(α, δ)-Neighborhood for Functions Associated with Sălăgean Differential Operator and Alexander Integral Operator

Kazuyuki Kugita, Kazuo Kuroki and Shigeyosi Owa

Department of Mathematics, Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
ib3mi3@bma.biglobe.ne.jp
freedom@sakai.zaq.ne.jp
owa@math.kindai.ac.jp

Abstract

By using Sălăgean differential operator and Alexander integral operator for analytic functions \( f(z) \) with \( f(0) = 0 \) and \( f'(0) = 1 \) in the open unit disk \( U \) which are the inverse operator each other, a generalized operator combining the both operators is introduced. Defining the class of \((\alpha, \delta)\)-neighborhood for analytic functions with the generalized operator, some interesting properties for \((\alpha, \delta)\)-neighborhood are discussed.

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1 Introduction and definitions

Let \( \mathcal{A} \) be the class of functions \( f(z) \) of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]

that are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). For \( f(z) \in \mathcal{A} \), Sălăgean [4] has introduced the following operator \( D^j f(z) \) which is called Sălăgean differential operator.
$$D^0 f(z) = f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

$$D^1 f(z) = Df(z) = zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n,$$

$$D^j f(z) = D(D^{j-1} f(z)) = z + \sum_{n=2}^{\infty} n^j a_n z^n \quad (j = 1, 2, 3, \cdots).$$

Also, Alexander [1] has defined the following Alexander integral operator

$$D^{-1} f(z) = \int_{0}^{z} \frac{f(\zeta)}{\zeta} d\zeta = z + \sum_{n=2}^{\infty} n^{-1} a_n z^n.$$ 

Further, we introduce

$$D^{-j} f(z) = D^{-1}(D^{-(j-1)} f(z)) = z + \sum_{n=2}^{\infty} n^{-j} a_n z^n \quad (j = 1, 2, 3, \cdots).$$

Therefore, combining Sălăgean differential operator and Alexander integral operator, we introduce the operator $D^j f(z)$ by

$$D^j f(z) = z + \sum_{n=2}^{\infty} n^j a_n z^n$$

for any integer $j$. Applying the above operator, we consider the subclass $(\alpha, \delta) - N_{m+1}^{j+1}(g)$ of $\mathcal{A}$ as follows. A function $f(z) \in \mathcal{A}$ is said to be in the class $(\alpha, \delta) - N_{m+1}^{j+1}(g)$ if it satisfies

$$\left| \frac{D^{j+1} f(z)}{z} - e^{i\alpha} \frac{D^{m+1} g(z)}{z} \right| < \delta \quad (z \in \mathbb{U})$$

for some $-\pi \leq \alpha \leq \pi$, $\delta > \sqrt{2(1 - \cos \alpha)}$, and for some $g(z) \in \mathcal{A}$.

**Remark 1.1** If $j = m = 0$, then

$$(\alpha, \delta) - N_1^1(g) \equiv (\alpha, \delta) - N(g)$$

and if $j = m = -1$, then

$$(\alpha, \delta) - N_0^0(g) \equiv (\alpha, \delta) - M(g).$$

The classes $(\alpha, \delta) - N(g)$ and $(\alpha, \delta) - M(g)$ were considered by Orhan, Kadioğlu and Owa [4].
2 Main theorem

Let us define $g(z) \in \mathcal{A}$ by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

through this paper. Our first result of $f(z)$ for $(\alpha, \delta) - N_{m+1}^{j+1}(g)$ is contained in

**Theorem 2.1** If $f(z) \in \mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty} n|n^j a_n - e^{i\alpha} n^m b_n| \leq \delta - \sqrt{2(1 - \cos \alpha)}$$

for some $-\pi \leq \alpha \leq \pi$, $\delta > \sqrt{2(1 - \cos \alpha)}$ and for some $g(z) \in \mathcal{A}$, then $f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g)$.

**Proof.** Note that

$$\left| \frac{D^{j+1}f(z)}{z} - e^{i\alpha} \frac{D^{m+1}g(z)}{z} \right| = \left| (1 - e^{i\alpha}) + \sum_{n=2}^{\infty} n(n^j a_n - e^{i\alpha} n^m b_n)z^{n-1} \right|$$

$$\leq \left| (1 - e^{i\alpha}) \right| + \sum_{n=2}^{\infty} n|n^j a_n - e^{i\alpha} n^m b_n||z|^{n-1}$$

$$< \sqrt{2(1 - \cos \alpha)} + \sum_{n=2}^{\infty} n|n^j a_n - e^{i\alpha} n^m b_n|.$$

If

$$\sum_{n=2}^{\infty} n|n^j a_n - e^{i\alpha} n^m b_n| \leq \delta - \sqrt{2(1 - \cos \alpha)},$$

then we see that

$$\left| \frac{D^{j+1}f(z)}{z} - e^{i\alpha} \frac{D^{m+1}g(z)}{z} \right| < \delta \quad (z \in \mathbb{U}).$$

This gives us that $f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g)$.

**Example 2.2** For given $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$, we consider $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ with
\[ a_n = \frac{\delta - \sqrt{2(1 - \cos \alpha)}}{n^{j+2}(n-1)} e^{i\gamma} + e^{i\alpha} n^{m-j} b_n \quad (n = 2, 3, 4, \cdots). \]

Then, we have that

\[
\sum_{n=2}^{\infty} n |n^j a_n - e^{i\alpha} n^m b_n| = \sum_{n=2}^{\infty} n \left| n^j \frac{\delta - \sqrt{2(1 - \cos \alpha)}}{n^{j+2}(n-1)} e^{i\gamma} \right|
\]

\[
= \left( \delta - \sqrt{2(1 - \cos \alpha)} \right) \left( \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \right)
\]

\[
= \left( \delta - \sqrt{2(1 - \cos \alpha)} \right) \left\{ \sum_{n=2}^{\infty} \left( \frac{1}{n-1} - \frac{1}{n} \right) \right\}
\]

\[
= \delta - \sqrt{2(1 - \cos \alpha)}.
\]

Therefore, \( f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g) \).

**Corollary 2.3** Let \( f(z) \in \mathcal{A} \) satisfy

\[
\sum_{n=2}^{\infty} n |n^j a_n - n^m b_n| \leq \delta - \sqrt{2(1 - \cos \alpha)}
\]

for some \(-\pi \leq \alpha \leq \pi, \delta > \sqrt{2(1 - \cos \alpha)}\) and for \( g(z) \in \mathcal{A} \) with \( \arg a_n - \arg b_n = \alpha \) \((n = 2, 3, 4, \cdots)\), then \( f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g) \).

**Proof.** By Theorem 2.1, we have that if \( f(z) \in \mathcal{A} \) satisfies

\[
\sum_{n=2}^{\infty} n |n^j a_n - e^{i\alpha} n^m b_n| \leq \delta - \sqrt{2(1 - \cos \alpha)},
\]

then \( f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g) \). Since \( \arg a_n - \arg b_n = \alpha \), if \( \arg a_n = \varphi_n \), then \( \arg b_n = \varphi_n - \alpha \). Therefore, we see that

\[
n^j a_n - e^{i\alpha} n^m b_n = n^j |a_n| e^{i\varphi_n} - e^{i\alpha} n^m b_n e^{i(\varphi_n - \alpha)} = (n^j |a_n| - n^m |b_n|) e^{i\varphi_n},
\]

that is, that

\[
|n^j a_n - e^{i\alpha} n^m b_n| = |n^j |a_n| - n^m |b_n||.
\]

This completes the proof of the corollary. \( \square \)

**Remark 2.4** If we take \( j = m = 0 \), or \( j = m = -1 \) in Theorem 2.1, then we can obtain the results for \((\alpha, \delta) - N(g)\) and \((\alpha, \delta) - M(g)\), respectively, given by Orhan, Kadioğlu and Owa \([4]\).
Next, we discuss the necessary conditions for neighborhoods.

**Theorem 2.5** If \( f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g) \) with
\[
\arg(n^j a_n - e^{i\alpha} n^m b_n) = (n - 1)\varphi \quad (\varphi \in \mathbb{R}),
\]
for \( n = 1, 2, 3, \ldots \), then,
\[
\sum_{n=2}^{\infty} n |n^j a_n - e^{i\alpha} n^m b_n| \leq \delta + \cos \alpha - 1.
\]

**Proof.** For \( f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g) \), we have
\[
\left|\frac{D^{j+1}f(z)}{z} - e^{i\alpha} \frac{D^{m+1}g(z)}{z}\right| = \left|1 - e^{i\alpha} + \sum_{n=2}^{\infty} n(n^j a_n - e^{i\alpha} n^m b_n)z^{n-1}\right|
\]
\[
= \left|1 - e^{i\alpha} + \sum_{n=2}^{\infty} n |n^j a_n - e^{i\alpha} n^m b_n| e^{i(n-1)\varphi} z^{n-1}\right| < \delta
\]
for all \( z \in \mathbb{U} \). If we consider a point \( z \in \mathbb{U} \) such that \( \arg z = -\varphi \), then
\[
z^{n-1} = |z|^{n-1} e^{-i(n-1)\varphi}.
\]
For such a point \( z \in \mathbb{U} \), we have
\[
\left|\frac{D^{j+1}f(z)}{z} - e^{i\alpha} \frac{D^{m+1}g(z)}{z}\right| = \left|1 - e^{i\alpha} + \sum_{n=2}^{\infty} n |n^j a_n - e^{i\alpha} n^m b_n| |z|^{n-1}\right|
\]
\[
= \left(1 + \sum_{n=2}^{\infty} n |n^j a_n - e^{i\alpha} n^m b_n| |z|^{n-1} - \cos \alpha\right)^2 + \sin^2 \alpha < \delta.
\]
It follows from the above that
\[
(1 - \cos \alpha) + \sum_{n=2}^{\infty} n |n^j a_n - e^{i\alpha} n^m b_n| |z|^{n-1} < \delta.
\]
If we take \( |z| \to 1^- \), then
\[
\sum_{n=2}^{\infty} n |n^j a_n - e^{i\alpha} n^m b_n| \leq \delta + \cos \alpha - 1.
\]

**Remark 2.6** If we take \( j = m = 0 \), or \( j = m = -1 \) in Theorem 2.5, then we obtain the corresponding results for \((\alpha, \delta) - N(g)\) and \((\alpha, \delta) - M(g)\), respectively, given by Orhan, Kadioğlu and Owa [4].
3 Applications of Jack’s lemma

We give some applications of Jack’s lemma. The following lemma was given by Jack [2] (see, also [3]).

**Lemma 3.1** Let the function \( f(z) \) be analytic in \( U \) with \( w(z) = 0 \). If there exists a point \( z_0 \in U \) such that \( \max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| \), then \( z_0 w'(z_0) = kw(z_0) \), where \( k \) is real and \( k \geq 1 \).

Applying Lemma 3.1, we derive

**Theorem 3.2** If \( f(z) \in A \) satisfies

\[
\left| \frac{D^{j+1}f(z)}{z} - e^{i\alpha} \frac{D^{m+1}g(z)}{z} \right| < 2\delta - \sqrt{2(1 - \cos \alpha)} \quad (z \in U)
\]

for some \(-\pi \leq \alpha \leq \pi\), \( \delta > \sqrt{2(1 - \cos \alpha)} \) and for some \( g(z) \in A \), then

\[
\left| \frac{D^j f(z)}{z} - e^{i\alpha} \frac{D^m g(z)}{z} \right| < \delta + \sqrt{2(1 - \cos \alpha)} \quad (z \in U).
\]

**Proof.** We define the function \( w(z) \) by

\[
\frac{D^j f(z)}{z} - e^{i\alpha} \frac{D^m g(z)}{z} - 1 + e^{i\alpha} = \delta w(z).
\]

Then \( w(z) \) is analytic in \( U \) with \( w(0) = 0 \). In view of the condition, we say that

\[
\left| \frac{D^{j+1}f(z)}{z} - e^{i\alpha} \frac{D^{m+1}g(z)}{z} \right| = \left| 1 - e^{i\alpha} + \delta w(z) \left( 1 + \frac{zw'(z)}{w(z)} \right) \right| < 2\delta - \sqrt{2(1 - \cos \alpha)} \quad (z \in U).
\]

Let us suppose that there is a point \( z_0 \in U \) such that \( \max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \). Then, by Lemma 3.1, we can write that

\[
w(z_0) = e^{i\theta} \quad \text{and} \quad \frac{z_0 w'(z_0)}{w(z_0)} = k \geq 1.
\]

Therefore, we see that

\[
\left| \frac{D^{j+1}f(z_0)}{z_0} - e^{i\alpha} \frac{D^{m+1}g(z_0)}{z_0} \right| = \left| 1 - e^{i\alpha} + \delta e^{i\theta}(1 + k) \right|
\]

\[
\geq \delta(1 + k) - |1 - e^{i\alpha}|
\]

\[
\geq 2\delta - \sqrt{2(1 - \cos \alpha)}.
\]
This contradicts our condition in Theorem 3.2. Thus, there is no point \( z_0 \in \mathbb{U} \) such that \( |w(z_0)| = 1 \). This means that \( |w(z)| < 1 \) for all \( z \in \mathbb{U} \). Therefore, we have that

\[
\left| \frac{D^j f(z)}{z} - e^{i\alpha} \frac{D^m g(z)}{z} \right| = \left| 1 - e^{i\alpha} + \delta w(z) \right| \\
\leq \left| 1 - e^{i\alpha} \right| + \delta \left| w(z) \right| \\
< \delta + \sqrt{2}(1 - \cos \alpha) \quad (z \in \mathbb{U}).
\]

\( \square \)

Letting \( \alpha = \frac{\pi}{2} \) in Theorem 3.2, we have the following the corollary.

**Corollary 3.3** If \( f(z) \in \mathcal{A} \) satisfies

\[
\left| \frac{D^{j+1} f(z)}{z} - i \frac{D^{m+1} g(z)}{z} \right| < 2 \delta - \sqrt{2} \quad (z \in \mathbb{U})
\]

for some \( \delta > \sqrt{2} \) and for some \( g(z) \in \mathcal{A} \), then

\[
\left| \frac{D^j f(z)}{z} - i \frac{D^m g(z)}{z} \right| < \delta + \sqrt{2} \quad (z \in \mathbb{U}).
\]

**Remark 3.4** By means of Theorem 3.1, we see that

\[
\left| \frac{D^{j+1} f(z)}{z} - e^{i\alpha} \frac{D^{m+1} g(z)}{z} \right| < 2 \delta - \sqrt{2}(1 - \cos \alpha)
\]

\[
\Rightarrow \left| \frac{D^j f(z)}{z} - e^{i\alpha} \frac{D^m g(z)}{z} \right| < \delta + \sqrt{2}(1 - \cos \alpha)
\]

\[
\Rightarrow \left| \frac{D^{j-1} f(z)}{z} - e^{i\alpha} \frac{D^{m-1} g(z)}{z} \right| < \delta_1 + \sqrt{2}(1 - \cos \alpha)
\]

\[
\Rightarrow \left| \frac{D^{j-2} f(z)}{z} - e^{i\alpha} \frac{D^{m-2} g(z)}{z} \right| < \delta_2 + \sqrt{2}(1 - \cos \alpha)
\]

\[\vdots\]

\[
\Rightarrow \left| \frac{D f(z)}{z} - e^{i\alpha} \frac{D^{m-j+1} g(z)}{z} \right| < \delta_{j-1} + \sqrt{2}(1 - \cos \alpha)
\]

\[
\Rightarrow \left| \frac{f(z)}{z} - e^{i\alpha} \frac{D^{m-j} g(z)}{z} \right| < \delta_j + \sqrt{2}(1 - \cos \alpha),
\]
where
\[ \delta_\ell = \frac{\delta + 2(2^\ell - 1)\sqrt{2(1 - \cos \alpha)}}{2^\ell} \quad (\ell = 0, 1, 2, \ldots). \]

Thus, we have that
\[
\left| \frac{D^{j+1}f(z)}{z} - e^{i\alpha} \frac{D^{m+1}g(z)}{z} \right| < 2\delta - \sqrt{2(1 - \cos \alpha)}
\]
\[ \implies \left| \frac{D^{j-\ell}f(z)}{z} - e^{i\alpha} \frac{D^{m-\ell}g(z)}{z} \right| < \delta_\ell + \sqrt{2(1 - \cos \alpha)} \quad (\delta_0 = \delta). \]

Next, we derive

**Theorem 3.5**  If \( f(z) \in \mathcal{A} \) satisfies
\[
\text{Re} \left( \frac{D^{j+1}f(z)}{z} - e^{i\alpha} \frac{D^{m+1}g(z)}{z} \right) > 1 - \cos \alpha - \frac{3}{4} \delta \quad (z \in \mathbb{U})
\]
for some \(-\pi \leq \alpha \leq \pi, \delta > 0\) and for some \( g(z) \in \mathcal{A} \), then
\[
\text{Re} \left( \frac{D^{j}f(z)}{z} - e^{i\alpha} \frac{D^{m}g(z)}{z} \right) > 1 - \cos \alpha - \frac{1}{2} \delta \quad (z \in \mathbb{U}).
\]

**Proof.** We define the function \( w(z) \) by
\[
\frac{D^{j}f(z)}{z} - e^{i\alpha} \frac{D^{m}g(z)}{z} = 1 + e^{i\alpha} = \frac{w(z)}{1 - w(z)} \quad (w(z) \neq 1).
\]

Then \( w(z) \) is analytic in \( \mathbb{U} \) with \( w(0) = 0 \) and satisfies
\[
\frac{D^{j+1}f(z)}{z} - e^{i\alpha} \frac{D^{m+1}g(z)}{z} = 1 - e^{i\alpha} + \delta \frac{w(z)}{1 - w(z)} + \frac{zw'(z)}{(1 - w(z))^2}.
\]

Therefore, we have that
\[
\text{Re} \left( 1 - e^{i\alpha} + \delta \frac{w(z)}{1 - w(z)} + \delta \frac{zw'(z)}{(1 - w(z))^2} \right) > 1 - \cos \alpha - \frac{3}{4} \delta \quad (z \in \mathbb{U}).
\]

If there is a point \( z_0 \in \mathbb{U} \) such that \( \max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \), then, by Lemma 3.1,
\[
w(z_0) = e^{i\theta} \quad \text{and} \quad z_0w'(z_0) = ke^{i\theta} \quad (k \geq 1).
\]
Therefore, for such a point $z_0 \in U$, we have that
\[
\operatorname{Re}\left( \frac{D^{j+1}f(z_0) - e^{i\alpha}D^{m+1}g(z_0)}{z_0} \right) = \operatorname{Re}\left( 1 - e^{i\alpha} + \delta \frac{e^{i\theta}}{1 - e^{i\theta}} + \delta \frac{ke^{i\theta}}{(1 - e^{i\theta})^2} \right) \\
\leq 1 - \cos \alpha - \frac{1}{2}\delta - \frac{1}{4}\delta \\
= 1 - \cos \alpha - \frac{3}{4}\delta,
\]
which contradicts our condition in Theorem 3.5. Thus, there is no point $z_0 \in U$ such that $|w(z_0)| = 1$. This shows that $|w(z)| < 1$ for all $z \in U$, that is, that
\[
\operatorname{Re}\left( \frac{w(z)}{1 - w(z)} \right) > -\frac{1}{2} \quad (z \in U).
\]
Consequently, we obtain
\[
\operatorname{Re}\left( \frac{D^j f(z) - e^{i\alpha}D^m g(z)}{z} \right) > 1 - \cos \alpha - \frac{1}{2}\delta \quad (z \in U).
\]

If we take $\alpha = \frac{\pi}{2}$, then we see

**Corollary 3.6** If $f(z) \in \mathcal{A}$ satisfies
\[
\operatorname{Re}\left( \frac{D^{j+1}f(z) - D^{m+1}g(z)}{z} \right) > 1 - \frac{3}{4}\delta \quad (z \in U)
\]
for some $\delta > 0$ and for some $g(z) \in \mathcal{A}$, then
\[
\operatorname{Re}\left( \frac{D^j f(z) - D^m g(z)}{z} \right) > 1 - \frac{1}{2}\delta \quad (z \in U).
\]
Furthermore, if $\delta = 2(1 - \beta)$ $(0 \leq \beta < 1)$, then
\[
\operatorname{Re}\left( \frac{D^{j+1}f(z) - D^{m+1}g(z)}{z} \right) > \frac{3}{4}\beta - \frac{1}{2} \quad (z \in U)
\]
implies that
\[
\operatorname{Re}\left( \frac{D^j f(z) - D^m g(z)}{z} \right) > \beta \quad (z \in U).
\]
References


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