Hypercyclicity Criterion of Multiple Weighted Composition Operators

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“Dedicated to Mola Ali”

Abstract
In this paper we give some sufficient conditions for the adjoint of the multiple weighted composition operators acting on some function spaces satisfying the Hypercyclicity Criterion.

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1 Introduction
Let $H$ be a Hilbert space of functions analytic on a plane domain $G$ such that for each $\lambda$ in $G$ the linear functional of evaluation at $\lambda$ given by $f \mapsto f(\lambda)$ is a bounded linear functional on $H$. By the Riesz representation theorem there is a vector $K_\lambda$ in $H$ such that $f(\lambda) = < f, K_\lambda >$. We call $K_\lambda$ the reproducing kernel at $\lambda$.

Let $T$ be a bounded linear operator on $H$. For $x \in H$, the orbit of $x$ under $T$ is the set of images of $x$ under the successive iterates of $T$: $\text{orb}(T, x) = \{ x, Tx, T^2x, \ldots \}$. The vector $x$ is called hypercyclic for $T$ if $\text{orb}(T, x)$ is dense in $H$. Also a hypercyclic operator is one that has a hypercyclic vector.

The holomorphic self maps of $U$ are divided into classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in $U$. It is well known that this map is conjugate to a rotation $z \mapsto \lambda z$ for some complex number $\lambda$ with $|\lambda| = 1$. The maps of that are not elliptic are called of non-elliptic type. The iterate of a non-elliptic map can be characterized by the Denjoy-Wolff Iteration Theorem ([2]).
A complex-valued function \( \psi \) on \( G \) is called a multiplier of \( H \) if \( \psi H \subset H \). The operator of multiplication by \( \psi \) is denoted by \( M_\psi \) and is given by \( f \rightarrow \psi f \).

If \( w \) is a multiplier of \( H \) and \( \varphi \) is a mapping from \( G \) into \( G \) such that \( f \circ \varphi \in H \) for all \( f \in H \), then \( C_\varphi \) (defined on \( H \) by \( C_\varphi f = f \circ \varphi \)) and \( M_w C_\varphi \) are called composition and weighted composition operator respectively. We define the iterates \( \varphi_n = \varphi \circ \varphi \circ \ldots \circ \varphi \) (\( n \) times). For some topics see [1–4].

## 2 Main Results

A nice criterion namely the Hypercyclicity Criterion is used in the proof of our main theorem.

**Definition 2.1 (The Hypercyclicity Criterion)** Suppose \( X \) is a separable Banach space and \( T \) is a continuous linear mapping on \( X \) i.e. \( T \in B(X) \). If there exist two dense subsets \( Y \) and \( Z \) in \( X \) and a sequence \( \{n_k\} \) such that:

1. \( T^{n_k} y \rightarrow 0 \) for every \( y \in Y \), and
2. There exist functions \( S_{n_k} : Z \rightarrow X \) such that for every \( z \in Z, S_{n_k} z \rightarrow 0 \), and \( T^{n_k} S_{n_k} z \rightarrow z \),

then we say that \( T \) satisfies the Hypercyclicity Criterion.

Throughout this section let \( H \) be a Hilbert space of analytic functions on the open unit disc \( D \) such that \( H \) contains constants and the functional of evaluation at \( \lambda \) is bounded for all \( \lambda \) in \( D \). Also let \( w_i : D \rightarrow C \) be non-constant multipliers of \( H \) for \( i = 1, 2 \), and \( \varphi \) be an analytic univalent map from \( D \) onto \( D \). By \( \varphi_n^{-1} \) we mean the \( n \)th iterate of \( \varphi^{-1} \).

**Lemma 2.2** Let \( \varphi(z) = e^{i\theta} z \) for some \( \theta \in [0, 2\pi] \) and every \( z \in D \). Also, let \( w_i : D \rightarrow C \) be such that the sets

\[
E_1 = \{ \lambda \in D : \lim_{\to \infty} \prod_{j=0}^{n-1} w_1(e^{(2j+1)i\theta})w_2(e^{2ji\theta}) = 0 \}
\]

and

\[
E_{-1} = \{ \lambda \in D : \lim_{\to \infty} \left( \prod_{j=1}^{n} w_1(e^{-2j-1)i\theta})w_2(e^{-2ji\theta}) \right)^{-1} = 0 \}
\]

have limit points in \( D \). Then \( T = (M_{w_2} C_\varphi M_{w_1} C_\varphi)^* \) satisfies the Hypercyclicity Criterion.

**Proof.** For all \( n \in N \) and all \( \lambda \) in \( D \) we can see that

\[
T^n K_\lambda = \left( \prod_{j=0}^{n-1} \overline{w_1(e^{(2j+1)i\theta})}w_2(e^{2ji\theta}) \right) K_{e^{2ni\theta}} \lambda.
\]
Put \( H_{E_m} = \text{span}\{ K_\lambda : \lambda \in E_m \} \) for \( m = -1, 1 \). Since the sets \( E_m \) have limit points in \( D \), thus the sets \( H_{E_m} \) are dense in \( H \) for \( m = -1, 1 \). Note that for each \( \lambda \) in \( D \), the sequence \( \{ e^{ijm\theta} \lambda \}_j \) is a subset of the compact set \( \{ z : |z| = |\lambda| \} \) for \( m = -1, 1 \). Now if \( f \in H \), then the set \( \{ f(e^{ijm\theta} \lambda) : j \in N \} \) is bounded and so by the Banach-Steinhaus Theorem the sequence \( \{ K_{e^{ijm\theta} \lambda} \}_j \) is bounded for \( m = -1, 1 \). Hence, \( T^n \to 0 \) pointwise on \( H_{E_1} \). Now to find the right inverse of \( T^n \), first consider the special case where the collection of linear functionals of point evaluations \( \{ K_\lambda : \lambda \in E_{-1} \} \) is linearly independent.

Define \( S_n : H_{E_{-1}} \to H \) by extending the definition

\[
S_n K_\lambda = \left( \prod_{j=1}^{n} i_{\lambda_j}(e^{-(2j-1)i\theta} \lambda), i_{\lambda_j}(e^{-2ji\theta} \lambda) \right)^{-1} K_{e^{-2ni\theta} \lambda},
\]

where \( \lambda \in E_{-1} \) and \( n \in N \). Now, clearly we can see that \( T^n S_n \) is identity on the dense subset \( H_{E_{-1}} \) of \( H \). Note that if \( \lambda \in E_{-1} \), then \( S_n \to 0 \) pointwise on \( H_{E_{-1}} \) that is dense in \( H \). Thus \( T \) satisfies the Hypercyclicity Criterion. In the case that linear functionals of point evaluations are not linearly independent, by the same way we can use a standard method as in Theorem 4.5 in [1] to complete the proof.

For the proof of the following theorem, we apply the method that we have used in the proof of Theorem 3 in [3].

**Theorem 2.3** Let \( \varphi \) be an elliptic automorphism with interior fixed point \( p \) and \( w_i : D \to C \) satisfies the inequality: \( |w_k(p)| < 1 \leq \liminf \frac{|w_k(z)|}{|z|^{-1}} \) for \( k = 1, 2 \). Then \( T = (M_{w_2} C_{\varphi} M_{w_1} C_{\varphi})^* \) satisfies the Hypercyclicity Criterion.

**Proof.** Put \( \Phi = \alpha_p \circ \varphi \circ \alpha_p \) and \( W_k = w_k \circ \alpha_p \) for \( k = 1, 2 \), where \( \alpha_p(z) = \frac{p-z}{1-pz} \). Since \( \Phi \) is an automorphism with \( \Phi(0) = 0 \), thus \( \Phi \) is a rotation \( z \to e^{i\theta} z \) for some \( \theta \in [0, 2\pi] \) and every \( z \in D \). Since \( |\alpha_p(z)| \to 1^- \) when \( |z| \to 1^- \), so

\[
\liminf_{|z|\to1^-} |w_k(z)| \leq \liminf_{|z|\to1^-} |w_k \circ \alpha_p(z)| \quad \text{for} \quad k = 1, 2.
\]

Thus \( |W_k(0)| < 1 \leq \liminf_{|z|\to1^-} |W_k(z)| \) for \( k = 1, 2 \). Therefore there exist some constants \( \lambda_1 \) and \( \lambda_2 \) and positive numbers \( \delta_1 < 1 \) and \( \delta_2 < 1 \) such that \( |W_k(z)| < \lambda_1 < 1 \) when \( |z| < \delta_1 \), and \( |W_k(z)| > \lambda_2 > 1 \) when \( |z| > 1 - \delta_2 \) for \( k = 1, 2 \). Set \( F_1 = \{ z : |z| < \delta_1 \} \) and \( F_{-1} = \{ z : |z| > 1 - \delta_2 \} \). So if \( z \in F_1 \), then for each positive integer \( n \), \( |W_k(\Phi_n(z))| < \lambda_1 < 1 \), and if \( z \in F_{-1} \), then \( |W_k((\Phi_n)(z))| > \lambda_2 > 1 \). Hence for \( m = -1, 1 \), \( F_m \) is a subset of \( E_m \) where \( E_m \) is defined as in Lemma 2.2 depending on \( W_k \) instead of \( w_k \). This says that the sets \( E_m \) have limit points in \( D \) for \( m = -1, 1 \) and so by Lemma 2.2, the operator \( S = (M_{W_2} C_{\varphi} M_{W_1} C_{\varphi})^* \) satisfies the Hypercyclicity Criterion. But \( T^* C_{\alpha_p} = C_{\alpha_p} S^* \), thus by similarity \( T \) satisfies the Hypercyclicity Criterion.

From now on we suppose that for some \( n \geq 1 \), \( \varphi_n = \varphi_0 \) where \( \varphi_0 \) is the identity mapping on \( D \). For simplicity, we will use the following notation:
Theorem 2.4 Let \( h(z) = \prod_{i=0}^{n-1} w_1 \circ \varphi_{2i+1}(z) \cdot w_2 \circ \varphi_{2i}(z) \). If \( \text{ran } h \) intersects the unit circle, then \( T^*_{w_2,\varphi,w_1} \) satisfies the Hypercyclicity Criterion.

**Proof.** For all \( n \in \mathbb{N} \) and all \( \lambda \) in \( D \) we have

\[
T^*_{n,\varphi,w_1} K_\lambda = \left( \prod_{i=0}^{n-1} w_1 \circ \varphi_{2i+1}(\lambda) \cdot w_2 \circ \varphi_{2i}(\lambda) \right) K_{\varphi_{2n}(\lambda)}
\]

Thus \( T^*_{n,\varphi,w_1} = M^*_h \). Since \( \text{ran } h \) intersects the unit circle, \( M^*_h \) and so \( T^*_{n,\varphi,w_1} \) satisfies the Hypercyclicity Criterion ([1]). This completes the proof.

Theorem 2.5 Let \( h(z) = \prod_{i=0}^{n-1} w_1 \circ \varphi_{2i+1}(z) \cdot w_2 \circ \varphi_{2i}(z) \). If \( T^*_{w_2,\varphi,w_1} \) satisfies the Hypercyclicity Criterion, then the closure of \( \text{ran } h \) intersects the unit circle.

**Proof.** First note that \( T^*_{n,\varphi,w_1} = M^*_h \). Hence \( M^*_h \) is hypercyclic and so it’s spectrum intersects the unit circle ([1]). This completes the proof.

**References**


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