Periodicity and Transformation of Difference Equations

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Abstract

In this research we develop a method to construct periodic difference equations with desired period. Also, we investigate the transformation of periodicity of difference equations under invertible maps. These techniques are applied to generalize and extend some results that have been obtained recently. One of the interesting results is derivation of some generalized Lyness’ equations.

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1 Introduction and Preliminaries

Let $D \subseteq \mathbb{R}$, and consider the difference equation

$$x_{n+k} = f(x_n, ..., x_{n+k-1}) \quad n = 0, 1, 2, ...$$

where $f$ is a continuous function such that $f : D^k \to D$, and $x_0, ..., x_{k-1} \in D$, and $k$ is a nonnegative integer. The study of periodicity of difference equations has received a great deal of attention [1, 2], [6], [8 – 11] and others. Our objective in this article is to develop a constructive method introduced in [2] and then to furnish a parametric form for the function $f$ for which Eq.(1) is periodic in global sense. This problem is motivated in [9]. Although the theorem establishes an endless applications. It is applied to generalize and extend several results that have been obtained recently, see for example, [4], [11] and references therein. It is worth mentioning that our method is applied to derive some generalized Lyness’ equations.
For the sake of completeness, we recall the following definitions and some notations and preliminaries that will be used in the present work.

A sequence \( \{x_n\} \) is called periodic, \((\tau - \text{period})\) if there is an integer \( \tau \in \mathbb{N} \) such that \( x_{n+\tau} = x_n \) for all \( n \in \mathbb{N} \), [6]. If a periodic sequence has a smallest period \( p \) (>0), this period is called the fundamental period. A difference equation of the form (1) is said to be global periodic of period \( \tau \) (or for a simple \( \tau - \text{period} \)) if every its solution sequence is \( \tau - \text{period} \) regardless of the initial terms \( x_0, ..., x_{k-1} \).

**Definition 1** A function \( I(x_1, ..., x_k) \) is said to be symmetric if it is invariant under any permutation of \((x_1, ..., x_k)\).

**Definition 2** A function \( I(x_1, ..., x_k) \) is said to be isovertible if it is an invertible function in each of its independent variables, i.e.,

\[
I_1(\lambda) = I(\lambda, x_2, ..., x_k), \quad I_2(\lambda) = I(x_1, \lambda, x_3, ..., x_k), ..., I_k(\lambda) = I(x_1, ..., x_{k-1}, \lambda)
\]

are invertible functions. The inverse of \( I_j \) is denoted by \( I^{-j} \), i.e., \( I^{-j} = I_j^{-1} \). In other words, \( I^{-j} : D \to D \), is defined as follows, for any \( j \in \{1, ..., k\}, \lambda \in D \),

\[
I^{-j}(x_1, ..., x_{j-1}, \lambda, x_{j+1}, ..., x_k) = x_j \quad (2)
\]

if and only if

\[
I(x_1, ..., x_{j-1}, x_j, x_{j+1}, ..., x_k) = \lambda
\]

**Remark 3** Observe that if \( g, h : D \to D \) are two invertible functions, and \( I : D^k \to D \) is symmetric and isovertible, then the function \( F \) defined by \( F(x_1, ..., x_k) = g(I(h(x_1), ..., h(x_k))) \).

This is due to the fact that it is invariant under any permutation of \((x_1, ..., x_k)\) and it is invertible.

We introduce the following notation that will play an important role in what follows

**Definition 4** Given a nonnegative integer \( r \),
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\[ I_r(x_1, \ldots, x_k) = \begin{cases} 
1 & \text{if } r = 0 \\
\sum_{1 \leq i_1 < \ldots < i_r \leq k} x_{i_s} & \text{if } 0 < r \leq k \\
0 & \text{if } r > k 
\end{cases} \]

For example, for \( k = 2 \) we have \( I_1 = x_1 + x_2 \), \( I_2 = x_1 x_2 \), for \( k = 3 \) we have \( I_1 = x_1 + x_2 + x_3 \), \( I_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \), \( I_3 = x_1 x_2 x_3 \), and so on.

**Lemma 5** Let \( r \) be a nonnegative integer such that \( 0 < r \leq k \) and \( I_r \) be as in Definition 4, then

(a) \( I_r \) is symmetric and isovertible.

(b) For every \( j = 1, \ldots, k \), and \( 0 < r \leq k \), \( I_{r}^{- (j)} \) is given by

\[
I_{r}^{- (j)}(x_1, \ldots, x_j, \ldots, x_k) = \frac{\lambda - I_r(x_1, \ldots, x_j-1, x_{j+1}, \ldots, x_k)}{I_{r-1}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k)}
\]

**Proof.** Part (a) is straightforward, and Part (b) follows from the following argument,

\[
\lambda = I_r(x_1, \ldots, x_j, \ldots, x_k) = \sum_{1 \leq i_1 < \ldots < i_r \leq k} x_{i_s}
\]

\[
= \sum_{1 \leq i_1 < \ldots < i_r \leq k} x_j \quad \sum_{1 \leq i_1 < \ldots < i_r \leq k} \prod_{s=1}^{r-1} x_{i_s}
\]

\[
= I_r(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k) + x_j I_{r-1}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k).
\]

Now, the result follows by solving for \( x_j \). \( \blacksquare \)

There is some relationship between the symmetric and isovertible function \( I(x_1, \ldots, x_k) \) and the invariants and then Liapunov function [9].

The paper is organized as follows. In the current Section, we present some definitions, preliminaries and the adopted notations. In the next sections, we introduce our main results. These results are of three issues. The first one is to establish, under restriction on the desired period \( p \), a constructive method that generates one-parameter families of functions for which all solutions are periodic of period \( p \) and this will be given in Section 2. The second issue is to construct from a given periodic difference equation of period \( p \), a new difference equation of period \( lp \), where \( p, l \) are positive integers and these periods are fundamental periods and this is given in Section 3. In the third issue we investigate a form of periodic difference equation that is transformed from a given periodic difference equation under invertible map which is given in Section 4. In each section we provide applications and some examples to illustrate the applicability of the results.
2 Global Periodicity

**Theorem 6** Let \( p \) be a positive integer such that \( p > k \). Suppose that \( p - k \) divides \( k \) with \( L = k/(p - k) \). Then all solutions of Equation (1) are periodic of period \( p \) if

\[
f(x_n, \ldots, x_{n+k-1}) = I^{-(n+k)}(x_n, x_{n+l}, \ldots, x_{n+(L-1)l}, \lambda)
\]

for some symmetric and isovertible function \( I : D^{L+1} \to D \) and arbitrary constant \( \lambda \in D \).

**Proof.** From Eq.(1) and the hypothesis, we get

\[
x_{n+k} = f(x_n, \ldots, x_{n+k-1}) = I^{-(n+k)}(x_n, x_{n+l}, \ldots, x_{n+(L-1)l}, \lambda)
\]

Hence

\[
I(x_n, x_{n+l}, \ldots, x_{n+(L-1)l}, x_{n+k}) = \lambda
\]

From invertibility in the \((n + k)\)st independent variable and the fact that \( l = p - k \), we have

\[
I(x_n, x_{n+(p-k)}, \ldots, x_{n+(L-(p-k))}, x_{n+L(p-k)}) = \lambda \tag{3}
\]

This is for all \( n \geq 0 \), in particular for \( n + p - k \), we get

\[
I(x_{n+(p-k)}, x_{n+2(p-k)}, \ldots, x_{n+L(p-k)}, x_{n+(L+1)(p-k)}) = \lambda
\]

Now, by symmetry of \( I \), this equation can be written in the form,

\[
I(x_{n+(L+1)(p-k)}, x_{n+(p-k)}, \ldots, x_{n+(L-1)(p-k)}, x_{n+L(p-k)}) = \lambda \tag{4}
\]

From Eq.(3) and Eq.(4) and invertibility of \( I \), if we equate the corresponding first terms, we obtain that,

\[
x_n = x_{n+(L+1)(p-k)} = x_{n+p}
\]

since \( L(p - k) = k \) and hence the result follows. \( \blacksquare \)

Using Lemma 5, Part b and Theorem 6, we obtain the following result.

**Corollary 7** Suppose that \( k, p \) are positive integers such that \( p > k \geq 2 \) and \( L = k/(p - k) \) is a positive integer. Then Eq.(1) is periodic with period \( p \) if

\[
f(x_n, x_{n+l}, \ldots, x_{n+(L-1)l}) = \lambda - I_r(x_n, x_{n+l}, \ldots, x_{n+(L-1)l}) \frac{I_{r-1}(x_n, x_{n+l}, \ldots, x_{n+(L-1)l})}{I_r(x_n, x_{n+l}, \ldots, x_{n+(L-1)l})}, \quad r = 1, 2, \ldots, \frac{p}{p - k}
\]

where \( \lambda \) is an arbitrary real number.

In order to illustrate the applicability of Theorem 6 and Corollary 7, we present the following applications.
2.1 Applications

In this subsection we will illustrate the use of our result by choosing different values for $k$ and $p$ in order to obtain parametric $p$–periodic difference equations of two parameters $\lambda$ (arbitrary) and $r (= 1, 2, \ldots, \frac{p}{p-k})$.

2.1.1 Example 1: Case $k=2$

There are only two possible values, 3 and 4, for $p$ for which $\frac{k}{p-k}$ is positive integer.

Thus, for $p = 3$, we have $l = p - k = 1$ and $L = \frac{k}{p-k} = 2$. So from Corollary 7, equation (1) will be in the form

$$x_{n+2} = f(x_n, x_{n+1}) = \frac{\lambda - I_r(x_{n+1}, x_n)}{I_{r-1}(x_{n+1}, x_n)}, \quad r = 1, 2, 3$$

where $\lambda$ is an arbitrary constant, and these difference equations are periodic of period 3. In this case, we have

$$for \ r = 1 : \quad x_{n+2} = \frac{\lambda - I_1(x_n, x_{n+1})}{I_0(x_n, x_{n+1})} \quad (5)$$

But from the Definition 4 of $I_r$ we have $I_0(x_{n+1}, x_n) = 1$ and $I_1(x_{n+1}, x_n) = x_{n+1} + x_n$. Thus equation (5) will be of the form

$$x_{n+2} = \lambda - x_{n+1} - x_n$$

Similarly, for other values of $r$.

$$for \ r = 2 : \quad x_{n+2} = \frac{\lambda - I_2(x_n, x_{n+1})}{I_1(x_n, x_{n+1})} = \frac{\lambda - x_n x_{n+1}}{x_n + x_{n+1}} \quad (6)$$

$$for \ r = 3 : \quad x_{n+2} = \frac{\lambda - I_3(x_{n+1}, x_n)}{I_2(x_{n+1}, x_n)} = \frac{\lambda}{x_n x_{n+1}} \quad (7)$$

Now, for $p = 4$, we have $l = p - k = 2$ and $L = 1$, and so all solutions of the difference equations

$$x_{n+2} = \frac{\lambda - I_r(x_n)}{I_{r-1}(x_n)}, \quad r = 1, 2$$
where $\lambda$ is an arbitrary constant, are periodic of period 4. In this case, we have the difference equations:

$$\text{for } r = 1 : \quad x_{n+2} = \frac{\lambda - I_1(x_n)}{I_0(x_n)} = \lambda - x_n$$  \hspace{1cm} (8)

$$\text{for } r = 2 : \quad x_{n+2} = \frac{\lambda - I_2(x_n)}{I_1(x_n)} = \frac{\lambda}{x_n}$$  \hspace{1cm} (9)

### 2.1.2 Example 2: Case k=3

There are two possible values for $p$ in order to have $\frac{k}{p-k}$ positive integer, these values are 4 and 6.

For $p = 4$, we have $l = p - k = 1$ and $L = \frac{k}{p-k} = 3$, so all solutions of the difference equations

$$x_{n+3} = \frac{\lambda - I_r(x_n, x_{n+1}, x_{n+2})}{I_{r-1}(x_n, x_{n+1}, x_{n+2})} \quad r = 1, 2, 3, 4$$

are periodic of period 4. In this case, we have the following difference equations:

$$r = 1 : \quad x_{n+3} = \lambda - x_n - x_{n+1} - x_{n+2}$$  \hspace{1cm} (10)

$$r = 2 : \quad x_{n+3} = \frac{\lambda - x_n x_{n+1} - x_n x_{n+2} - x_{n+1} x_{n+2}}{x_n + x_{n+1} + x_{n+2}}$$  \hspace{1cm} (11)

$$r = 3 : \quad x_{n+3} = \frac{\lambda - x_n x_{n+1} x_{n+2}}{x_n x_{n+1} + x_n x_{n+2} + x_{n+1} x_{n+2}}$$  \hspace{1cm} (12)

$$r = 4 : \quad x_{n+3} = \frac{\lambda}{x_n x_{n+1} x_{n+2}}$$  \hspace{1cm} (13)
For \( p = 6 \), we have \( l = 3 \), and \( L = 1 \), so all solutions of the difference equations

\[
x_{n+3} = \frac{\lambda - I_r(x_n)}{I_{r-1}(x_n)} \quad r = 1, 2
\]

are periodic of period 6. In this case, we have the difference equations:

\[
\begin{align*}
    r &= 1 : \quad x_{n+3} = \lambda - x_n \\
    r &= 2 : \quad x_{n+3} = \frac{\lambda}{x_n}
\end{align*}
\]

3 Transformation of Periodicity

In order to widen the area of applicability of Theorem 6 and Corollary 7, we introduce the following Proposition

**Proposition 8** Eq.(1) is periodic with fundamental period \( p \) if and only if the difference equation

\[
y_{n+l_k} = f(y_n, y_{n+l}, \ldots, y_{n+(k-1)l})
\]

is periodic with fundamental period \( lp \).

**Proof.** The change of variables

\[
x_{n+i} = y_{n+il}
\]

for \( i = 0, 1, 2, \ldots \) and \( l = 1, 2, 3, \ldots \) reduces Eq.(1) to Eq.(16). The fact, \( p \) is period of Eq.(1) if and only if \( lp \) is period of Eq.(16), is directly proved from (17).

In order to show that, \( p \) is fundamental if and only if \( lp \) is fundamental, we firstly assume that \( p \) is fundamental for Eq.(1), but \( lp \) is not, then we establish the proof by contradiction.

Assume that

\[
x_{n+p} = x_n \quad \text{for all } n
\]

Hence

\[
y_{n+lp} = y_n \quad \text{for all } n
\]
since, by assumption, \( lp \) is not the smallest such number, then there is some \( \alpha, \ 0 < \alpha < p \) such that
\[
y_n = y_{n+l(p-\alpha)} \quad \text{for all } n
\]
so
\[
x_n = x_{n+p-\alpha} \quad \text{for all } n
\]
This contradicts the assumption. Hence \( lp \) must be fundamental for Eq. (16). The proof of the other direction is similar and will be omitted. \( \blacksquare \)

3.1 Applications

The following applications generalize some results published recently.

3.1.1 Example 1

The difference equation
\[
x_{n+l} = \frac{1}{x_n} \quad l = 1, 2, ...
\]
is periodic with period \( 2l \), recalling that the difference equation
\[
x_{n+1} = \frac{1}{x_n}
\]
is periodic with period \( 2 \).

3.1.2 Example 2

The difference equation
\[
x_{n+2l} = \frac{\max\{x_{n+l}, 1\}}{x_n} \quad l = 1, 2, ...
\]
is periodic with period \( 5l \), recalling that the difference equation
\[
x_{n+2} = \frac{\max\{x_{n+1}, 1\}}{x_n}
\]
is periodic with period \( 5 \), see [4].
3.1.3 Example 3

A generalized Lyness’ equation

\[ x_{n+2l} = \frac{x_{n+l} + 1}{x_n} \quad l = 1, 2, \ldots \]

is periodic with period \(5l\), recalling that, Lyness’ equation

\[ x_{n+2} = \frac{x_{n+1} + 1}{x_n} \]  

is periodic with period \(5\), see [6]. In the next section we present another generalization for this well known equation.

4 Transformation of Periodic Equations

**Theorem 9** Let \( T : D \to D \) be an invertible transformation. If Eq. (1) is periodic with fundamental period \( p \), then the difference equation

\[ y_{n+k} = T(f(T^{-1}(y_n), \ldots, T^{-1}(y_{n+k-1}))) \]  

where \( y_n = T(x_n) \), is periodic with fundamental period \( p \).

**Proof.** By the given transformation, we have

\[ y_{n+p} = T(x_{n+p}) = T(x_n) = y_n \quad \text{for } n \in \mathbb{N} \]

which implies the result.

Now, if we combine Theorem 6 and Theorem 9 together, we obtain the following theorem of more general statement

**Theorem 10** Let \( p \in \mathbb{N} \), such that \( \frac{k}{p-k} = L \in \mathbb{N} \). Let \( I : D^{L+1} \to D \) be a symmetric, isovertible and continuous function, and \( x_n = T(y_n) \) be an invertible transformation, then Eq. (1) is periodic with period \( p \) if

\[ f(x_n, \ldots, x_{n+k-1}) = T(I^{-k}(T^{-1}(x_n), \ldots, T^{-1}(x_{n+(L-1)l})), \lambda) \]

where \( \lambda \) is an arbitrary real number independent of \( n \) and \( l = p - k \).

4.1 Applications

The applications of these above theorems are endless. However, in the following, we choose some particular periodic equations that have been derived in previous sections in order to illustrate the applicability of the theorems using the particular invertible transformation,

\[ y_n = \frac{ax_n + b}{cx_n + 1}, \quad a \neq bc \]  

This transformation will blow up the form of the equations inserting parameters \( a, b \) and \( c \) in the form of the equation, see the following examples;
4.1.1 Example 1

If we apply the transformation (20) in the Eq.(5) and simplify, we obtain that the difference equation

\[ y_{n+2} = \frac{A_1 + A_2(y_n + y_{n+1}) + A_3y_ny_{n+1}}{A_2 + A_3(y_n - y_{n+1}) + A_4y_ny_{n+1}} \]

where

\[
\begin{align*}
A_1 &= a^2(\lambda a + 3b) \\
A_2 &= -a(\lambda ac + 2bc + a) \\
A_3 &= c(\lambda ac + 2a + bc) \\
A_4 &= c^2(\lambda c + 3)
\end{align*}
\]

and \( a \neq bc, \ \lambda \) is an arbitrary constant, is periodic of period 3.

4.1.2 Example 2

If we apply the transformation (20) in the Eq.(8) and simplify, we obtain that the difference equation

\[ y_{n+2} = \frac{A_1 + A_2y_n}{-A_2 + A_3y_n} \]

where

\[
\begin{align*}
A_1 &= a(\lambda a + 2b) \\
A_2 &= -c(\lambda c + a + bc) \\
A_3 &= -c(\lambda c + 2)
\end{align*}
\]

and \( a \neq bc, \ \lambda \) is an arbitrary constant, is periodic with period 4.

4.1.3 Example 3

If we apply the transformation (20) in Lyness’equation (18) and simplify, we obtain that \( y_n \) satisfies the difference equation;

\[ y_{n+2} = \frac{a(1 - c)y_{n+1} + by_n + a^2 - ab - b^2}{c(1 - c)y_{n+1} + y_n + ac - bc - b} \] \hspace{1cm} (21)

where \( a, b \) and \( c \) are any numbers such that \( a \neq bc \).

This parametric difference equation is periodic with period 5. In particular, for \( c = 0 \), Eq(21) reduces to

\[ y_{n+2} = \frac{ay_{n+1} + by_n + a^2 - ab - b^2}{y_n - b} \quad a \neq 0 \]

which is a generalized Lynness’ equation [6].
5 Conclusion

We conclude the paper by mentioning that Th.(10) can be applied with other different forms of transformations $T$, for instance, $T(x) = \log_a x$, $a > 0$; $T(x) = x^m$, $m$ is an odd integer; and $T(x) = a^x$, $a > 0$, ...etc. or any combination of these injective maps. The cases that have been applied in this paper are nothing but a particular examples whereas the applications of the theorems are endless.

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