Near Approximations in Topological Spaces

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Abstract

Most real life situations need some sort of approximation to fit mathematical models. The beauty of using topology in approximation is achieved via obtaining approximation for qualitative subsets without coding or using assumption. The aim of this paper is to apply near concepts in the approximation spaces. The basic notions of near approximations are introduced and sufficiently illustrated. Near approximations considered as mathematical tools to modify the approximations of sets. Moreover, proved results, examples and counter examples are provided.

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1 Introduction

One of the most powerful notions in system analysis is the concept of
topological structures [7] and their generalizations. Many works have appeared
recently for example in structural analysis [8], in chemistry [17], and physics [5].
The purpose of the present work is to put a starting point for the applications of
abstract topological theory into rough set analysis. Rough set theory, introduced
by Pawlak in 1982 [14], is a mathematical tool that supports also the uncertainty
reasoning but qualitatively. In this paper, we shall integrate some ideas in terms of
concepts in topology. Topology is a branch of mathematics, whose concepts exist
not only in almost all branches of mathematics, but also in many real life
applications. We believe that topological structure will be an important base for
modification of knowledge extraction and processing.

2 Preliminaries

This section presents a review of some fundamental notions of topological
spaces and rough set theory.

A topological space [7] is a pair \((X, \tau)\) consisting of a set \(X\) and family \(\tau\)
of subsets of \(X\) satisfying the following conditions:

\(\tau\) is closed under arbitrary union.

\(\tau\) is closed under finite intersection.

Throughout this paper \((X, \tau)\) denotes a topological space, the elements of
\(X\) are called points of the space, the subsets of \(X\) belonging to \(\tau\) are called open
sets in the space, the complement of the subsets of \(X\) belonging to \(\tau\) are called
closed sets in the space, and the family of all \(\tau\)-closed subsets of \(X\) is denoted
by \(\tau^c\); the family \(\tau\) of open subsets of \(X\) is also called a topology for \(X\).

A family \(\tau\subseteq \mathcal{B}\) is called a base for \((X, \tau)\) iff every nonempty open subset
of \(X\) can be represented as a union of subfamilies of \(\mathcal{B}\). Clearly, a topological
space can have many bases. A family \(\mathcal{S}\subseteq \tau\) is called a subbase iff the family of
all finite intersections is a base for \((X, \tau)\).

The \(\tau\)-closure of a subset \(A\subseteq X\) is denoted by \(A^-\) and is given by
\(A^- = \cap \{F \subseteq X : A \subseteq F \text{ and } F \in \tau^c\}\). Evidently, \(A^-\) is the smallest closed subset
of \(X\) which contains \(A\). Note that \(A\) is closed iff \(A = A^-\).

The \(\tau\)-interior of a subset \(A\subseteq X\) is denoted by \(A^o\) and is given by
\(A^o = \cup \{G \subseteq X : G \subseteq A \text{ and } G \in \tau\}\). Evidently, \(A^o\) is the union of all open subsets
of \(X\) which contained in \(A\). Note that \(A\) is open iff \(A = A^o\). The boundary of a
subset \(A\subseteq X\) is denoted by \(A^b\) and is given by \(A^b = A^- - A^o\).
Motivation for rough set theory has come from the need to represent subsets of a universe in terms of equivalence classes of a partition of that universe. The partition characterizes a topological space, called approximation space $K = (X, R)$, where $X$ is a set called the universe and $R$ is an equivalence relation [10, 15]. The equivalence classes of $R$ are also known as the granules, elementary sets or blocks; we shall use $R_x \subseteq X$ to denote the equivalence class containing $x \in X$, and $X/R$ to denote the set of all elementary sets of $R$. In the approximation space, we consider two operators, the upper and lower approximations of subsets: Let $A \subseteq X$, then the upper approximation (resp. the lower approximation) of $A$ is given by

$$\overline{A} = \{ x \in X : R_x \cap A \neq \emptyset \} \quad (\text{resp. } \underline{A} = \{ x \in X : R_x \subseteq A \}).$$

In an approximation space $K = (X, R)$, if $A$ and $B$ are two subsets of $X$, then directly from the definitions of lower and upper approximations, we can get the following properties of the $R-$lower and $R-$upper approximations [15]:

1) $\underline{A} \subseteq A \subseteq \overline{A}$.
2) $\overline{\phi} = \overline{\phi} = \phi$ and $R X = \overline{R X} X = X$.
3) $\overline{R(A \cup B)} = \overline{R A} \cup \overline{R B}$.
4) $\overline{R(A \cap B)} = R A \cap R B$.
5) If $A \subseteq B$, then $\overline{R A} \subseteq \overline{R B}$.
6) If $A \subseteq B$, then $\overline{R A} \subseteq \overline{R B}$.
7) $\overline{R(A \cup B)} \supseteq \overline{R A} \cup \overline{R B}$.
8) $\overline{R(A \cap B)} \subseteq \overline{R A} \cap \overline{R B}$.
9) $\overline{A^c} = [\overline{R A}]^c$.
10) $\overline{R(A^c)} = [\overline{R A}]^c$.
11) $\overline{R R A} = \overline{R R A} = \overline{R A}$.
12) $\overline{R R A} = \overline{R R A} = \overline{R A}$.

3 Generalization of Pawlak Approximation Space

Pawlak noted that [16] the approximation space $K = (X, R)$ with equivalence relation $R$ defines a uniquely topological space $(X, \tau_k)$ where $\tau_k$ is the family of all clopen sets in $(X, \tau_k)$ and $X/R$ is a base of $\tau_k$. Moreover the lower ( resp. upper ) approximation of any subset $A \subseteq X$ is exactly the interior ( resp. closure ) of the subset $A$. In this section we shall generalize Pawlak’s concepts in the case of general relations. Hence the approximation space $K = (X, R)$ with general relation $R$ defines a uniquely topological space $(X, \tau_k)$.
where $\tau_\kappa$ is the topology associated to $K$ (i.e. $\tau_\kappa$ is the family of all open sets in $(X, \tau_\kappa)$ and $X/R$ is a subbase of $\tau_\kappa$). We shall give this hypothesis in the following definition.

**Definition 3.1.** Let $K = (X, R)$ be an approximation space with general relation $R$ and $\tau_\kappa$ is the topology associated to $K$. Then the triple $\kappa = (X, R, \tau_\kappa)$ is called a topologized approximation space.

The following definition introduces lower and upper approximations in a topologized approximation space $\kappa = (X, R, \tau_\kappa)$.

**Definition 3.2.** Let $\kappa = (X, R, \tau_\kappa)$ be a topologized approximation space. If $A \subseteq X$, then the lower approximation (resp. upper approximation) of $A$ is defined by

$$\tag{3.7} R A = A^- \quad \text{(resp. } \overline{R} A = A^+).$$

The following definition introduces new concepts of definability for a subset $A \subseteq X$ in a topologized approximation space $\kappa = (X, R, \tau_\kappa)$.

**Definition 3.3.** Let $\kappa = (X, R, \tau_\kappa)$ be a topologized approximation space. If $A \subseteq X$, then

i) $A$ is totally $R-$definable (exact) set if $R A = A = \overline{R} A$,

ii) $A$ is internally $R-$definable set if $A = \overline{R} A$, $A \neq \overline{R} A$,

iii) $A$ is externally $R-$definable set if $A \neq R A$, $A = \overline{R} A$,

iv) $A$ is $R-$indefinable (rough) set if $A \neq R A$, $A \neq \overline{R} A$.

**Proposition 3.1.** Let $\kappa = (X, R, \tau_\kappa)$ be a topologized approximation space. If $A$ and $B$ are two subsets of $X$, then

1) $R A \subseteq A \subseteq \overline{R} A$.

2) $\overline{\phi} = \overline{R} \phi = \phi$ and $RX = \overline{R} X = X$.

3) $\overline{R} (A \cup B) = \overline{R} A \cup \overline{R} B$.

4) $\overline{R} (A \cap B) = \overline{R} A \cap \overline{R} B$.

5) If $A \subseteq B$, then $\overline{R} A \subseteq \overline{R} B$.

6) If $A \subseteq B$, then $\overline{R} A \subseteq \overline{R} B$.

7) $\overline{R} (A \cup B) \supseteq \overline{R} A \cup \overline{R} B$.

8) $\overline{R} (A \cap B) \subseteq \overline{R} A \cap \overline{R} B$.

9) $\overline{R} A^c = [\overline{R} A]^c$.

10) $\overline{R} (A^c) = [\overline{R} A]^c$. 
Proof. By using properties of interior and closure, the proof is obvious. □

The following example illustrates that properties (11 and 12) which introduced in Section 2 can not be applied for this new generalization.

Example 3.1. Let $\kappa = (X, R, \tau_k)$ be a topologized approximation space such that $X = \{a, b, c, d\}$ and $X / R = \{\{c\}, \{b, d\}\}$. Then

$$S = \{\{c\}, \{b, d\}\}, \quad B = \{X, \phi, \{c\}, \{b, d\}\},$$

$\tau_k = \{X, \phi, \{c\}, \{b, d\}\},$ and $\tau_k^* = \{\phi, X, \{a, b, d\}, \{a, c\}, \{a\}\}.$

Let $A = \{a, b, c\}$ and $B = \{a, b, d\}$. Then

$$\overline{R \cap A} = \{d\}, \quad \overline{R \cap A} = \{a, c\},$$

thus $\overline{R \cap A} = \{d\}$ and $\overline{R \cap B} = \{a, b, d\},$. Also

$$\overline{R \cap B} = \overline{R \cap B} = \{a, b, d\}, \quad \overline{R \cap B} = \{b, d\},$$

thus $\overline{R \cap B} = \{b, d\}$.

Lemma 3.1 [6]. Let $(X, \tau)$ be a topological space. Then

$$(A^*)^c = (A^c)^*$$

for all $A \subseteq X$.

Lemma 3.2 [11]. Let $A$ and $B$ be two subsets of $X$ in a topological space $(X, \tau)$. If $A$ is open, then $A \cap B^c \subseteq (A \cap B)^c$.

Proposition 3.2. $\kappa = (X, R, \tau_k)$ be a topologized approximation space. If $A$ and $B$ are two subsets of $X$, then

1) $R(A^{-}B) \subseteq R(A \cap B)$.

2) $\overline{R(A^{-}B)} \supseteq \overline{R(A \cap B)}$.

Proof.

1) We need to show that $(A^{-}B)^c \subseteq A^c \cap B^c$. Now,

$A^{-}B = A \cap B^c$, then

$$(A^{-}B)^c = (A \cap B^c)^c = A^c \cap (B^c)^c.$$ Thus by Lemma 3.1 we have

$$(A^{-}B)^c = A^c \cap (B^c)^c = A^c - B^c \subseteq A^c \cap B^c.$$ Therefore

$$R(A^{-}B) = (A^{-}B)^c \subseteq A^c - B^c = R(A \cap B).$$

2) We need to show that $(A^{-}B)^c \supseteq A^{-} - B^{-}$. Now,

$A^{-}B^{-} = A^{-} \cap (B^{-})^c$, then by Lemma 3.1 we have

$A^{-} - B^{-} = A^{-} \cap (B^{-})^c$. Hence by Lemma 3.2 we have
\[ A^- B^- = A^- \cap (B^c)^c \subseteq (A \cap (B^c)) = (A \cap (B^-)^c) = (A - B^-)^c, \text{ thus} \]

\[ A^- - B^- \subseteq (A - B)^c. \text{ Therefore} \]

\[ \bar{R}(A - B) = (A - B)^c \supseteq A^- - B^- = \bar{R}A - \bar{R}B. \] □

4 Near Lower and Near upper Approximations

In this section, we study approximation spaces from topological view. We obtain some rules to find lower and upper approximations in several ways in approximation spaces with general relations. We shall recall some definitions about some classes of near open sets which are essential for our present study. Some forms of near open sets are introduced in the following definition.

**Definition 4.1.** Let \((X, \tau)\) be a topological space, then the subset \(A \subseteq X\) is called:

i) Regular open [18] (briefly \(r\)-open) if \(A = A^-\).

ii) Semi-open [9] (briefly \(s\)-open) if \(A \subseteq A^-\).

iii) Pre-open [12] (briefly \(p\)-open) if \(A \subseteq A^-\).

iv) \(\gamma\)-open [4] (= b-open [3]) if \(A \subseteq A^- \cup A^\sim\).

v) \(\alpha\)-open [13] if \(A \subseteq A^- \cup A^\sim\).

vi) \(\beta\)-open [1] (= semi-pre-open [2]) if \(A \subseteq A^- \cup A^\sim\).

**Notice 4.1.**

i) The complement of an \(r\)-open (resp. \(s\)-open, \(p\)-open, \(\gamma\)-open, \(\alpha\)-open and \(\beta\)-open) set is called \(r\)-closed (resp. \(s\)-closed, \(p\)-closed, \(\gamma\)-closed, \(\alpha\)-closed and \(\beta\)-closed) set.

ii) The family of all \(r\)-open (resp. \(s\)-open, \(p\)-open, \(\gamma\)-open, \(\alpha\)-open and \(\beta\)-open) sets of \((X, \tau)\) is denoted by \(RO(X)\) (resp. \(SO(X)\), \(PO(X)\), \(\gamma O(X)\), \(\alpha O(X)\) and \(\beta O(X)\)).

iii) The family of all \(r\)-closed (resp. \(s\)-closed, \(p\)-closed, \(\gamma\)-closed, \(\alpha\)-closed and \(\beta\)-closed) sets of \((X, \tau)\) is denoted by \(RC(X)\) (resp. \(SC(X)\), \(PC(X)\), \(\gamma C(X)\), \(\alpha C(X)\) and \(\beta C(X)\)).

The aim of the following example is to illustrate the existence of spaces in which the above classes of near open sets and near closed sets are not coincided and are not the discrete structure.
Example 4.1. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a, b\}, \{d\}, \{a, b, d\}\}$. Then the classes of near open sets are

- $RO(X) = \{X, \phi, \{d\}, \{a, b\}\}$
- $SO(X) = \{X, \phi, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}\}$
- $PO(X) = \{X, \phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}$
- $\gamma O(X) = \{X, \phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$
- $\alpha O(X) = \{X, \phi, \{d\}, \{a, b\}, \{a, b, d\}\}$
- $\beta O(X) = \{X, \phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$

From known results $[1, 4]$ we have the following two remarks. We shall use the symbol $\supseteq$ instead of $\subseteq$ in the implications between sets.

Remark 4.1. In a topological space $(X, \tau)$, the implications between $\tau$ and the families of near open sets are given in the following diagram.

$RO(X) \supseteq \tau \supseteq \alpha O(X) \supseteq \gamma O(X) \supseteq \beta O(X) \supseteq PO(X)$

Remark 4.2. In a topological space $(X, \tau)$, the implications between $\tau^*$ and the families of near closed sets are given in the following diagram.

$RC(X) \supseteq \tau^* \supseteq \alpha C(X) \supseteq \gamma C(X) \supseteq \beta C(X) \supseteq PC(X)$

The following definition is given to introduce the near interior of a subset $A$ of $X$ in a topological space $(X, \tau)$.

Definition 4.2. Let $(X, \tau)$ be a topological space and $A \subseteq X$, then the near interior (briefly $j-$interior) of $A$ is denoted by $A^{j\circ}$ for all $j \in \{r, s, p, \gamma, \alpha, \beta\}$ and is defined by

$$A^{j\circ} = \bigcup \{G \subseteq X : G \subseteq A, G \text{ is a } j \text{-open set} \}.$$
The aim of the following definition is to introduce the near closure of a subset $A$ of $X$ in a topological space $(X, \tau)$.

**Definition 4.3.** Let $(X, \tau)$ be a topological space and $A \subseteq X$, then the near closure (briefly $j$-closure) of $A$ is denoted by $A^j_{-}$ for all $j \in \{r, s, p, \gamma, \alpha, \beta\}$ and is defined by

$$A^j_{-} = \bigcap \{ H \subseteq X : A \subseteq H, H \text{ is a } j \text{-closed set} \}.$$

The following two definitions introduce near lower and near upper approximations in a topologized approximation space $\kappa = (X, R, \tau_K)$.

**Definition 4.4.** Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. If $A \subseteq X$, then the near lower approximation (briefly $j$-lower approximation) of $A$ is denoted by $A^j_{\ell}$ and is defined by

$$A^j_{\ell} = A^j_{-}, \text{ where } j \in \{r, p, s, \gamma, \alpha, \beta\}.$$

**Definition 4.5.** Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. If $A \subseteq X$, then the near upper approximation (briefly $j$-upper approximation) of $A$ is denoted by $A^j_{u}$ and is defined by

$$A^j_{u} = A^j_{-}, \text{ where } j \in \{r, p, s, \gamma, \alpha, \beta\}.$$

**Proposition 4.1.** Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. If $A \subseteq X$, then

$$A^j_{\ell} \subseteq A^j_{-} \subseteq A \subseteq \overline{A}^j_{-} \subseteq \overline{A}^j_{\ell} \subseteq \overline{A}, \quad \forall j \in \{s, p, \gamma, \alpha, \beta\} \text{ and } j \neq r.$$

**Proof.** We shall prove the proposition in the case of $j = p$ and the other cases can be proved similarly. Now,

$$R \bar{A} = A^\ell = \bigcup \{ G \in \tau : G \subseteq A \} \subseteq \bigcup \{ G \in PO(X) : G \subseteq A \} = A^p_{\ell} = R_{p} A \subseteq A. \quad (4.1)$$

$$\overline{R} A = A^{\ell} = \bigcap \{ F \in \tau^* : A \subseteq F \} \supseteq \bigcap \{ F \in PC(X) : A \subseteq F \} = A^p_{\ell} = \overline{R}_{p} A \supseteq A. \quad (4.2)$$

From (4.1) and (4.2) we get $R \bar{A} \subseteq R_{p} A \subseteq A \subseteq \overline{R}_{p} A \subseteq \overline{R} A$. \qed

In general the above proposition is not true in the case of $j = r$ as the following example illustrates.
Example 4.2. Let \( \kappa = (X, R, \tau_k) \) be a topologized approximation space such that \( X = \{a, b, c\} \) and \( X / R = \{\{a, b\}, \{a, c\}\} \); Then
\[
S = \{\{a, b\}, \{a, c\}\}, \quad B = \{\{b\}, \{a, b\}, \{a, c\}\} \quad \text{and} \quad \tau_k = \{X, \phi, \{b\}, \{a, b\}, \{a, c\}\};
\]
Hence \( RO(X) = \{X, \phi\} \) and \( RC(X) = \{\phi, X\} \). If \( A = \{a, c\} \), then
\[
\overline{R} A = A^* = \phi, \quad \overline{R} A = A^* = \{a, c\}, \quad \overline{R} A = A^* = \phi \quad \text{and} \quad \overline{R} A = A^* = X.
\]
Therefore \( \overline{R} A = \overline{R} A \) and \( \overline{R} A \subseteq \overline{R} A \).

Proposition 4.2. Let \( \kappa = (X, R, \tau_k) \) be a topologized approximation space. If \( A \) and \( B \) are two subsets of \( X \), then for all \( j \in \{r, p, \gamma, \alpha, \beta\} \),
\[
\begin{align*}
1) & \quad \overline{R} \phi = \overline{R} \phi = \phi \quad \text{and} \quad \overline{R} X = \overline{R} X = X. \\
2) & \quad \text{If} \quad A \subseteq B, \quad \text{then} \quad \overline{R} A \subseteq \overline{R} B. \\
3) & \quad \text{If} \quad A \subseteq B, \quad \text{then} \quad \overline{R} A \subseteq \overline{R} B. \\
4) & \quad \overline{R} (A \cup B) \supseteq \overline{R} A \cup \overline{R} B. \\
5) & \quad \overline{R} (A \cup B) \supseteq \overline{R} A \cup \overline{R} B. \\
6) & \quad \overline{R} (A \cap B) \subseteq \overline{R} A \cap \overline{R} B. \\
7) & \quad \overline{R} (X \cap B) \subseteq \overline{R} A \cap \overline{R} B. \\
8) & \quad \overline{R} A^c = \left[\overline{R} A \right]^c. \\
9) & \quad \overline{R} (A^c) = \left[\overline{R} A \right]^c.
\end{align*}
\]

Proof. By using properties of \( j \)-interior and \( j \)-closure for all \( j \in \{r, p, \gamma, \alpha, \beta\} \), the proof is obvious. □

In general, properties (3 and 4) which introduced in Section 2 can not be applied for \( j \)-lower and \( j \)-upper approximations, where \( j \in \{p, \gamma, \alpha, \beta\} \). The following example illustrates this fact in the case of \( j = \beta \).

Example 4.3. Let \( \kappa = (X, R, \tau_k) \) be a topologized approximation space such that \( X = \{a, b, c, d\} \) and \( X / R = \{\{d\}, \{a, b\}\} \). Then
\[
S = \{\{d\}, \{a, b\}\}, \quad B = \{X, \phi, \{a\}, \{a, b\}\}, \quad \tau_k = \{X, \phi, \{a\}, \{a, b\}, \{a, b, d\}\}, \quad \text{and hence} \\
\beta O(X) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{b, c\}, \{d, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\},
\]
\[
\beta C(X) = \{\phi, X, \{b, c, d\}, \{a, c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, d\}, \{b, c\}, \{a, d\}, \{a, c\}, \{a, b\}, \{d\}, \{c\}, \{b\}, \{a\}\}.
\]
If \( A = \{a, c\} \) and \( B = \{b, c\} \). Then
\[
\overline{R} \beta A \cap \overline{R} \beta B = \{a, c\} \cap \{b, c\} = \{c\}, \quad \text{but} \quad \overline{R} \beta (A \cap B) = \phi.
\]
Thus \( R_\beta(A \cap B) \neq R_\beta A \cap R_\beta B \). Also, If \( A = \{a, b\} \) and \( B = \{a, d\} \). Then
\[
\overline{R}_\beta A \cup \overline{R}_\beta B = \{a, b\} \cup \{a, d\} = \{a, b, d\},
\]
but \( \overline{R}_\beta (A \cup B) = X \).
Thus \( \overline{R}_\beta (A \cup B) \neq \overline{R}_\beta A \cup \overline{R}_\beta B \).

In general, properties (11 and 12) which introduced in Section 2 can not be applied for \( j \)-lower and \( j \)-upper approximations, where \( j \in \{p, s, \gamma, \beta\} \). The following example illustrates this fact in the case of \( j = \beta \).

**Example 4.4.** Let \( \kappa = (X, R, \tau_\kappa) \) be the topologized approximation space which is given in Example 4.3. If \( A = \{a, b, d\} \) and \( B = \{c\} \). Then
\[
\overline{R}_\beta \overline{R}_\beta A = \overline{R}_\beta A = \{a, b, d\}, \quad \overline{R}_\beta \overline{R}_\beta A = X,
\]
thus \( \overline{R}_\beta \overline{R}_\beta A = \overline{R}_\beta A \neq \overline{R}_\beta \overline{R}_\beta A \). Also
\[
\overline{R}_\beta \overline{R}_\beta B = \overline{R}_\beta B = \{c\}, \quad \overline{R}_\beta \overline{R}_\beta B = \phi,
\]
hence \( \overline{R}_\beta \overline{R}_\beta B = \overline{R}_\beta B \neq \overline{R}_\beta \overline{R}_\beta B \).

**Lemma 4.1.** Let \( A \) be any subset of \( X \) in a topological space \((X, \tau)\). Then
\[
\left(A^c\right)^j = \left(A^j\right)^c
\]
for all \( j \in \{r, p, s, \gamma, \alpha, \beta\} \).

**Proof.**

Let \( A \subseteq X \), then for all \( j \in \{r, p, s, \gamma, \alpha, \beta\} \) we get
\[
\left(A^j\right)^c = X - A^j = X - \cap \{F \subseteq X : F \text{ is a } j \text{-closed set and } A \subseteq F\}
\]
\[
= \cup \{X - F \subseteq X : X - F \text{ is a } j \text{-open set and } X - F \subseteq X - A\}
\]
\[
= (X - A)^j.
\]
Thus \( \left(A^c\right)^j = \left(A^j\right)^c \). □

**Proposition 4.3.** Let \( \kappa = (X, R, \tau_\kappa) \) be a topologized approximation space. If \( A \) and \( B \) are two subsets of \( X \), then
\[
R_j (A - B) \subseteq R_j A - R_j B, \text{ for all } j \in \{r, p, s, \gamma, \alpha, \beta\}.
\]

**Proof.**

We need to show that \( (A - B)^j \subseteq A^j - B^j \). Now,
\[
A - B = A \cap B^c, \text{ then}
\]
\[
\left(A - B\right)^j = \left(A \cap B^c\right)^j \subseteq A^j \cap \left(B^c\right)^j.
\]
Thus by Lemma 4.1 we have
\[
\left(A - B\right)^j \subseteq A^j \cap \left(B^j\right)^c = A^j - B^j \subseteq A^j - B^j.
\]
Therefore
\[
R_j (A - B) = \left(A - B\right)^j \subseteq A^j - B^j = R_j A - R_j B. \quad \square
\]
In general, part (2) in Proposition 3.2 cannot be applied for $j – \text{upper approximation}$ for all $j \in \{r, p, s, \gamma, \beta\}$. Example 4.5 (resp. Example 4.6) illustrates that part (2) in Proposition 3.2 cannot be applied in the case of $j = \beta$ (resp. $j = r$).

**Example 4.5.** Let $\kappa = (X, R, \tau_\kappa)$ be the topologized approximation space which is given in Example 4.3. If $A = \{a, b, d\}$ and $B = \{a, b\}$. Then

$$\overline{R}_\beta (A - B) = \overline{R}_\beta \{d\} = \{d\},$$

but $\overline{R}_\beta A - \overline{R}_\beta B = X - \{a, b\} = \{c, d\}$. Hence $\overline{R}_\beta (A - B) \subseteq \overline{R}_\beta A - \overline{R}_\beta B$.

**Example 4.6.** Let $\kappa = (X, R, \tau_\kappa)$ be the topologized approximation space which is given in Example 4.3. If $A = \{a\}$ and $B = \{c\}$. Then

$$\overline{R}_\gamma (A - B) = \overline{R}_\gamma \phi = \phi,$$

but $\overline{R}_\gamma A - \overline{R}_\gamma B = \{a, b, c\} - \{c\} = \{a, b\}$. Hence $\overline{R}_\gamma (A - B) \subseteq \overline{R}_\gamma A - \overline{R}_\gamma B$.

### 5 Conclusions

In this paper, we used topological concepts to introduce a generalization of Pawlak approximation space. Concepts of definability for subsets in topologized approximation spaces are introduced. Several types of approximations which called near approximations are mathematical tools to modify the approximations. The suggested methods of near approximations open way for constructing new types of lower and upper approximations.

### References


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