On Schur Complement in $k$-Kernel Symmetric Matrices

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Abstract

In this paper we give the Necessary and sufficient conditions for a Schur complement in a $k$-Kernel Symmetric matrix to be $k$-Kernel symmetric.

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1 Introduction

Let $\mathcal{F}_{mn}$ denotes the set of all $m \times n$ matrices over the fuzzy algebra $\mathcal{F}$. In short $\mathcal{F}_{nn}$ is denoted as $\mathcal{F}_n$. A development of the theory of fuzzy matrices were made by Kim and Roush[3] analogous to that of Boolean matrices.Throughout we deal with fuzzy matrices that is, matrices over a fuzzy algebra $\mathcal{F}$ with support $[0,1]$ under max-min operations.For $a,b \in \mathcal{F}$, $a+b = \max\{a,b\}$ and $a.b = \min\{a,b\}$.For $A \in \mathcal{F}_n$, let $A^T, R(A)$ and $N(A)$ denote the transpose, Rowspace and Nullspace of $A$ respectively.We denote a solution $X$ of the equation $AXA = A$ by $A^-$.For a complex matrix $M$ partitioned in the form $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ a Schur complement of $A$ in $M$ denoted by $M/A$ is defined as $D - CA^{-1}B$. This is called generalized Schur complement for complex matrices [1,7]. A matrix $A \in C_{n \times n}$ is said to be $k$-EP if it satisfies the condition

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\[ Ax=0 \iff KA^*Kx = 0 \] or equivalently \( N(A) = N(A^*K) \). For further properties of \( k \)-EP matrices one may refer [6]. For \( x = (x_1, x_2, \ldots x_n) \in \mathcal{F}_{1\times n} \) and let \( \kappa(x) = (x_{k(1)}, x_{k(2)}, \ldots x_{k(n)}) \). Let \( k \)-be a fixed product of disjoint transpositions in \( S_n = 1, 2, \ldots, n \) and \( K \) be the associated permutation matrix. In this paper we discuss when a Schur complement in \( k \)-Kernel symmetric fuzzy matrix will be \( k \)-Kernel symmetric which includes the results found in [6] as a particular case and analogous to that of the results on complex matrices found in [7]. The paper is organized as follows. In section 2, we give some basic definitions and results. In section 3 we discuss Schur complement in \( k \)-Kernel Symmetric fuzzy matrices.

2. Preliminaries

We shall recall the definition of \( k \)-kernel symmetric matrices studied in our earlier work[5].

Definition 2.1

A matrix \( A \in \mathcal{F}_n \) is said to be \( k \)-Kernel symmetric if \( N(A) = N(KA^T K) \).

In the sequel, we shall make use of the following results.

Definition 2.2[[4],P.119]

For \( A \in \mathcal{F}_n \) is Kernel symmetric if \( N(A) = N(A^T) \), where \( N(A) = \{ x/ \forall xA = 0 \text{ and } x \in \mathcal{F}_{1n} \} \).

Lemma 2.3[[4],P.125]

For \( A, B \in \mathcal{F}_n \) and \( P \) be a Permutation matrix \( N(A) = N(B) \iff N(PAP^T) = N(PBP^T) \).

3. Schur Complement in \( k \)-Kernel symmetric Matrices

Throughout we are concerned with a block fuzzy matrix of the form

\[
M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\] (3.1)

with respect to this partitioning a Schur complement of \( A \) in \( M \) is a fuzzy matrix of the form \( M/A = D - CA^{-1}B \). where \( A \) and \( D \) are square matrices. Here \( M/A \) is a fuzzy matrix if and only if \( D \geq CA^{-1}B \) , that is, \( D = D + CA^{-1}B \). A partitioned matrix \( M \) of the form (3.1) is \( k \)-Kernel symmetric then it is not true in general that a Schur complement of \( A \) in \( M \), \( M/A \) is \( k \)-Kernel symmetric. Here the necessary and sufficient conditions for \( M/A \) to be \( k \)-Kernel
symmetric are obtained. Throughout this section let \( k_1 \) and \( k_2 \) be the product of disjoint transpositions in \( S_{2n} \) defined as follows:

For \( x = (x_1, x_2, \ldots, x_{n+1}, \ldots, x_{2n}) \)

\[
k_1(x) = \{\kappa(x), x_{n+1}, \ldots, x_{2n}\}
\]

\[
k_2(x) = \{x_1, x_2, \ldots, x_{n}, \kappa(x)\}, x_{n+1} \rightarrow x_{k(1)}, x_{n+2} \rightarrow x_{k(2)}, \ldots, x_{2n} \rightarrow x_{k(n)}
\]

If \( \tilde{k} = k_1 k_2 \) then \( \tilde{k}(x) = (k(x), k(x)) \)

The permutation matrices associated with \( k_1, k_2 \) and \( \tilde{k} \) are

\[
K_1 = \begin{bmatrix} K & 0 \\ 0 & I_n \end{bmatrix}, K_2 = \begin{bmatrix} I_n & 0 \\ 0 & K \end{bmatrix} \text{ and } K = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}
\]

respectively.

In this section, \( k \)-Kernel symmetric matrices of a block matrix is discussed.

**Theorem 3.2**

Let \( M \) be a matrix of the form (3.1) with \( N(A) \subseteq N(B) \) and \( N(M/A) \subseteq N(C) \), then the following are equivalent.

1. \( M \) is \( \tilde{k} \)-Kernel symmetric matrix with \( \tilde{k} = k_1 k_2 \).
2. \( A \) is \( k \)-Kernel symmetric, \( M/A \) is \( k \)-Kernel symmetric,
   \( N(A^T) \subseteq N(C^T) \) and \( N((M/A)^T) \subseteq N(B^T) \)
3. Both the matrices \( \begin{bmatrix} A & 0 \\ C & M/A \end{bmatrix} \) and \( \begin{bmatrix} A & B \\ 0 & M/A \end{bmatrix} \) are \( \tilde{k} \)-Kernel symmetric.

**Proof**

(1) \( \Rightarrow \) (2):

To prove \( A \) is \( k \)-Kernel symmetric, \( M/A \) is \( k \)-Kernel symmetric.

Let \( x_1 \in N(A) \) and \( x_2 \in N(M/A) \). Hence \( x_1 A = 0 \) and \( x_2 (M/A) = 0 \) \quad (3.3)

Define \( x = [x_1 \ x_2] \)

we claim that \( xM = [x_1 \ x_2] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = 0 \)

Since \( N(M/A) \subseteq N(C) \), \( x_2 (M/A) = 0 \Rightarrow x_2 C = 0 \)
\[ N(A) \subseteq N(B), \ x_1A = 0 \Rightarrow x_1B = 0 \]

Hence, \( x_1A + x_2C = 0 \) and \( x_1B + x_2D = 0 \).

Therefore \( xM = 0 \) that is \( x \in N(M) \).

Since \( M \) is \( \tilde{k} \)-Kernel symmetric, \( N(M) = N(KM^TK) \)

Therefore, \( xKM^TK = 0 \)

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
\end{bmatrix}
\begin{bmatrix}
  K & 0 \\
  0 & K \\
\end{bmatrix}
\begin{bmatrix}
  A^T & C^T \\
  B^T & D^T \\
\end{bmatrix}
\begin{bmatrix}
  K & 0 \\
  0 & K \\
\end{bmatrix}
= 0
\]

\[ \Rightarrow x_1KA^TK + x_2KB^TK = 0 \Rightarrow x_1KA^TK = 0 \text{ and } x_2KB^TK = 0 \]

and \( x_1KC^TK + x_2KD^TK = 0 \Rightarrow x_1KC^TK = 0 \text{ and } x_2KD^TK = 0 \)

Hence \( x_1 \in N(KA^TK) \), \( x_2 \in N(KB^TK) \) and \( x_2 \in N(KD^TK) \)

Since \( x_1 \in N(A) \) and \( x_2 \in N(M/A) \) it follows that

\( N(A) \subseteq N(KA^TK) \), \( N(M/A) \subseteq N(KB^TK) \) and

\( N(M/A) \subseteq N(KD^TK) \) implies \( N(M/A) \subseteq N(K(M/A)^TK) \)

Similarly it can be shown that \( N(KA^TK) \subseteq N(A) \)

Thus \( A \) is \( k \)-Kernel symmetric.

Since, \( x_1 \in N(KC^TK) \) and \( A \) is \( k \)-Kernel symmetric \( N(A) = N(KA^TK) \subseteq N(KC^TK) \)

By using Lemma(2.3) \( N(A^T) \subseteq N(C^T) \).

By Definition \( M/A = D - CA^{-1}B \) implies \( N(M/A) \subseteq N(K(M/A)^TK) \)

Similarly it can be shown that \( N(K(M/A)^TK) \subseteq N(M/A) \).

Therefore \( N(M/A) = N(K(M/A)^TK) \). Hence \( M/A \) is \( k \)-Kernel Symmetric.
Since, $N(M/A) \subseteq N(KB^TK)$ and $M/A$ is $k$-Kernel Symmetric, $N(K(M/A)^TK) \subseteq N(KB^TK)$. Therefore by using Lemma(2.3) $N((M/A)^T) \subseteq N(B^T)$.

Thus (1) $\Rightarrow$ (2) holds.

(2) $\Rightarrow$ (3):

Let $x \in N(M_1)$. Partition $x$ in conformity with that of $M_1$ as $x = [x_1 \ x_2]$ then,

$$(x_1 \ x_2) \begin{bmatrix} A & 0 \\ C & M/A \end{bmatrix} = 0$$

$x_1A = 0, \ x_2C = 0; \ x_2C = 0$ and $x_2D = 0 \Rightarrow x_2(M/A) = 0$. Since $A$ and $M/A$ are $k$ - Kernel symmetric,

$x_1 \in N(A) = N(KA^TK) \Rightarrow x_1KA^TK = 0$

$x_2 \in N(M/A) = N(K(M/A)^TK)) \Rightarrow x_2(K(M/A)^TK) = 0$.

Since, $N(A^T) \subseteq N(C^T)$,

By Lemma 2.3 $N(KA^TK) \subseteq N(KC^TK) \Rightarrow x_1KC^TK = 0$.

Now, by using $x_1KA^TK = 0, \ x_1KC^TK = 0$ and $x_2K(M/A)^TK = 0$ it can be verified that

$$(x_1 \ x_2) \begin{bmatrix} KA^TK & KC^TK \\ 0 & K(M/A)^TK \end{bmatrix} = 0.$$

Thus $N(M_1) \subseteq N(KM_1^TK)$. By using Lemma(2.3) $N(KM_1^TK) \subseteq N(M_1)$.

Therefore $N(M_1) = N(KM_1^TK)$.

Hence $M_1$ is $\bar{k}$-Kernel symmetric.

In the same manner, it can be proved that $M_2$ is $k$-Kernel symmetric.

Thus (2) $\Rightarrow$ (3) holds.

(3) $\Rightarrow$ (1):

$M_1$ is $k$ - Kernel symmetric $\Rightarrow N(M_1) = N(KM_1^TK)$

$M_2$ is $\bar{k}$ - Kernel symmetric $\Rightarrow N(M_2) = N(KM_2^TK)$
To prove, $M$ is $k$-Kernel Symmetric that is $N(M) = N(KM^T K)$.
Let $x \in N(M) \Rightarrow xM = 0$.

Partition $x$ in conformity with that of $M$ as $x = [x_1 \ x_2]$ then,

$$
\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = 0.
$$

$x_1A + x_2C = 0 \Rightarrow x_1A = 0$ and $x_2C = 0$

$x_1B + x_2D = 0 \Rightarrow x_1B = 0$ and $x_2D = 0$

From the definition of $M/A = D - CA^{-1}B$ we have,

$x_2D = 0$ and $x_2C = 0 \Rightarrow x_2(M/A) = 0$

$x_1A + x_2C = 0$ and $x_2(M/A) = 0$ \quad (3.4)

And $x_1A = 0$, $x_1B + x_2(M/A) = 0$ \quad (3.5)

From (3.4), $x \in N(M_1) \Rightarrow x \in N(KM^T_1 K)$.

From (3.5), $x \in N(M_2) \Rightarrow x \in N(KM^T_2 K)$. Hence $x \in N(KM^T K)$.

$N(M) \subseteq N(KM^T K)$. Similarly $N(KM^T K) \subseteq N(M)$.

Therefore $M$ is $\tilde{k}$ - Kernel symmetric matrix. Hence the theorem.

**Theorem 3.6**

Let $M$ be a matrix of the form (3.1) with $N(A^T) \subseteq N(C^T)$ and $N((M/A)^T) \subseteq N(B^T)$, then the following are equivalent.

1. $M$ is $\tilde{k}$ - Kernel symmetric matrix with $\tilde{k} = k_1k_2$.

2. $A$ is $k$ - Kernel symmetric, $M/A$ is $k$ - Kernel symmetric further $N(A) \subseteq N(B)$ and $N(M/A) \subseteq N(C)$

3. Both the matrices $\begin{bmatrix} A & 0 \\ C & M/A \end{bmatrix}$ and $\begin{bmatrix} A & B \\ 0 & M/A \end{bmatrix}$ are $\tilde{k}$ - Kernel symmet-
Theorem 3.2 and from the fact that $\tilde{k}$ - Kernel symmetric $\iff M^T$ is $\tilde{k}$ - Kernel symmetric.

**Corollary 3.7**

Let $M$ be a matrix of the form $\begin{bmatrix}A & C^T \\ C & D\end{bmatrix}$ with $N(A) \subseteq N(C^T)$ and $N((M/A)) \subseteq N(C)$, then the following are equivalent.

1. $M$ is $\tilde{k}$ - Kernel symmetric matrix with $\tilde{k} = k_1k_2$.
2. $A$ is $k$ - Kernel symmetric, $M/A$ is $k$ - Kernel symmetric
3. The matrix $\begin{bmatrix}A & 0 \\ C & M/A\end{bmatrix}$ is $\tilde{k}$ - Kernel symmetric.

**Remark 3.8**

The condition taken on $M$ in Theorem 3.2 and Theorem 3.6 are essential. This is illustrated in the following example.

Example: Let $M = \begin{bmatrix}1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1\end{bmatrix}$ and $K = \begin{bmatrix}0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0\end{bmatrix}$.

For this $M$, since $M$ has no zero rows and no zero columns $N(M) = \{0\}$.

$N(KM^TK) = \{0\}$. Thus $N(M) = N(KM^TK) \Rightarrow M$ is $\tilde{k}$ - Kernel Symmetric.

$A^0 = \begin{bmatrix}1 & 0 \\ 1 & 1\end{bmatrix}$ is a g-inverse, with respect to $A^-$.

$M/A = D - CA^-B = \begin{bmatrix}0 & 0 \\ 1 & 0\end{bmatrix}$

$M/A$ is $k$ - Kernel Symmetric, since $N(M/A) = N(K(M/A)^TK) = \{0\}$.

$A$ is $k$ - Kernel Symmetric, since $N(A) = N(KA^TK) = \{0\}$ for all $K$. 

$N(A) \subseteq N(B)$ and $N(A^T) \subseteq N(C^T)$.

Here, $N(M/A) = \{(0, x_2) : x_2 \in \mathcal{F}\} = N((M/A)^T)$, $N(C) = \{0\}$, $N(B^T) = \{0\}$.

$N(M/A)$ not contained in $N(C)$ and $N((M/A)^T)$ not contained in $N(B^T)$.

Further, $M_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, $N(M_1) = \{0\}$. $N(KM_1^T K) = \{(0, 0, 0, x_4) : x_4 \in \mathcal{F}\}$.

$\Rightarrow M_1$ is not $k$- Kernel Symmetric.

$M_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. $M_2$ is not $k$- Kernel Symmetric.

Thus condition (1) of Theorem (3.2) holds but condition (2) and (3) of the Theorem (3.2) fail. Thus condition (1) of Theorem (3.6) holds but condition (2) and (3) fail.

**Remark 3.9**

For a $k$ - Kernel symmetric matrix $M$ of the form $M_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $k = k_1k_2$, the following are equivalent

$N(A) \subseteq N(B)$, $N(M/A) \subseteq N(C)$

$N(A^T) \subseteq N(C^T)$, $N((M/A)^T) \subseteq N(B^T)$

However this fails if we omit the condition that $M$ is $k$ -Kernel symmetric.

Example:
On Schur complement

\[ M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \]

\[ N(M) = \{(0, 0, x_3, 0) : x_3 \in \mathcal{F}\} \neq N(KM^TK). \]

Therefore \( M \) is not \( k \)-Kernel symmetric.

\[ M/A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \]

Here \( N(A) \subseteq N(B) \), \( N(M/A) \subseteq N(C) \) but \( N(A^T) \) is not contained in \( N(C^T) \), \( N((M/A)^T) \) is not contained in \( N(B^T) \).

References


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