The Convergence of Family of Integral Operators with Positive Kernel

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Abstract
The aim of this study is to investigate the convergence of family of integral operators with positive kernel in the space $L_1(-\infty, \infty)$ generalized as:

$$L_{\lambda}(f, x) = \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} P_{k,\lambda} f(x + \alpha_{k,\lambda} t) K_{\lambda}(t) \, dt$$

within this integral operator, when $P_{k,\lambda}$ and $\alpha_{k,\lambda}$ are real numbers for $\forall \lambda \in \Lambda \subset \mathbb{R}$, $\lambda \geq 0$ parameter

$$\sum_{k=1}^{\infty} |p_{k,\lambda}| \leq M < \infty$$

$M$ is independent of $\lambda$ and for $\forall \lambda \in \Lambda$,

$$\sum_{k=1}^{\infty} p_{k,\lambda} = 1 \quad \text{and} \quad \sup_{k,\lambda} \{\alpha_{k,\lambda}\} = \alpha^* < \infty.$$ 

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1 Introduction

The family of the integral operators with positive kernel,

$$L_{\lambda}(f, x) = \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} P_{k,\lambda} f(x + \alpha_{k,\lambda} t) K_{\lambda}(t) \, dt$$

(1)
had been handled and asymptotic behaviour in the class of derivative functions had been investigated in \[3\].

In this study, using a new modulus of continuity, the convergence of the family of the integral operators \((1)\) to the function \(f\) in the norm of \(L_1(-\infty, \infty)\) has been investigated.

In the family of the integral operators \((1)\) which we have investigated, if \(\lambda\) parameter and \(P_{k,\lambda}\) and \(\alpha_{k,\lambda}\) numbers are chosen properly, a great number of well-known operators can be obtained (see, \[1\], \[2\] and \[7\]). For example, if

\[
P_{k,\lambda} = \begin{cases} 1, & k = 1 \\ 0, & k = 2, 3, \ldots \end{cases} \quad \text{and} \quad \alpha_{k,\lambda} = \begin{cases} 1, & k = 1 \\ 0, & k = 2, 3, \ldots, \end{cases}
\]

is taken particularly and if we select \(K_\lambda(t) = \frac{\lambda}{\sqrt{\pi}} e^{-\lambda^2 t^2}\) satisfying the condition

\[
\int_{-\infty}^{\infty} K_\lambda(t) \, dt = 1
\]

then

\[
L_\lambda(f, x) = \int_{-\infty}^{\infty} f(x + t) \frac{\lambda}{\sqrt{\pi}} e^{-\lambda^2 t^2} \, dt
\]

Gauss-Weierstrass family of the integral operators is obtained and some different examples can be found \[9\]. This family of the integral operators that transform from \(L_1\)-space into \(L_1\)-space converge to the function \(f\) in the norm of \(L_1(-\infty, \infty)\).

### 2 Convergence in \(L_1(-\infty, \infty)\) Space

Let us first show the existence of the function which is defined following series

\[
\sum_{k=1}^{\infty} p_{k,\lambda} f(x + \alpha_{k,\lambda} t)
\]

for \(f \in L_1(-\infty, \infty)\). Let \(\varphi\) be the function satisfying the inequality

\[
|f(x)| < \varphi(x), \quad \varphi \in L_1.
\]
It is obvious that,

\[
\left| \sum_{k=1}^{\infty} p_{k,\lambda} f (x + \alpha_{k,\lambda} t) \right| \leq \sum_{k=1}^{\infty} |p_{k,\lambda} f (x + \alpha_{k,\lambda} t)|
\]

\[
\leq \sum_{k=1}^{\infty} |p_{k,\lambda} f (x + \alpha_{k,\lambda} t)| \frac{\varphi (x + \alpha_{k,\lambda} t)}{\varphi (x)}
\]

\[
\leq \varphi (x) \sum_{k=1}^{\infty} |p_{k,\lambda}| \frac{f (x + \alpha_{k,\lambda} t)}{\varphi (x + \alpha_{k,\lambda} t)} \frac{\varphi (x + \alpha_{k,\lambda} t)}{\varphi (x)}
\]

\[
\leq M \varphi (x) \mu (\alpha^* t)
\]

where \(\alpha^* = \sup \{\alpha_{k,\lambda}\} < \infty\) and \(\sum_{k=1}^{\infty} |p_{k,\lambda}| \leq M < \infty\).

Therefore the series \(\sum_{k=1}^{\infty} p_{k,\lambda} f (x + \alpha_{k,\lambda} t)\) is convergent for each \(t \in R\). Here,

\[
\mu (t) = \sup_{-\infty < x < \infty, |y| \leq t} \frac{\varphi (x + y)}{\varphi (x)} < \infty
\]

additionally, due to the fact that for \(f \in L_1 (-\infty, \infty)\)

\[
\int_{-\infty}^{\infty} \left| \sum_{k=1}^{\infty} p_{k,\lambda} f (x + \alpha_{k,\lambda} t) \right| dx \leq \int_{-\infty}^{\infty} M \varphi (x) \mu (\alpha^* t) dx
\]

\[
\leq M \mu (\alpha^* t) \int_{-\infty}^{\infty} \varphi (x) dx
\]

\[
= M \mu (\alpha^* t) \| \varphi \|_{L_1}.
\]

Hence the function \(\sum_{k=1}^{\infty} p_{k,\lambda} f (x + \alpha_{k,\lambda} t)\) belongs to \(L_1 (-\infty, \infty)\) space.

Luzin’s theorem from which we’ll benefit to prove our theorems is as follows;

**Theorem 2.1 (Luzin Theorem)** If \(f \in L_1 [a, b]\), then a continuous function \(\varphi\) is obtained according to each \(\varepsilon > 0\) number by getting the following;

\[
\|f - \varphi\|_{L_1 [a, b]} < \varepsilon.
\]

(See [8]).

Modulus of continuity used in previous studies (for example [4], [5] and [6]), such as

\[
\omega_1 (f, \delta) = \sup_{|t| \leq \delta} \int_{-\infty}^{\infty} |f (x + t) - f (x)| dx.
\]
This modulus of continuity is not so beneficial to be used to estimate convergence with the help of the family of the integral operator with positive kernel:

\[ L_\lambda (f, x) = \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} p_{k,\lambda} f(x + \alpha_{k,\lambda} t) \, K_\lambda (t) \, dt. \]

Because it is a series which is under integral sign in this operator, it is too difficult to displace the integral sign with summation sign. For this reason a more useful modulus of continuity has been defined in this paper as the following:

\[ \omega^*_{L_1} (f, \delta) = \sup_{|\alpha_{k,\lambda} t| \leq \delta} \left| \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} p_{k,\lambda} [f(x + \alpha_{k,\lambda} t) - f(x)] \, dx \right|. \]

The function \( \omega^*_{L_1} (f, \delta) \) defines a modulus of continuity in \( L_1 (-\infty, \infty) \).

It is obvious that \( \omega^*_{L_1} (f, \delta) \) function is positive and monotone increasing function compared to \( \delta \).

Certain basic properties of this modulus of continuity has been given with below mentioned theorem.

**Theorem 2.2 a)**

\[ \lim_{\delta \to 0} \omega^*_{L_1} (f, \delta) = 0. \]

**b)** Let \( m \in \mathbb{N} \),

\[ \omega^*_{L_1} (f, m\delta) \leq m \, \omega^*_{L_1} (f, \delta). \]

**c)** Where \( \lambda > 0 \) is an arbitrary real number,

\[ \omega^*_{L_1} (f, \lambda\delta) \leq (1 + \lambda) \, \omega^*_{L_1} (f, \delta). \]

**Proof. a)** Because of the fact that \( f \) and \( \sum_{k=1}^{\infty} p_{k,\lambda} f(x + \alpha_{k,\lambda} t) \) functions belong to \( L_1 (-\infty, \infty) \) space, for \( \forall \varepsilon > 0 \), there exists a real number \( a \) that will give the inequalities shown below as follows:

\[
\begin{align*}
a) \int_{-\infty}^{a} \left| \sum_{k=1}^{\infty} p_{k,\lambda} f(x + \alpha_{k,\lambda} t) \right| \, dx &< \frac{\varepsilon}{4}, \\
b) \int_{a}^{\infty} \left| \sum_{k=1}^{\infty} p_{k,\lambda} f(x + \alpha_{k,\lambda} t) \right| \, dx &< \frac{\varepsilon}{4} \\
c) \int_{-\infty}^{a} \left| \sum_{k=1}^{\infty} p_{k,\lambda} f(x) \right| \, dx &< \frac{\varepsilon}{4}, \\
d) \int_{a}^{\infty} \left| \sum_{k=1}^{\infty} p_{k,\lambda} f(x) \right| \, dx &< \frac{\varepsilon}{4}.
\end{align*}
\]
Further we any $\delta > 0$, from (2)
\[
\int_{-\infty}^{-a-\delta} \sum_{k=1}^{\infty} p_{k,\lambda} f(x + \alpha_{k,\lambda}t) \, dx < \frac{\varepsilon}{4}, \quad \int_{a+\delta}^{\infty} \sum_{k=1}^{\infty} p_{k,\lambda} f(x + \alpha_{k,\lambda}t) \, dx < \frac{\varepsilon}{4}
\]
\[
\int_{-\infty}^{-a-\delta} \sum_{k=1}^{\infty} p_{k,\lambda} f(x) \, dx < \frac{\varepsilon}{4}, \quad \int_{a+\delta}^{\infty} \sum_{k=1}^{\infty} p_{k,\lambda} f(x) \, dx < \frac{\varepsilon}{4}.
\]

We can write these inequalities as follows;
\[
\int_{-\infty}^{a+\delta} \sum_{k=1}^{\infty} p_{k,\lambda} (f(x + \alpha_{k,\lambda}t) - f(x)) \, dx
\]
\[
\leq \int_{-a-\delta}^{a+\delta} \sum_{k=1}^{\infty} p_{k,\lambda} (f(x + \alpha_{k,\lambda}t) - f(x)) \, dx
\]
\[
+ \int_{-\infty}^{-a-\delta} \sum_{k=1}^{\infty} p_{k,\lambda} (f(x + \alpha_{k,\lambda}t) - f(x)) \, dx
\]
\[
+ \int_{a+\delta}^{\infty} \sum_{k=1}^{\infty} p_{k,\lambda} (f(x + \alpha_{k,\lambda}t) - f(x)) \, dx
\]
\[
\leq \int_{-a-\delta}^{a+\delta} \sum_{k=1}^{\infty} p_{k,\lambda} (f(x + \alpha_{k,\lambda}t) - f(x)) \, dx + \varepsilon
\]

then
\[
\sup_{|\alpha_{k,\lambda}t| \leq \delta} \int_{-\infty}^{a+\delta} \sum_{k=1}^{\infty} p_{k,\lambda} (f(x + \alpha_{k,\lambda}t) - f(x)) \, dx
\]
\[
\leq \sup_{|\alpha_{k,\lambda}t| \leq \delta} \int_{-a-\delta}^{a+\delta} \sum_{k=1}^{\infty} p_{k,\lambda} (f(x + \alpha_{k,\lambda}t) - f(x)) \, dx + \varepsilon.
\]

Thus, the proof is completed if we can show that
\[
\sup_{|\alpha_{k,\lambda}t| \leq \delta} \int_{-a-\delta}^{a+\delta} \sum_{k=1}^{\infty} p_{k,\lambda} (f(x + \alpha_{k,\lambda}t) - f(x)) \, dx < \varepsilon.
\] (3)
It has been known by Luzin’s theorem that for an arbitrary number \( a \), there exists a continuous function \( \Psi \) such that
\[
\| f - \Psi \|_{L_1} < \varepsilon
\]
in the interval \([-a - 2\delta, a + 2\delta]\). Integral defined in (3) can be separated into three integrals as follows:

\[
\int_{-a-\delta}^{a+\delta} \left( \sum_{k=1}^{\infty} |p_{k,\lambda}| \| f(x + \alpha_{k,\lambda}t) - f(x) \| \right) dx
\]
\[
\leq \int_{-a-\delta}^{a+\delta} \sum_{k=1}^{\infty} |p_{k,\lambda}| \| f(x + \alpha_{k,\lambda}t) - \Psi(x + \alpha_{k,\lambda}t) \| dx
\]
\[
+ \int_{-a-\delta}^{a+\delta} \sum_{k=1}^{\infty} |p_{k,\lambda}| \| \Psi(x + \alpha_{k,\lambda}t) - \Psi(x) \| dx
\]
\[
+ \int_{-a-\delta}^{a+\delta} \sum_{k=1}^{\infty} |p_{k,\lambda}| \| f(x) - \Psi(x) \| dx
\]
\[
= I_1(t) + I_2(t) + I_3(t)
\]

Let us investigate each integral one by one. First, let us take \( I_1(t) \) integral,
\[
I_1(t) = \int_{-a-\delta}^{a+\delta} \sum_{k=1}^{\infty} |p_{k,\lambda}| \| f(x + \alpha_{k,\lambda}t) - \Psi(x + \alpha_{k,\lambda}t) \| dx
\]
when supremum of both sides are taken according to \( |\alpha_{k,\lambda}t| \leq \alpha^*t \leq \delta \), then
\[
\sup_{|\alpha_{k,\lambda}t| \leq \alpha^*t} \int_{-a-\delta}^{a+\delta} \sum_{k=1}^{\infty} |p_{k,\lambda}| \| f(x + \alpha_{k,\lambda}t) - \Psi(x + \alpha_{k,\lambda}t) \| dx
\]
\[
\leq \int_{-a-\delta}^{a+2\delta} \sum_{k=1}^{\infty} |p_{k,\lambda}| \| f(x) - \Psi(x) \| dx
\]
\[
\leq M \int_{-a-2\delta}^{a+2\delta} |f(x) - \Psi(x) | dx
\]
\[
\leq \varepsilon M.
\]
Hence, if we take limit for $\delta \to 0$, the following is obtained.

$$\lim_{\delta \to 0} \sup_{|\alpha_{k,\lambda}t| \leq \delta} I_1(t) = 0.$$ 

By following similar ways for $I_3$, we get

$$\lim_{\delta \to 0} \sup_{|\alpha_{k,\lambda}t| \leq \delta} I_3(t) = 0.$$ 

If we benefit from the continuity of function $\Psi$ to prove $I_2(t)$, then we have the following inequality,

$$I_2(t) = \int_{-\frac{a}{\delta}}^{\frac{a}{\delta}} \sum_{k=1}^{\infty} \left| p_{k,\lambda} \right| \left| \Psi(x + \alpha_{k,\lambda}t) - \Psi(x) \right| dx \leq \varepsilon M (2a + 2\delta).$$

When we take limit for $\delta \to 0$, we obtain

$$\lim_{\delta \to 0} I_2(t) = 0.$$ 

In consequence,

$$\sup_{|\alpha_{k,\lambda}t| \leq \delta} \int_{-\frac{a}{\delta}}^{\frac{a}{\delta}} \sum_{k=1}^{\infty} \left| p_{k,\lambda} \right| \left| f(x + \alpha_{k,\lambda}t) - f(x) \right| dx < \varepsilon.$$ 

Hence the proof is completed for a).

b) and c) can be easily obtained.

Now, we are going to prove the family of the integral operators with positive kernel,

$$L_{\lambda}(f, x) = \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} P_{k,\lambda} f(x + \alpha_{k,\lambda}t) K_{\lambda}(t) dt$$

converges to the function $f \in L_1(-\infty, \infty)$ in the norm of $L_1(-\infty, \infty)$.

**Theorem 2.3** For each function $f \in L_1(-\infty, \infty)$ satisfying the following inequality

$$|f(x)| < \varphi(x), \quad \varphi \in L_1(-\infty, \infty)$$

and for $\forall \delta > 0$,

$$\lim_{\lambda \to \infty} \int_{-\delta}^{\delta} \mu(\alpha^* t) K_{\lambda}(t) dt = 0, \quad \lim_{\lambda \to \infty} \int_{-\delta}^{\delta} K_{\lambda}(t) dt = 0$$
and when $K_{\lambda}(t)$ is non-negative even function satisfying the condition, $\int_{-\infty}^{\infty} K_{\lambda}(t) \, dt = 1$ then for $\lambda \to \infty$,

$$\|L_{\lambda}f - f\|_{L_1} \to 0.$$  

Where

$$L_{\lambda}(f, x) = \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} p_{k,\lambda} f(x + \alpha_{k,\lambda} t) K_{\lambda}(t) \, dt$$

is the family of the integral operators with positive kernel.

**Proof.** Owing to the fact that $\int_{-\infty}^{\infty} K_{\lambda}(t) \, dt = 1$ it comes as follows

\[ f(x) \int_{-\infty}^{\infty} K_{\lambda}(t) \, dt = f(x) \]

then

\[ L_{\lambda}(f, x) - f(x) = \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} p_{k,\lambda} [f(x + \alpha_{k,\lambda} t) - f(x)] K_{\lambda}(t) \, dt \]

If we pass to the norm of $L_1(-\infty, \infty)$ on both sides, it can be written as follows;

\[
\int_{-\infty}^{\infty} |L_{\lambda}(f, x) - f(x)| \, dx \\
\leq \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} p_{k,\lambda} (f(x + \alpha_{k,\lambda} t) - f(x)) K_{\lambda}(t) \, dt \right| \, dx \\
\leq \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} p_{k,\lambda} (f(x + \alpha_{k,\lambda} t) - f(x)) \, dx \right) K_{\lambda}(t) \, dt \\
\leq \int_{-\infty}^{-\delta} \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} p_{k,\lambda} (f(x + \alpha_{k,\lambda} t) - f(x)) \, dx \, K_{\lambda}(t) \, dt \\
+ \int_{-\delta}^{\delta} \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} p_{k,\lambda} (f(x + \alpha_{k,\lambda} t) - f(x)) \, dx \, K_{\lambda}(t) \, dt \\
+ \int_{\delta}^{\infty} \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} p_{k,\lambda} (f(x + \alpha_{k,\lambda} t) - f(x)) \, dx \, K_{\lambda}(t) \, dt \\
= I_1(t) + I_2(t) + I_3(t)
\]
First let us take $I_2(t)$ integral.

$$ I_2(t) = \int_{-\delta}^{\delta} \left( \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} p_{k,\lambda} (f(x + \alpha_{k,\lambda}t) - f(x)) \, dx \right) K_\lambda(t) \, dt $$

$$ \leq \int_{-\delta}^{\delta} \left( \sup_{|\alpha_{k,\lambda}t| \leq \delta} \int_{-\infty}^{\infty} \left| \sum_{k=1}^{\infty} p_{k,\lambda} (f(x + \alpha_{k,\lambda}t) - f(x)) \right| \, dx \right) K_\lambda(t) \, dt $$

$$ \leq \int_{-\delta}^{\delta} \omega_{L_1}(f, \delta) \, K_\lambda(t) \, dt = \omega_{L_1}(f, \delta) \int_{-\delta}^{\delta} K_\lambda(t) \, dt \leq \omega_{L_1}(f, \delta) $$

where

$$ I_2(t) \leq \omega_{L_1}(f, \delta). $$

If we take limits of both sides first for $\lambda \to \infty$ and then for $\delta \to 0$, then

$$ \lim_{\lambda \to \infty} I_2(t) = 0. $$

Now, let us take $I_3(t)$ integral.

$$ I_3(t) = \int_{-\delta}^{\delta} \left( \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} p_{k,\lambda} (f(x + \alpha_{k,\lambda}t) - f(x)) \, dx \right) K_\lambda(t) \, dt $$

$$ \leq \int_{-\delta}^{\delta} \left( \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} p_{k,\lambda} (f(x + \alpha_{k,\lambda}t)) \, dx + \int_{-\infty}^{\infty} |f(x)| \, dx \right) K_\lambda(t) \, dt $$

$$ \leq \int_{-\delta}^{\delta} \left( \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} \left| p_{k,\lambda} \right| \frac{f(x + \alpha_{k,\lambda}t)}{\varphi(x + \alpha_{k,\lambda}t)} \varphi(x) \, dx \right) $$

$$ + \int_{-\infty}^{\infty} |f(x)| \, dx \right) K_\lambda(t) \, dt $$

$$ \leq \int_{-\delta}^{\delta} \left( \int_{-\infty}^{\infty} M \mu(\alpha^*t) \varphi(x) \, dx + \int_{-\infty}^{\infty} |f(x)| \, dx \right) K_\lambda(t) \, dt $$

$$ = M \|\varphi\|_{L_1} \int_{-\delta}^{\delta} \mu(\alpha^*t) \, K_\lambda(t) \, dt + \|f\|_{L_1} \int_{-\delta}^{\delta} K_\lambda(t) \, dt $$
in consequence, when

$$I_3(t) \leq M \|\varphi\|_{L_1} \int_{\delta}^{\infty} \mu(\alpha^* t) K_\lambda(t) dt + \|f\|_{L_1} \int_{\delta}^{\infty} K_\lambda(t) dt.$$ 

If the limit of both sides of the inequality above is taken for $\lambda \to \infty$,

$$\lim_{\lambda \to \infty} I_3(t) = 0$$

is obtained.

By using similar methods,

$$\lim_{\lambda \to \infty} I_1(t) = 0$$

equality is obtained for $I_1(t)$.

Hence, for $\lambda \to \infty$ we get

$$\|L_\lambda f - f\|_{L_1} \to 0.$$ 

The proof is completed.

References


The convergence of family of integral operators


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