
(iii) $\Rightarrow$ (i) Let $F$ be a closed set of $(X, \tau)$. By assumption, $f(F)$ is $rg\alpha$-closed in $(Y, \sigma)$. But $f(F) = (f^{-1})^{-1}(F)$ and therefore $f^{-1}$ is continuous.

Theorem 2.18. If a map $f : (X, \tau) \rightarrow (Y, \sigma)$ is $rg\alpha$-open, then $f(int(A)) \subseteq rga-int(f(A))$ for every subset $A$ of $(X, \tau)$.

Proof. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a open map and $A$ be any subset of $(X, \tau)$. Then $int(A)$ is open in $(X, \tau)$ and so $f(int(A))$ is $rg\alpha$-open in $(Y, \sigma)$. We have $f(int(A)) \subseteq f(A)$. Therefore by Theorem 2.2 (iii) in [29], $f(int(A)) \subseteq rga-int(f(A))$.

Remark 2.8. The converse of the above Theorem need not be true in general as seen from the following example.
Example 2.16 Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{a, c\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Let $f$ be the identity map. In $(Y, \sigma)$, $\mathrm{rga}\text{-}\mathrm{int}(f(A)) = f(A)$ for every subset $A$ of $(X, \tau)$. So $f(\mathrm{int}(A)) \subset f(A) = \mathrm{rga}\text{-}\mathrm{int}(f(A))$ for every subset $A$ of $X$. But $f$ is not $\mathrm{rga}$-open, since for the open set $\{a, c\}$ of $(X, \tau)$, $f(\{a, c\}) = \{a, c\}$ which is not $\mathrm{rga}$-open in $(Y, \sigma)$.

Theorem 2.19. If a map $f : (X, \tau) \to (Y, \sigma)$ is $\mathrm{rga}$-open, then for each neighbourhood $U$ of $x$ in $(X, \tau)$, there exists a $\mathrm{rga}$-neighbourhood $W$ of $f(x)$ in $(Y, \sigma)$ such that $W \subset f(U)$.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a $\mathrm{rga}$-open map. Let $x \in X$ and $U$ be an arbitrary neighbourhood of $x$ in $(X, \tau)$. Then there exists an open set $G$ in $(X, \tau)$ such that $x \in G \subset U$. Now $f(x) \in f(G) \subset f(U)$ and $f(G)$ is $\mathrm{rga}$-open in $(Y, \sigma)$, as $f$ is a $\mathrm{rga}$-open map. By Theorem 3.8 in [28], $f(G)$ is a $\mathrm{rga}$-neighbourhood of each of its points. Taking $f(G) = W$, $W$ is a $\mathrm{rga}$-neighbourhood of $f(x)$ in $(Y, \sigma)$ such that $W \subset f(U)$.

Theorem 2.20. A map $f : (X, \tau) \to (Y, \sigma)$ is $\mathrm{rga}$-open if and only if for any subset $S$ of $(Y, \sigma)$ and any closed set of $(X, \tau)$ containing $f^{-1}(S)$, there exists a $\mathrm{rga}$-closed set $K$ of $(Y, \sigma)$ containing $S$ such that $f^{-1}(K) \subset F$.

Proof. Suppose $f$ is $\mathrm{rga}$-open map. Let $S \subset Y$ and $F$ be a closed set of $(X, \tau)$ such that $f^{-1}(S) \subset F$. Now $X - F$ is an open set in $(X, \tau)$. Since $f$ is $\mathrm{rga}$-open map, $f(X - F)$ is $\mathrm{rga}$-open set in $(Y, \sigma)$. Then $K = Y - f(X - F)$ is a $\mathrm{rga}$-closed set in $(Y, \sigma)$. Note that $f^{-1}(S) \subset F$ implies $S \subset K$ and $f^{-1}(K) = X - f^{-1}(X - F) \subset X - (X - F) = F$. That is $f^{-1}(K) \subset F$.

For the converse, let $U$ be an open set of $(X, \tau)$. Then $f^{-1}(f(U))^c \subset U^c$ and $U^c$ is a closed set in $(X, \tau)$. By hypothesis, there exists a $\mathrm{rga}$-closed set $K$ of $(Y, \sigma)$ such that $(f(U))^c \subset K$ and $f^{-1}(K) \subset U^c$ and so $U \subset (f^{-1}(K))^c$. Hence $K^c \subset f(U) \subset f((f^{-1}(K))^c) \subset K^c$ which implies $f(U) = K^c$. Since $K^c$ is a $\mathrm{rga}$-open, $f(U) \subset \mathrm{rga}$-open in $(Y, \sigma)$ and therefore $f$ is $\mathrm{rga}$-open map.

Theorem 2.21. If a function $f : (X, \tau) \to (Y, \sigma)$ is $\mathrm{rga}$-open, then $f^{-1}(\mathrm{rga}\text{-}\mathrm{cl}(B)) \subset \mathrm{cl}(f^{-1}(B))$ for each subset $B$ of $(Y, \sigma)$.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a $\mathrm{rga}$-open map and $B$ be any subset of $(Y, \sigma)$. Then $f^{-1}(B) \subset \mathrm{cl}(f^{-1}(B))$ and $\mathrm{cl}(f^{-1}(B))$ is closed set in $(X, \tau)$. By above Theorem 2.20., there exists a $\mathrm{rga}$-closed set $K$ of $(Y, \sigma)$ such that $B \subset K$ and $f^{-1}(K) \subset \mathrm{cl}(f^{-1}(B))$. Now $\mathrm{rga}\text{-}\mathrm{cl}(B) \subset \mathrm{rga}\text{-}\mathrm{cl}(K) = K$, by Theorems 2.9 and 2.10 in [28], as $K$ is $\mathrm{rga}$-closed set of $(Y, \sigma)$. Therefore $f^{-1}(\mathrm{rga}\text{-}\mathrm{cl}(B)) \subset f^{-1}(K)$ and so $f^{-1}(\mathrm{rga}\text{-}\mathrm{cl}(B)) \subset f^{-1}(K) \subset \mathrm{cl}(f^{-1}(B))$. Thus $f^{-1}(\mathrm{rga}\text{-}\mathrm{cl}(B)) \subset \mathrm{cl}(f^{-1}(B))$ for each subset of $B$ of $(Y, \sigma)$. ■
Remark 2.9. The converse of the above Theorem need not be true in general as seen from the following example.

Example 2.17 Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{b\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$. Let $f$ be the identity map. In $(Y, \sigma)$, $\text{rg}\alpha\text{-cl}(B) = B$ for every subset $B$ of $(Y, \sigma)$. So $f^{-1}(\text{rg}\alpha\text{-cl}(B)) = f^{-1}(B) \subset \text{cl}(f^{-1}(B))$ for every subset $B$ of $(Y, \sigma)$. But $f$ is not $\text{rg}\alpha$-open map, since for the open set $\{b, c\}$ of $(X, \tau)$, $f(\{b, c\}) = \{b, c\}$ which is not $\text{rg}\alpha$-open in $(Y, \sigma)$.

We define another new class of maps called $\text{rg}\alpha^*$-closed maps which are stronger than $\text{rg}\alpha$-closed maps.

Definition 2.3. A map $f : (X, \tau) \to (Y, \sigma)$ is said to be $\text{rg}\alpha^*$-closed map if the image $f(A)$ is $\text{rg}\alpha$-closed in $(Y, \sigma)$ for every $\text{rg}\alpha$-closed set $A$ in $(X, \tau)$.

Theorem 2.22. Every $\text{rg}\alpha^*$-closed map is $\text{rg}\alpha$-closed map but not conversely.

Proof. The proof follows from the definitions and fact that every closed set is $\text{rg}\alpha$-closed.

The converse of the above Theorem is not true in general as seen from the following example.

Example 2.18 Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then $f$ is $\text{rg}\alpha$-closed map but not $\text{rg}\alpha^*$-closed map. Since $\{a\}$ is $\text{rg}\alpha$-closed set in $(X, \tau)$, but its image under $f$ is $\{a\}$, which is not $\text{rg}\alpha$-closed in $(Y, \sigma)$.

Theorem 2.23. If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ are $\text{rg}\alpha^*$-closed maps, then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is also $\text{rg}\alpha^*$-closed.

Proof. Let $F$ be a $\text{rg}\alpha$-closed set in $(X, \tau)$. Since $f$ is $\text{rg}\alpha^*$-closed map, $f(F)$ is $\text{rg}\alpha$-closed set in $(Y, \sigma)$. Since $g$ is $\text{rg}\alpha^*$-closed map, $g(f(F))$ is $\text{rg}\alpha$-closed set in $(Z, \eta)$. Therefore $g \circ f$ is $\text{rg}\alpha^*$-closed map.

Analogous to $\text{rg}\alpha^*$-closed map, we define another new class of maps called $\text{rg}\alpha^*$-open maps which are stronger than $\text{rg}\alpha$-open maps.

Definition 2.4. A map $f : (X, \tau) \to (Y, \sigma)$ is said to be $\text{rg}\alpha^*$-open map if the image $f(A)$ is $\text{rg}\alpha$-open set in $(Y, \sigma)$ for every $\text{rg}\alpha$-open set $A$ in $(X, \tau)$.

Remark 2.10. Since every open set is a $\text{rg}\alpha$-open set, we have every $\text{rg}\alpha^*$-open map is $\text{rg}\alpha$-open map. The converse is not true in general as seen from the following example.
Example 2.19 Let \( X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\} \).
Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity map. Then \( f \) is \( rga\)-open map but not \( rga^*\)-open map, since for the \( rga\)-open set \( \{a, c\} \) in \( (X, \tau) \), \( f(\{a, c\}) = \{a, c\} \) which is not \( rga\)-open set in \( (Y, \sigma) \).

Theorem 2.24. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) are \( rga^*\)-open maps, then their composition \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is also \( rga^*\)-open.

Proof. Proof is similar to the Theorem 2.23.

Theorem 2.25. For any bijection map \( f : (X, \tau) \rightarrow (Y, \sigma) \), the following statements are equivalent:
\( (i) \) \( f^{-1} : (Y, \sigma) \rightarrow (X, \tau) \) is \( rga \) irresolute
\( (ii) \) \( f \) is \( rga^*\)-open map
\( (iii) \) \( f \) is \( rga^*\)-closed map.

Proof. Proof is similar to that of Theorem 2.17.

References


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