An $L^p$ Version of Hardy Uncertainty Principle for the $q$-Dunkl Transform on the Real Line

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Abstract

This paper deals with an $L^p$ version of the Hardy’s theorem for the $q$-Dunkl transform introduced and studied in [2].

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1 Introduction

In harmonic analysis, the uncertainty principle states that a function and its Fourier transform can not simultaneously decrease very quickly. In the literature, this fact is in general given by the way of some inequalities involving a function $f$ and its Fourier transform $\hat{f}$. One of the famous formulations of the uncertainty principle is stated by the so-called Hardy’s theorem (see [4, 8, 9]), which asserts that if

$$\sup_{x \in \mathbb{R}} |e^{ax^2} f(x)| < \infty, \quad \sup_{\lambda \in \mathbb{R}} |e^{b\lambda^2} \hat{f}(\lambda)| < \infty \quad \text{and} \quad ab > \frac{1}{4},$$
then \( f \equiv 0 \).

In [3], M. G. Cowling and J. F. Price obtained an \( L^p \) version of the Hardy’s theorem by showing that for \( p, n \in [1, +\infty] \) with at least one of them is finite, if \( \| e^{ax^2} f(x) \|_p < \infty \), \( \| e^{a\lambda^2} \hat{f}(\lambda) \|_n < \infty \) and \( ab > \frac{1}{4} \), then \( f = 0 \).

Generalizations of these results in both classical and quantum analysis have been revealed (see [1], [5], [7], [11], [14], [15]) and many versions of Hardy uncertainty principles were obtained for several generalized Fourier transforms.

In [2], Bettaibi et al. introduced and studied a \( q \)-analogue of the classical Bessel-Dunkl transform (\( q \)-Dunkl transform). In particular they provided, for this transform, a Plancherel formula and proved an inversion theorem. In this paper, we state an \( L^p \) version of the Hardy’s theorem for the \( q \)-Dunkl transform \( F_{D}^{\alpha,q}(f) \).

This paper is organized as follows: in Section 2, we present some preliminaries notions and notations useful in the sequel. In Section 3, we recall some results and properties from the theory of the \( q \)-Dunkl operator and the \( q \)-Dunkl transform (see [2]). Section 4 is devoted to give a \( q \)-analogue of the \( L^p \) version of Hardy’s inequality for \( F_{D}^{\alpha,q} \).

### 2 Notations and preliminaries

Throughout this paper, we assume \( q \in ]0,1[ \), we refer to the general reference [6] for the definitions, notations and properties of the \( q \)-shifted factorials and the \( q \)-hypergeometric functions.

We write \( \mathbb{R}_q = \{ \pm q^n : n \in \mathbb{Z} \} \), \( \mathbb{R}_{q,+} = \{ q^n : n \in \mathbb{Z} \} \),

\[
[x]_q = \frac{1 - q^x}{1 - q}, \quad x \in \mathbb{C} \quad \text{and} \quad [n]_q! = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}.
\]

The \( q^2 \)-analogue differential operator is (see [12, 13])

\[
\partial_{q}(f)(z) = \begin{cases} 
\frac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z} & \text{if } z \neq 0 \\
\lim_{x \to 0} \partial_{q}(f)(x) & \text{in } \mathbb{R}_q \\
0 & \text{if } z = 0.
\end{cases}
\]

Remark that if \( f \) is differentiable at \( z \), then \( \lim_{q \to 1} \partial_{q}(f)(z) = f'(z) \).

A repeated application of the \( q^2 \)-analogue differential operator is denoted by:

\[
\partial_{q}^{0} f = f, \quad \partial_{q}^{n+1} f = \partial_{q}(\partial_{q}^{n} f).
\]

The following lemma lists some useful computational properties of \( \partial_{q} \):
Lemma 1

1) For all function \( f \) on \( \mathbb{R}_q \), \( \partial_q f(z) = \frac{f_e(q^{-1}z) - f_e(z)}{(1-q)z} + \frac{f_o(z) - f_o(qz)}{(1-q)z} \).

2) For two functions \( f \) and \( g \) on \( \mathbb{R}_q \), we have
   \begin{align*}
   \partial_q(fg)(z) &= q\partial_q(f)(qz)g(z) + f(qz)\partial_q(g)(z) = \partial_q(g)(z)f(z) + qg(qz)\partial_q(f)(qz); \\
   &\quad \text{if } f \text{ is even and } g \text{ is odd,} \\
   \partial_q(fg)(z) &= \partial_q(f)(z)g(q^{-1}z) + f(z)\partial_q(g)(z). \\
   &\quad \text{if } f \text{ and } g \text{ are even,}
   \end{align*}

Here, for a function \( f \) defined on \( \mathbb{R}_q \), \( f_e \) and \( f_o \) are its even and odd parts respectively.

The operator \( \partial_q \) induces a \( q \)-analogue of the classical exponential function (see [12, 13])

\[
e(z; q^2) = \sum_{n=0}^{\infty} a_n \frac{z^n}{[n]_q!}, \quad \text{with } a_{2n} = a_{2n+1} = q^{n(n+1)}. \tag{3}\]

The \( q \)-Jackson integrals are defined by (see [10])

\[
\int_0^a f(x)d_qx = (1-q)a \sum_{n=0}^{\infty} q^n f(aq^n), \quad \int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx,
\]

\[
\int_0^\infty f(x)d_qx = (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n),
\]

and

\[
\int_{-\infty}^\infty f(x)d_qx = (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n) + (1-q) \sum_{n=-\infty}^{\infty} q^n f(-q^n),
\]

provided the sums converge absolutely.

The \( q \)-Gamma function is given by (see [10])

\[
\Gamma_q(x) = \frac{(q; q)_{\infty}}{(q^x; q)_{\infty}}(1-q)^{1-x}, \quad x \neq 0, -1, -2, ...
\]

In what follows, we will need the following sets and spaces:

- \( L_\infty^q(\mathbb{R}_q) = \left\{ f : \|f\|_{\infty,q} = \sup_{x \in \mathbb{R}_q} |f(x)| < \infty \right\} \).
- \( L_p^{\alpha,q}(\mathbb{R}_q) = \left\{ f : \|f\|_{p,\alpha,q} = \left( \int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1}d_qx \right)^{\frac{1}{p}} < \infty \right\} \).
3 The $q$-Dunkl operator and the $q$-Dunkl transform

In this section we collect some basic properties of the $q$-Dunkl operator and the $q$-Dunkl transform introduced in [2], useful in the sequel.

For $\alpha \geq -\frac{1}{2}$, the $q$-Dunkl operator is defined by

$$\Lambda_{\alpha,q}(f)(x) = \partial_q [H_{\alpha,q}(f)](x) + [2\alpha + 1]_q \frac{f(x) - f(-x)}{2x},$$

where

$$H_{\alpha,q} : f = f_e + f_o \longmapsto f_e + q^{2\alpha+1}f_o.$$

It was shown in [2] that for each $\lambda \in \mathbb{C}$, the function

$$\psi^\alpha_{\lambda,q} : x \mapsto j_{\alpha}(\lambda x; q^2) + \frac{i\lambda x}{[2\alpha + 2]_q} j_{\alpha+1}(\lambda x; q^2)$$

is the unique solution of the $q$-differential-difference equation:

$$\left\{ \begin{array}{lcl} \Lambda_{\alpha,q}(f) = i\lambda f \\ f(0) = 1, \end{array} \right.$$ 

where $j_{\alpha}(\cdot; q^2)$ is the normalized third Jackson's $q$-Bessel function given by

$$j_{\alpha}(x; q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^2; q^2)_n (q^{2(\alpha+1)}; q^2)_n} ((1 - q)x)^n.$$ 

The function $\psi^\alpha_{\lambda,q}(x)$, has an unique extension to $\mathbb{C} \times \mathbb{C}$ and we have the following properties.

• $\psi^\alpha_{a\lambda,q}(x) = \psi^\alpha_{\lambda,q}(ax) = \psi^\alpha_{ax,q}(\lambda)$, $\forall a, x, \lambda \in \mathbb{C}$.

• For $\alpha = -\frac{1}{2}$, $\psi^\alpha_{\lambda,q} = e(i\lambda x; q^2)$ and for $\alpha > -\frac{1}{2}$, $\psi^\alpha_{\lambda,q}$ has the following $q$-integral representation of Mehler type

$$\psi^\alpha_{\lambda,q}(x) = \frac{(1+q)\Gamma_{q^2}(\alpha+1)}{2\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\alpha+\frac{1}{2})} \int_{-1}^{1} \frac{(e^{2q^2}q^2)_\infty}{(t^2q^{4\alpha+1}q^2)_\infty} (1 + t)e(i\lambda xt; q^2) d_q t.$$ 

• For all $x, \lambda \in \mathbb{R}_q$,

$$| \psi^\alpha_{\lambda,q}(x) | \leq \frac{4}{(q; q)_\infty}.$$ 


The $q$-Dunkl transform $F^\alpha,q_D$ is defined on $L^1_{\alpha,q}(\mathbb{R}_q)$ (see [2]) by

$$F^\alpha,q_D(f)(\lambda) = \frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} f(x) \psi_{-\lambda}^{\alpha,q}(x)|x|^{2\alpha+1}dq_x,$$

where

$$c_{\alpha,q} = \frac{(1+q)^{-\alpha}}{\Gamma_q(\alpha+1)}.$$

It satisfies the following properties:

- For $\alpha = -\frac{1}{2}$, $F^\alpha,q_D$ is the $q^2$-analogue Fourier transform $\widehat{f}(\cdot;q^2)$ given by (see [13, 12])

$$\widehat{f}(\lambda; q^2) = \frac{(1 + q)^{1/2}}{2\Gamma_q(\frac{1}{2})} \int_{-\infty}^{+\infty} f(x)e(-i\lambda x; q^2)dx.$$

- On the even functions space, $F^\alpha,q_D$ coincides with the $q$-Bessel transform given by (see [2])

$$F^\alpha,q_{\alpha,q}(f)(\lambda) = c_{\alpha,q} \int_{0}^{+\infty} f(x) j_{\alpha}(\lambda x; q^2)x^{2\alpha+1}dq_x.$$

- For all $f \in L^1_{\alpha,q}(\mathbb{R}_q)$, we have:

$$\|F^\alpha,q_D(f)\|_{\infty,q} \leq \frac{2c_{\alpha,q}}{(q; q)_{\infty}} \|f\|_{1,\alpha,q}.$$  \hspace{1cm} (8)

- The $q$-Dunkl transform $F^\alpha,q_D$ is an isomorphism from $L^2_{\alpha,q}(\mathbb{R}_q)$ onto itself and satisfies the following Plancherel formula and the inversion theorem:

$$\forall f \in L^2_{\alpha,q}(\mathbb{R}_q), \quad \|F^\alpha,q_D(f)\|_{2,\alpha,q} = \|f\|_{2,\alpha,q} \quad (9)$$

and

$$(F^\alpha,q_D)^{-1}(f)(\lambda) = F^\alpha,q_D(f)(-\lambda), \quad \lambda \in \mathbb{C}.$$

### 4 $q$-$L^p$-version of Hardy’s uncertainty principle

In this section, we shall state an $L^p$ version of Hardy’s theorem for the $q$-Dunkl transform $F^\alpha,q_D$. We begin by the two following lemmas. The first is a deduction from the Mehler type relation (6 ) and [[1], Proposition 1], and the second can be proved using the same steps as in [3].
Lemma 2 For all $z \in \mathbb{C}$ and all $t \in \mathbb{R}$, we have
$$|\psi_z^{\alpha,q}(t)| \leq 2e^{k|z|},$$
with $k = 1 + \sqrt{q}$.

Lemma 3 Let $n \in [1, +\infty]$ and $h$ be an entire function on $\mathbb{C}$ such that:

- $\forall z \in \mathbb{C}, \ |h(z)| \leq Me^{a(R(z))^2}$, for some constants $a, M > 0$;
- $\|h\|_{n,a} = \left(\int_{-\infty}^{+\infty} h(t)|t|^{2\alpha+1}dt\right)^{1/n} < \infty$.

Then $h = 0$ on $\mathbb{C}$.

Using Lemma 2, we can state the following result.

Theorem 1 Let $p \in [1, +\infty]$, $a > 0$ and $f$ be a function defined on $\mathbb{R}$ such that
$$\|e^{at^2}f\|_{p,a,q} < \infty.$$ Then $F_D^{\alpha,q}(f)$ is entire on $\mathbb{C}$ and for all $b \in ]0, a[$, we have:
$$\forall z \in \mathbb{C}, \ |F_D^{\alpha,q}(f)(z)| \leq C_2 e^{\frac{k^2|z|^2}{4q}}$$
for some positive constant $C_2$.

Proof. Since $|\psi_z^{\alpha,q}(t)| \leq 2e^{(1+\sqrt{q})|t|}$, $t \in \mathbb{R}$ and $z \in \mathbb{C}$, then from the hypothesis, the Hölder’s inequality and the analyticity theorem, one deduces that $F_D^{\alpha,q}(f)$ is entire on $\mathbb{C}$ and for all $z \in \mathbb{C}$, we have
$$|F_D^{\alpha,q}(f)(z)| \leq \frac{c_{\alpha,q}}{2} \int_{-\infty}^{\infty} |\psi_z^{\alpha,q}(t)||f(t)||t|^{2\alpha+1}d_qt \leq c_{\alpha,q} \int_{-\infty}^{\infty} e^{k|z||t|-at^2} e^{a|t^2|} |f(t)||t|^{2\alpha+1}d_qt$$
$$\leq c_{\alpha,q} \left(\int_{-\infty}^{\infty} e^{n(k|z||t|-at^2)}|t|^{2\alpha+1}d_qt\right)^{1/n} \|e^{at^2}f\|_{p,a,q},$$
where $n$ is the real satisfying $\frac{1}{p} + \frac{1}{n} = 1$.

Now, for $b \in ]0, a[$, we have
$$\left(\int_{-\infty}^{\infty} e^{n(k|z||t|-at^2)}|t|^{2\alpha+1}d_qt\right)^{1/n} = \left(\int_{-\infty}^{\infty} e^{n(k|z||t|-bt^2)}e^{-n(a-b)t^2}|t|^{2\alpha+1}d_qt\right)^{1/n}$$
$$\leq \left(\sup_{t \in [0,\infty)} e^{n(k|z||t|-bt^2)}\right)^{1/n} \left(\int_{-\infty}^{\infty} e^{-n(a-b)t^2}|t|^{2\alpha+1}d_qt\right)^{1/n}$$
$$= C_2 e^{\frac{k^2|z|^2}{4q}},$$
with $C_2 = \left(\int_{-\infty}^{\infty} e^{-n(a-b)t^2}|t|^{2\alpha+1}d_qt\right)^{1/n}$. 

$\blacksquare$
Theorem 2 Let \( p \in [1, +\infty] \), \( n \in [1, +\infty] \), \( a, b > 0 \) and \( f \) be a function defined on \( \mathbb{R}_q \) such that
\[
\| e^{at^2} f(t) \|_{p,q} < \infty \tag{12}
\]
and
\[
\| e^{b\lambda^2} F_D^{\alpha,q}(f)(\lambda) \|_{n} < \infty. \tag{13}
\]
If \( ab > \frac{k^2}{4} \) then \( f = 0 \) on \( \mathbb{R}_q \).

Proof. Let \( a, b > 0 \) satisfying the conditions of the theorem such that \( ab > \frac{k^2}{4} \) and \( a_1 \in \left[ \frac{k^2}{16}, a \right] \).

From Theorem 1 and the relation (12), we have \( F_D^{\alpha,q}(f) \) is entire on \( \mathbb{C} \) and
\[
\forall z \in \mathbb{C}, \quad |F_D^{\alpha,q}(f)(z)| \leq C_2 e^{\frac{k^2}{2a_1}|z|^2}.
\]
On the other hand, it is easy to see that the function \( h(z) = e^{b|z|^2} F_D^{\alpha,q}(f)(z) \) is entire on \( \mathbb{C} \) and we have:
\[
\forall z \in \mathbb{C}, \quad |h(z)| \leq C_1 e^{\frac{k^2}{2a_1}(\Re(z))^2}
\]
and
\[
\| h \|_{n} \leq \| e^{b\lambda^2} F_D^{\alpha,q}(f)(\lambda) \|_{n} < \infty.
\]
So, by Lemma 3, \( h \) is the zero function on \( \mathbb{C} \), which implies that \( F_D^{\alpha,q}(f) \) is the zero function on \( \mathbb{R}_q \). The Plancherel formula finishes the proof.

The following result gives a \( q \)-analogue of the Hardy’s theorem.

Corollary 1 If \( f \) is a function defined on \( \mathbb{R}_q \) such that
\[
\| e^{at^2} f(t) \|_{\infty, q} < \infty, \quad \| e^{b\lambda^2} F_D^{\alpha,q}(f)(\lambda) \|_{\infty} < \infty \quad \text{and} \quad ab > \frac{k^2}{4}, \tag{14}
\]
then \( f = 0 \) on \( \mathbb{R}_q \).

Proof. The function \( f \) satisfies for \( c \in \left[ \frac{k^2}{4a}, b \right] \), \( \| e^{at^2} f(t) \|_{\infty, q} < \infty \) and
\[
\| e^{c\lambda^2} F_D^{\alpha,q}(f)(\lambda) \|_{1} = \int_{-\infty}^{\infty} e^{c\lambda^2} |F_D^{\alpha,q}(f)(\lambda)||\lambda|^{2\alpha+1} d_q \lambda \leq \int_{-\infty}^{\infty} e^{c\lambda^2} |F_D^{\alpha,q}(f)(\lambda)||\lambda|^{2\alpha+1} d_q \lambda \leq \| e^{b\lambda^2} F_D^{\alpha,q}(f)(\lambda) \|_{\infty} \int_{-\infty}^{\infty} e^{(c-b)\lambda^2} |\lambda|^{2\alpha+1} d_q \lambda < \infty.
\]
So, the previous theorem gives the result.
References


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