On the Degree of Approximation of Function
Belonging to the Lipschitz Class by (C,2)(E,1)

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Abstract
In this paper, we obtained a theorem on the degree of approximation of functions belonging to the Lipschitz class by (C,2)(E,1) product means of its Fourier series.

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1. INTRODUCTION
The degree of approximation of functions belonging to Lip $\alpha$ by Cesaro means and Nörlund means has been discussed by a number of researchers like Alexits\textsuperscript{[3]}, Sahney and Goel\textsuperscript{[2]}, Chandra\textsuperscript{[7]}, Qureshi\textsuperscript{[5]} and Qureshi and Neha\textsuperscript{[6]}. In the paper Lal and Yadav\textsuperscript{[8]} obtained a theorem on the degree of approximation of the function belonging to the Lipschitz class by (C,2)(E,1) product means of its Fourier series.
This paper is developed from the Lal and Yadav’s paper[8]. Namely, we obtained a theorem on the degree of approximation of functions belonging to the Lipschitz class by (C,2)(E,1) product means of its Fourier series. The proof is done with different method from Lal and Yadav[8].

Let \( f(t) \) be periodic with period \( 2\pi \) and integrable in the sense of Lebesgue. The Fourier series of \( f(t) \) is given by

\[
f(t) \approx \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).
\] (1.1)

A function \( f \in Lip \alpha \) if

\[
f(x + t) - f(x) = O\left[\left|\alpha^\alpha\right|\right] \quad \text{for} \quad 0 < \alpha \leq 1.
\] (1.2)

The degree of approximation of a function \( f : R \to R \) by a trigonometric polynomial \( t_n \) of order \( n \) is defined by Zygmund[1; p.114]

\[
\|f_n - f\|_\infty = \text{Sup} \{\|t_n(x) - f(x)\| : x \in R\}.
\] (1.3)

If

\[
E_n^1 = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} s_{k} \to s, \quad \text{as} \quad n \to \infty
\] (1.4)

then an infinite series \( \sum_{k=0}^{\infty} a_k \) with the partial sums \( s_n \) is said to be summable (E,1) to the definite number \( s \), Hardy[4; p.70].

The (C,2) transform of the (E,1) transform \( E_n^1 \) defines the (C,2)(E,1) transform of the partial sums \( s_n \) of the series \( \sum_{k=0}^{n} a_k \). Thus if

\[
(C_2E)_n^1(x) = \sum_{k=0}^{n} \frac{2(n-k+1)}{(n+1)(n+2)} E_k^1 \to s \quad \text{as} \quad n \to \infty
\]

where \( E_n^1 \) denotes the (E,1) transform of \( s_n \), then the series \( \sum_{k=0}^{\infty} a_k \) is said to be summable by (C,2)(E,1) means or simply summable (C,2)(E,1) to \( s \). We shall use following notation:
\( \phi(t) = f(x + t) + f(x - t) - 2f(x) \)

2. MAIN RESULT

We shall prove the following main theorem.

**Theorem:** If \( f : R \rightarrow R \) is \( 2\pi \)-periodic, Lebesgue integrable on \([-\pi, \pi]\) and belonging to the Lipschitz class then the degree of approximation of \( f \) by the \((C,2)(E,1)\) product means of its Fourier series satisfies for \( n = 0, 1, 2, ..., \)

\[
\left\| (C, E)^{1/2}_n - f \right\|_\infty = O\left( \frac{1}{(n+1)^{\alpha/2}} \right), \quad 0 < \alpha < 1.
\]

**Proof:** The \( n \)th partial sum \( S_n(x) \) of the series (1.1) at \( t = x \) is written as

\[
S_n(x) = f(x) + \frac{1}{2\pi} \int_0^\pi \phi(t) \frac{\sin \left( \frac{n+1}{2} \right)}{\sin \frac{t}{2}} \, dt.
\]

So the \((E,1)\) means of the series (1.1) are

\[
E^1_n(x) = 2^{-n} \sum_{k=0}^n \binom{n}{k} S_k(x) \quad (n = 0, 1, 2, ...)
\]

\[
= f(x) + \frac{1}{2^{n+1} \pi} \int_0^\pi \phi(t) \left( \sum_{k=0}^n \frac{n}{k} \sin(k + \frac{1}{2} t) \right) \, dt
\]

\[
= f(x) + \frac{1}{2^{n+1} \pi} \int_0^\pi \phi(t) \operatorname{Im} \left( e^{it/2} (1 + e^{it})^n \right) \, dt
\]

\[
= f(x) + \frac{1}{2^{n+1} \pi} \int_0^\pi \phi(t) \operatorname{Im} \left( e^{it/2} (1 + \cos t + i \sin t)^n \right) \, dt
\]
= f(x) + \frac{1}{2^{n+1}\pi} \int_{0}^{\pi} \phi(t) \frac{t}{2} \left( \cos \frac{t}{2} + i \sin \frac{t}{2} \right)^n dt

= f(x) + \frac{1}{2^{n+1}\pi} \int_{0}^{\pi} \phi(t) 2^n \cos^n \left( \frac{t}{2} \right) \frac{t}{2} \left( \cos \frac{nt}{2} + i \sin \frac{nt}{2} \right)^n dt

= f(x) + \frac{1}{2\pi} \int_{0}^{\pi} \phi(t) \cos^n \frac{t}{2} \sin(n+1) \frac{t}{2} dt.

Therefore, (C,2)(E,1) product means of the series (1.1) are

\( (C_2 E)_n^1(x) = \sum_{k=0}^{n} \frac{2(n-k+1)}{(n+1)(n+2)} E_k^1(x) = \frac{2(n+1)}{(n+1)(n+2)} \sum_{k=0}^{n} E_k^1(x) - \frac{2(n+1)(n+2)}{n} \sum_{k=0}^{n} k E_k^1(x) \)

\( = f(x) + \frac{1}{\pi(n+2)} \int_{0}^{\pi} \phi(t) \left( \sum_{k=0}^{n} k \cos \frac{t}{2} \sin(k+1) \frac{t}{2} \right) dt \)

\( - \frac{1}{\pi(n+1)(n+2)} \int_{0}^{\pi} \phi(t) \left( \sum_{k=0}^{n} k \cos \frac{t}{2} \sin(k+1) \frac{t}{2} \right) dt \)

\( = f(x) + I_1 - I_2. \) (2.1)

Now, we estimate \( I_2. \)

\( I_2 = \frac{1}{\pi(n+1)(n+2)} \int_{0}^{\pi} \phi(t) \left( \sum_{k=0}^{n} k \cos \frac{t}{2} \sin(k+1) \frac{t}{2} \right) dt \)

\( = \frac{1}{\pi(n+1)(n+2)} \left( \int_{0}^{\frac{\pi}{2}} \phi(t) \sum_{k=0}^{n} k \cos \frac{t}{2} \sin(k+1) \frac{t}{2} dt \right) \)

\( = I_{21} + I_{22}. \) (2.2)
Applying the fact that $f \in \text{Lip } \alpha$ and $\sin \frac{t}{2} \geq \frac{t}{\pi}$, we have

$$|I_{21}| \leq \frac{n}{2\pi(n+2)} \int_0^{\sqrt{n+1}} \left| \phi(t) \frac{t}{\sin \frac{t}{2}} \right| dt = \frac{n}{2(n+2)} \int_0^{\sqrt{n+1}} \frac{O(t^\alpha)}{t} dt = O\left( \frac{1}{(n+1)^{\alpha/2}} \right) \quad (2.3)$$

By using $k \cos^k \left( \frac{t}{2} \right) = \frac{-2 \left( \cos^k \left( \frac{t}{2} \right) \right)'}{\sin \frac{t}{2}} \cos \frac{t}{2}$, we estimate $I_{22}$.

$$|I_{22}| \leq \frac{1}{\pi(n+1)(n+2)} \int_0^{\sqrt{n+1}} \left| \phi(t) \frac{\sum_{k=0}^{n-1} \cos^k \frac{t}{2}}{\sin \frac{t}{2}} \right| \cos \frac{t}{2} \sin(k+1) \frac{t}{2} dt$$

$$\leq \frac{-2\pi}{(n+1)(n+2)} \int_0^{\sqrt{n+1}} \frac{t^\alpha}{2} \left( \sum_{k=0}^{n-1} \cos^k \frac{t}{2} \right)' dt$$

$$= \frac{-2\pi}{(n+1)(n+2)} \int_0^{\sqrt{n+1}} t^{\alpha-2} \left( \cos \frac{t}{2} - 1 \right)' dt$$

$$= \frac{2}{\pi(n+2)} \int_0^{\sqrt{n+1}} t^{\alpha-2} \cos^\alpha \frac{t \cdot \sin t}{\cos t - 1} dt - \frac{2}{\pi(n+1)(n+2)} \int_0^{\sqrt{n+1}} t^{\alpha-2} \sin t \left( \frac{\cos^\alpha t - 1}{\cos t - 1} \right) dt$$

$$= A - B \quad (2.4)$$

$$|A| = \left| \frac{2}{\pi(n+2)} \int_0^{\sqrt{n+1}} t^{\alpha-2} \cos^\alpha \frac{t \cdot \sin t}{\cos t - 1} dt \right| \leq \frac{c_1}{n+2} \int_0^{\sqrt{n+1}} t^{\alpha-3} dt = O\left( \frac{1}{(n+1)^{\alpha/2}} \right) \quad (2.5)$$

where $c_1$ is a positive constant.
\[ |B| = \left| \frac{2}{\pi(n+1)(n+2)} \int_{\sqrt[n]{1}}^{\pi} t^{n-2} \sin t \frac{\cos^n t - 1}{(\cos t - 1)^2} dt \right| \]

\[ \leq \frac{C_2}{(n+1)(n+2)} \int_{\sqrt[n]{1}}^{\pi} t^{n-3} dt = O\left( \frac{1}{n^{1+\alpha/2}} \right) \quad (2.6) \]

In similar way, we can obtain

\[ I_1 = O\left( \frac{1}{(n+1)^\alpha} \right). \quad (2.7) \]

From (2.1), (2.2), (2.3), (2.4), (2.5), (2.6) and (2.7) we have

\[ \left| (C_2 E)^n_1 (x) - f(x) \right| = O\left( \frac{1}{(n+1)^{\alpha/2}} \right) \quad 0 < \alpha < 1. \]

Thus

\[ \left\| (C_2 E)^n_1 (x) - f(x) \right\|_e = \sup_{-\pi \leq x \leq \pi} \left| (C_2 E)^n_1 (x) - f(x) \right| = O\left( \frac{1}{(n+1)^{\alpha/2}} \right). \]

Theorem is proved.

References

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