On the Best $L^2$–Approximation of Entire Functions with Index-Pair $(p, q)$

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Abstract

In this paper, we have generalized the results of Reddy [8] to any compact Jordan region of positive transfinite diameter for any weight function, positive and continuous on this region. Moreover, we have obtained $(p, q)$–order and $(p, q)$–type in terms of best $L^2$–approximation error.

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1 Introduction

Let $E$ be compact Jordan region in complex plane with transfinite diameter $d > 0$. Let $w(z)$ be a positive, continuous function on $E$ and let $H_2(E)$ denote the Hilbert space of functions analytic in $E$ with inner product.

$$(f, g) = \int \int_E w(z)f(z)\overline{g(z)}dxdy \quad f, g \in H_2(E). \quad (1.1)$$

For any $f \in H_2(E)$, we have

$$\|f\| = \left[\int \int_E w(z)|f(z)|^2dxdy\right]^{1/2} < \infty. \quad (1.2)$$

1This work was done in the memory of Prof. H.S.Kasana, Senior Associate, ICTP, Trieste, Italy.
If \( A(E) \equiv \{ p_{n-1}(z) \}_{n=1}^{\infty}, p_n(z) \) being a polynomial of degree \( n \), is a complete orthonormal sequence in \( H_2(E) \). Such a sequence of polynomials always exists in \( H_2(E) \) as can be easily seen with the help of Faber polynomials ([2],[10]).

Let us define the \( L^2 \)-approximation error as follows:

\[
e_n(f) \equiv e_n(f, E) = \inf_{c_n}
\left[ \int_E w(z) |f(z) - C_0 - C_1p_1(z) - \cdots - C_np_n(z)|^2 \, dx \, dy \right]^{1/2}
\]

\[
a_n \equiv a_n(f, E) = \int_E w(z) f(z)p_n(z) \, dx \, dy, \quad n = 0, 1, 2, \cdots \tag{1.4}
\]

\( e_n(f, E) \) is called the minimum error of \( f \) in \( L^2 \)-norm with respect to the system \( A(E) \) and \( a_n \) is called the \( n^{th} \) Fourier coefficients of \( f \) with respect to the system \( A(E) \). Goffman and Pedrick [3] showed that if \( D \) denotes the unit disc and \( w(z) \equiv 1 \), then \( A(D) = \{ \sqrt{\frac{\pi}{n}} z^{n-1} \}_{n=1}^{\infty} \) forms a complete orthonormal sequence in \( H_2(D) \) and that if \( f(z) = \sum_{n=0}^{\infty} b_n z^n (|z| < 1) \) is in \( H_2(D) \) then

\[
b_n = \frac{\sqrt{n + 1}}{\pi} a_n(f, D). \tag{1.5}
\]

In the consequence of (1.5) it follows that if \( f \) can be extended to an entire function of order \( \rho \), lower order \( \lambda \) and type \( T \), then in all the results that give \( \rho, \lambda, T \) in terms of the coefficients \( b_n \)s ([1],[4],[7]), one can replace \( b_n \) by \( a_n \).

Rizvi and Juneja [9] showed that if \( f \in H_2(E) \), the Fourier series \( \sum_{k=0}^{\infty} a_k p_k(z) \) converges uniformly to \( f(z) \) on \( E \) and \( f \) can be extended to an entire function if and only if

\[
\lim_{n \to \infty} |a_n(f, E)|^{1/n} = 0. \tag{1.6}
\]

If \( f \) is order \( \rho(0 < \rho < \infty) \), type \( T \) then

\[
\rho = \lim sup_{n \to \infty} \frac{n \log n}{\log |a_n| - 1}, \tag{1.7}
\]

\[
Td^\rho = \lim sup_{n \to \infty} (n/e \rho)|a_n|^\rho/n. \tag{1.8}
\]

Reddy [8] has obtained relations involving \( e_n(f, D) \) and the order and type of entire function \( f \). The aim of the present paper is to generalize the results of Reddy to any compact Jordan region \( E \) for any weight function \( w(z) \), positive and continuous on \( E \). Moreover, for inclusion of entire functions of slow
growth and fast growth, their results will also be extendend to the \((p, q)\)-scale introduced by Juneja et.al. ([5],[6]).

Now, we need the concept of \((p, q)\)-scale, \(p \geq q \geq 1\), and certain notations which will be frequently used in the test:

\[
P(L(p, q)) = \begin{cases} 
L(p, q) & \text{if } q < p < \infty \\
1 + L(p, q) & \text{if } p = q = 2 \\
\max(1, L(p, q)) & \text{if } 3 \leq p = q \\
\infty & \text{if } p = q = \infty.
\end{cases}
\]

Let \(f(z) = \sum_{n=0}^{\infty} b_n z^n\) be an entire function. We set \(M(r, f) = \max_{|z|=r} |f(z)|\); \(M(r, f)\) is called the maximum modulus of \(f(z)\) on the circle \(|z|=r\).

**Definition 1.** An entire function \(f(z)\) is said to be of \((p, q)\)-order \(\rho(p, q)\) if it is of index-pair \((p, q)\) such that

\[
\limsup_{r \to \infty} \frac{\log^{[p]} M(r, f)}{\log^{[q]} r} = \rho(p, q)
\]

and the function \(f(z)\) having \((p, q)\)-order \(\rho(p, q)(b < \rho(p, q) < \infty)\) is said to be of \((p, q)\)-type \(T(p, q)\) if

\[
\limsup_{r \to \infty} \frac{\log^{[p-1]} M(r, f)}{\left(\log^{[q-1]} r\right)^{\rho(p, q)}} = T(p, q), \quad 0 \leq T(p, q) \leq \infty,
\]

where \(b = 1, \text{ if } p = q, b = 0 \text{ if } p > q\).

The coefficient characterizations are as follows:

\[
\rho(p, q) = \limsup_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q-1]} |a_n|^{-1/n}}, \tag{1.9}
\]

\[
\frac{T(p, q)}{BM(p, q)} = \limsup_{n \to \infty} \frac{\log^{[p-2]} n}{\log^{[q-1]} |a_n|^{-1/n}}, \tag{1.10}
\]

where \(B = d^{-p(p,q)}\) for \(q = 1\) and \(B = 1\) for \(q > 1\) and

\[
M(p, q) = \begin{cases} 
(p - 1)^{\rho-1}/\rho^\rho & \text{if } (p, q) = 2, 2 \\
1/e\rho & \text{if } (p, q) = 2, 1 \\
1 & \text{otherwise}.
\end{cases}
\]

These expression for \(\rho(p, q)\) and \(T(p, q)\) are due to Juneja et.al. ([5],[6]). The coefficients \(a_n\) can be replaced by \(b_n\) as shown previously.
2 Auxiliary Results

This section contains various results which we have proved to establish main theorems.

**Proposition 1.** Let $f \in H_2(E)$, then

$$e_n(f) = \left[ \sum_{n+1}^{\infty} |a_k|^2 \right]^{1/2}, \quad n = 1, 2, 3, \ldots \quad (2.1)$$

**Proof.** We have

$$e_n(f) \leq \left[ \int \int_E w(z) \left| f(z) - \sum_{k=0}^{n} a_k p_k(z) \right|^2 dx dy \right]^{1/2}$$

$$= \left[ \int \int_E w(z) \left| \sum_{n+1}^{\infty} a_k p_k(z) \right|^2 dx dy \right]^{1/2}$$

due to convergence of $\sum_{k=0}^{\infty} a_k(z)$ to $f(z)$ on $E$, which gives

$$e_n(f) \leq \left[ \sum_{n+1}^{\infty} |a_k|^2 \right]^{1/2}. \quad (2.2)$$

For any $(n+1)$ complex numbers $\{a'_0, a'_1, \ldots, a'_n\}$

$$\int \int_E w(z) \left| f(z) - a'_0 p_0(z) - a'_1 p_1(z) - \cdots - a'_n p_n(z) \right|^2 dx dy$$

$$= \int \int_E w(z) \left[ \sum_{k=0}^{n} (a_k - a'_k) p_k(z) + \sum_{k=n+1}^{\infty} a_k p_k(z) \right]^2 dx dy$$

$$= \sum_{k=0}^{n} |a_k - a'_k|^2 + \sum_{k=n+1}^{\infty} |a_k|^2 \geq \sum_{k=n+1}^{\infty} |a_k|^2.$$  

Since this is true for any $(n+1)$ complex numbers, we have

$$[e_n(f)]^2 \geq \sum_{n+1}^{\infty} |a_k|^2. \quad (2.3)$$

(2.2) and (2.3) together give (2.1).
Proposition 2. [11]. Let \( w = \varphi(\infty) = \infty \) and \( \varphi'(\infty) > 0 \) and let \( E_r = \{ z : |\varphi(z)|d = r \} \). Let \( M(r) = \max_{z \in E_r} |f(z)| \) and let \( f \) be an entire function of \((p, q)\)-order \( \rho \), \((p, q)(b < \rho(p, q) < \infty), (p, q)\)-type \( T(p, q) \) then

\[
\rho(p, q) = \limsup_{r \to \infty} \frac{\log[p] M(r)}{\log[q] r},
\]

(2.4)

\[
T(p, q) = \limsup_{r \to \infty} \frac{\log[p-1] M(r)}{\log[q-1] r^p(p,q)}.
\]

(2.5)

Proposition 3. Let \( f \) be an entire function, then for any \( \varepsilon > 0 \)

\[
e_n(f) \leq K \overline{M}(r) \left( \frac{de^\varepsilon}{r} \right)^n
\]

for \( r \geq r_0(\varepsilon) \), where \( K \) is a constant independent of \( n \) and \( r \).

Proof. Winiarski [11] has proved that there exist polynomials \( \{g_n(z)\}_{n=1}^\infty \) of respective degree less than or equal to \( n \), such that for all \( z \in E \)

\[
|f(z) - g_n(z)| \leq A \overline{M}(r) \left( \frac{de^\varepsilon}{r} \right)^n
\]

(2.6)

for \( n \geq n_0(\varepsilon, E) \) and \( r \geq r_0(\varepsilon) \) and where \( A \) is a constant independent of \( n \) and \( \varepsilon \) and \( E \). Since \( g_n(z) \) can be written as a linear combination of \( \{p_k(z)\}_{k=0}^n \), it follows that

\[
e_n(f) \leq \left[ \int \int_E w(z)|f(z) - g_n(z)|^2dxdy \right]^{1/2}.
\]

Using (2.6), we get

\[
e_n(f) \leq K \overline{M}(r) \left( \frac{de^\varepsilon}{r} \right)^n,
\]

where \( K \) is independent of \( n \) and \( r \).

Proposition 4. Let \( f \in H_2(E) \). Then \( f \) can be extended to an entire function, if and only if

\[
\lim_{n \to \infty} e_n(f)^{1/n} = 0.
\]

(2.7)

Proof. Suppose \( f \) can be extended to an entire function. Then, by (1.6), we have

\[
\lim_{n \to \infty} a_n^{1/n} = 0.
\]
For any \( \varepsilon \) such that \( 0 < \varepsilon < 1 \), there exists an \( n_0 = n_0(\varepsilon) \) such that

\[
a_n < \varepsilon^n \text{ for } n > n_0.
\]

In view of (2.1), we get

\[
e_n(f) \leq \frac{\varepsilon^n}{(1 - \varepsilon^2)^{1/2}} \Rightarrow \limsup_{n \to \infty} e_n(f)^{1/n} \leq \varepsilon
\]

or,

\[
\limsup_{n \to \infty} e_n^{1/n}(f) = 0.
\]

Conversely, if \( \lim_{n \to \infty} e_n(f)^{1/n} = 0 \), then by (2.1)

\[
e_n(f) > |a_{n+1}| \text{ for every } n.
\]

So,

\[
\lim_{n \to \infty} |a_n|^{1/n} = 0.
\]

Hence, in view of (1.6), \( f \) is entire.

### 3 Main Results

**Theorem 1.** Let \( f \in H_2(E) \), then \( f \) can be extended to an entire function of \( (p, q) \) order \( \rho(p, q) \), if and only if

\[
\rho(p, q) = P(L^*(p, q)),
\]

where

\[
L^*(p, q) = \limsup_{n \to \infty} \frac{\log^{|p-1|} n}{\log^{|q|} e_n(f)^{-1/n}}.
\]

**Proof.** Let (3.1) holds. For any \( \varepsilon > 0 \), there exists \( n_0 = n_0(\varepsilon) \) such that

\[
\frac{\log^{|p-1|}}{\log^{|q|} e_n(f)^{-1/n}} < L^*(p, q) + \varepsilon \text{ for } n \geq n_0,
\]

or

\[
e_n(f) < \exp \left\{ -n \exp^{[q-1]} \frac{\log^{|p-1|} n}{L^*(p, q) + \varepsilon} \right\}.
\]

(3.2)

Now, first we consider the case for \( (p, q) = (2, 1) \), (3.2) gives

\[
e_n(f) < n^{-n/L^*(2,1)+\varepsilon} \text{ for } n \geq n_0,
\]
or

\[ \lim_{n \to \infty} e_n(f)^{1/n} = 0. \]

For \((p, q) = (2, 2)\), we have

\[ e_n(f) < \exp \left\{ -n^{(L^*(2,2)+1+\varepsilon)/L^*(2,2)} \right\} \text{ for } n \geq n_0, \]

or

\[ e_n(f)^{1/n} < \exp \left\{ -n^{1/L^*(2,2)} \right\}, \]

or

\[ \lim_{n \to \infty} e_n(f)^{1/n} = 0. \]

Finally, for \((p, q) \neq (2, 1)\) and \((2, 2)\), i.e., for \(3 \leq q \leq p < \infty\), let \(p = q = 3\), we have

\[ e_n(f) < \exp \left\{ -n \exp^{[2]} \left( \log \log n \right)^{1/(L^*(p,q)+\varepsilon)} \right\} \]

or

\[ e_n(f)^{1/n} < \exp \left\{ -\exp(\log n)^{1/(L^*(p,q)+\varepsilon)} \right\}, \]

or

\[ \lim_{n \to \infty} e_n(f)^{1/n} = 0. \]

Similarly, we can see easily that \(\lim_{n \to \infty} e_n(f)^{1/n} = 0\), for \(3 \leq q \leq p < \infty\).

Hence, \(\lim_{n \to \infty} e_n(f)^{1/n} = 0\), for all index-pair \((p, q)\). So, in view of Proposition 3, \(f\) can be extended to an entire function. Let its \((p, q)\)-order be \(\rho(p, q)\).

By (2.1), we have

\[ e_n(f) > |a_{n+1}| \text{ for every } n, \]

which gives

\[ \limsup_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q]} e_n(f)^{-1/n}} \geq \limsup_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q]} |a_n|^{-1/n}} \]
In view of (1.9), we get

$$\rho(p, q) \leq P(L^*(p, q)).$$

(3.3)

Conversely, suppose that $f$ is an entire function of $(p, q)$—order $\rho(p, q)$, $(b < \rho(p, q) < \infty)$. Then (1.9) gives that for any $\varepsilon > 0$, there exists $n_0(\varepsilon)$ such that

$$|a_n| < \exp \left\{-n \exp^{q-1} \left( \frac{\log^{[p-1]} n}{L(p, q) + \varepsilon} \right) \right\}.$$

Using (2.1), we get

$$[e_n(f)]^2 \leq \sum_{k=n+1}^{\infty} \exp \left\{-2k \exp^{q-1} \left( \frac{\log^{[p-1]} k}{L(p, q) + \varepsilon} \right) \right\} \text{ for } n \geq n_0,$$

or

$$[e_n(f)]^2 \leq \exp \left\{-2(n + 1) \exp^{q-1} \left( \frac{\log^{[p-1]} (n + 1)}{L(p, q) + \varepsilon} \right) \right\} \left[1 + O(1) \right] \text{ as } n \to \infty.$$

i.e.,

$$\limsup_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q]} e_n(f)^{-1/n}} \leq L(p, q) + \varepsilon.$$

Since $\varepsilon$ is arbitrary, so we get

$$L^*(p, q) \leq L(p, q)$$

or

$$P(L^*(p, q)) \leq P(L(p, q)) = \rho(p, q).$$

(3.4)

Combining (3.3 and (3.4), we get required result.

**Theorem 2.** Let $f \in H_2^\rho(E)$. Then $f$ can be extended to an entire function of $(p, q)$—order $\rho(p, q)(b < \rho(p, q) < \infty)$ and generalized $(p, q)$—type $T(p, q)(0 < T(p, q) < \infty)$ if and only if

$$\frac{T(p, q)}{BM} = \limsup_{n \to \infty} \frac{\log^{[p-2]} n}{\log^{[q-1]} e_n(f)^{-1/n}} \rho(p, q)^{-A},$$

(3.5)

where $B, M$ and $A$ are defined as earlier.
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Proof. Let

$$\limsup_{n \to \infty} \frac{\log^{[p-2]} n}{\left( \log^{[q-1]} e_n(f) - 1/n \right) \rho(p, q) - A} = \alpha < \infty.$$ 

For given $\varepsilon > 0$ and for all $n > n(\varepsilon)$, we have

$$\log^{[q-2]} n < (\alpha + \varepsilon) \left( \log^{[p-1]} e_n(f) - 1/n \right)^{\rho(p,q) - A},$$

which gives

$$\frac{\log^{[p-1]} n}{\log^{[q]} e_n(f) - 1/n} \leq \rho(p, q) - A + O(1).$$

Proceeding to limits, we obtain

$$L^*(p, q) = - \limsup_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q]} e_n(f) - 1/n} \leq \rho(p, q) - A.$$

Thus $P(L^*(p, q)) \leq \rho(p, q)$ and it follows by Theorem 1 that $f$ is of $(p, q)$-order at most $\rho(p, q)$. Similarly, if $\alpha > 0$, $f$ is of $(p, q)$-order at least $\rho(p, q)$.

Let $f$ be of $(p, q)$-type $T(p, q) < \infty$. Then for any $\varepsilon, r_0 = r_0(\varepsilon)$ such that

$$\log M(r) < \exp^{[p-2]} \left\{ (T(p, q) + \varepsilon) \left( \log^{[q-1]} \rho(p,q) \right) \right\}$$

Using (2.6) and (3.6), we get

$$\log e_n(f) \leq \exp^{[p-2]} \left\{ (T(p, q) + \varepsilon) \left( \log^{[q-1]} r \right)^{\rho(p,q)} \right\}$$

$$-n \log r + n \log d + n \varepsilon + \log K, \text{ for } n \geq n_0 \text{ and } r \geq r_0. \quad (3.7)$$

Now for $(p, q) = (2, 1)$, choosing and such that

$$r = \left[ \frac{n}{\rho(2,1)(T(2,1) + \varepsilon)} \right]^{1/\rho(2,1)},$$

Using (3.7), we get

$$\log e_n(f) \leq \frac{n}{\rho(2,1)} - \frac{n}{\rho(2,1)} \left[ \log n - \log(T(2,1) + \varepsilon)\rho(2,1) \right] + n \log d + n \varepsilon + \log K,$$

or

$$ne_n(f)^{\rho(2,1)/n} \leq c\rho(2,1)d^{\rho(2,1)}(T(2,1) + \varepsilon)e^{\rho(2,1)\varepsilon} + 0(1).$$
Hence,

\[
\limsup_{n \to \infty} n e_n(f)^{\rho(2,1)/n} \leq e^{\rho(2,1)} \rho(2,1) T(p, q).
\]  

(3.8)

For \((p, q) = (2, 2)\), choosing \(r\) as

\[
r = \exp\left(\frac{n}{\rho(2,2)(T(2,2) + \varepsilon)}\right)^{1/(\rho(2,2)-1)},
\]

(3.7) gives

\[
\log e_n(f) \leq (T(2,2) + \varepsilon) \left(\frac{n}{\rho(2,2)(T(2,2) + \varepsilon)}\right)^{\rho(2,2)/(\rho(2,2)-1)}
\]

\[
- n \left(\frac{n}{\rho(2,2)(T(2,2) + \varepsilon)}\right)^{1/(\rho(2,2)-1)} + n \log d + n \varepsilon + \log K,
\]

or

\[
\log e_n(f)^{-1/n} \geq \left(\frac{n}{\rho(2,2)(T(2,2) + \varepsilon)}\right)^{1/(\rho(2,2)-1)}
\]

\[
\left[\frac{-1}{\rho(2,2)} + 1 - \left(\frac{\rho(2,2)(T(2,2) + \varepsilon)}{n}\right)^{1/(\rho(2,2)-1)} \log d + O(1)\right],
\]

or

\[
T(2,2) \rho(2,2)^{(\rho(2,2)/(\rho(2,2)-1))} (\rho(2,2) - 1) \geq \limsup_{n \to \infty} \frac{n}{(\log e_n(f)^{-1/n})^{\rho(2,2)-1}}.
\]

Now for \(3 \leq q \leq p < \infty\). For \(r > r_0\) and for all \(n\), let

\[
r = \exp^{[q-1]} \left(\frac{1}{T(p,q) + \varepsilon} \log^{[p-2]} n / \rho(p,q)\right)^{1/\rho(p,q)}.
\]

Using (3.7), we have

\[
\log e_n(f) < \frac{n}{\rho(p,q)} - n \exp^{[q-2]} \left(\frac{1}{T(p,q) + \varepsilon} \log^{[p-2]} n / \rho(p,q)\right)^{1/\rho(p,q)}
\]

+ \(n \log d + n \varepsilon + \log K\),

which gives

\[
T(p, q) \geq \limsup_{n \to \infty} \frac{\log^{[p-2]} n}{(\log^{[q-1]} e_n(f)^{-1/n})^{\rho(p,q)}}.
\]

(3.9)

Combining (3.7),(3.8) and (3.9), we get

\[
\frac{T(p, q)}{BM} \geq \limsup_{n \to \infty} \frac{\log^{[p-2]} n}{(\log^{[q-1]} e_n(f)^{-1/n})^{\rho(p,q)-A}}.
\]

(3.10)
For reverse inequality, we observe that, since
\[ e_n(f) \geq |a_{n+1}| \text{ for all } n. \]
So,
\[ \limsup_{n \to \infty} \frac{\log^{[p-2]} n}{(\log^{[q-1]} |a_n|^{-1/n})^{\rho(p,q)-A}} \leq \limsup_{n \to \infty} \frac{\log^{[p-2]} n}{(\log^{[q-1]} e_n(f)^{-1/n})^{\rho(p,q)-A}}. \]

Using (3.10), we get
\[ \frac{T(p,q)}{BM} \leq \limsup_{n \to \infty} \frac{\log^{[p-2]} n}{(\log^{[q-1]} e_n(f)^{-1/n})^{\rho(p,q)-A}}. \]  
(3.11)

Inequalities (3.10) and (3.11) gives (3.4).

References


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