Startified \( L \)-Locally Uniform Spaces

Debajit Hazarika and Dipak Kumar Mitra

Department of Mathematical Sciences, Tezpur University
Napam, Tezpur -784028, India
debajit@tezu.ernet.in

Abstract

We introduce stratified \( L \)-locally uniform spaces and show that an \( L \)-topology is stratified if and only if it is generated by a stratified \( L \)-local uniformity. It is also established that every typically \( T_1 \) regular \( L \)-topology is stratified. Further, every separable \( L \)-local uniformity is shown to be stratified.

Mathematics Subject Classification: 54E15, 54A20

Keywords: \( L \)-topology, stratified \( L \)-topology, \( L \)-local uniformity

1 Introduction

Lowen introduced the category of stratified fuzzy topological spaces [4] to remove certain shortcomings in Chang’s fuzzy topological spaces. The notion of uniformity was introduced by Hutton [2] in the category \( L\text{-TOP} \) of fixed basis fuzzy topological spaces and obtained a necessary and sufficient condition under which every \( L \)-topology (non-stratified) is uniformizable. Zhang [6] introduced stratified Hutton uniform spaces and established that every stratified Hutton uniform space generates a stratified \( L \)-topology, the converse of which was shown to be not true. In [5], we developed the notion of \( L \)-local uniformity as a generalisation of Hutton’s uniformity. In this paper, we take up the problem of stratification of an \( L \)-local uniformity in this paper and provide several solutions.

Throughout the paper \((L, \leq, \wedge, \vee, ')\) is a completely distributive lattice with order reversing involution \('\); \(0_L\) and \(1_L\) respectively are inf and sup in \( L \). \( X \) is an arbitrary set and \( L^X \) will denote the collection of all mappings \( A : X \to L \). Any member of \( L^X \) is called an \( L \)-fuzzy set. For any \( A \in L^X \), the set \( \text{supp}(A) = \{ x \in X \mid 0_L < A(x) \} \) is called as the support of \( A \). The \( L \)-fuzzy sets \( x_\alpha : X \to L \) defined by \( x_\alpha(y) = 0_L \) if \( x \neq y \) and \( x_\alpha(y) = \alpha \) if \( x = y \) are the \( L \)-fuzzy points. The mappings \( A : X \to L \) and \( B : X \to L \)
defined by $A(x) = 1_L, \forall x \in X$ and $B(x) = 0_L, \forall x \in X$ are denoted by $\mathbb{1}$ and $\mathbb{0}$ respectively. For any $A, B \in L^X$, the union and intersection of $A$ and $B$ are defined as $A \cup B = A(x) \bigvee_{x \in X} B(x)$ and $A \cap B = A(x) \bigwedge_{x \in X} B(x)$ respectively; we say $A \subseteq B$ iff $A(x) \leq B(x)$ and $x_\alpha \in A$ iff $\alpha < A(x)$, where $x_\alpha$ is an $L$-fuzzy point; complement of $A$ is defined as $A'(x) = A(x)'$. An $L$-topology $\mathcal{F}$ on $L^X$ is a subset of $L^X$ closed under finite intersection and arbitrary union. The elements of $\mathcal{F}$ are called open sets and their complements are the closed sets. $\mathcal{F}$ is called stratified if for any $\alpha \in L$ implies $\alpha \in \mathcal{F}$, where $\alpha : X \rightarrow L$ is a mapping such that $\alpha(x) = \alpha, \forall x \in X$. An $L$-topological space $(L^X, \mathcal{F})$ is said to be typically $T_1$ if and only if every $L$-fuzzy point in $(L^X, \mathcal{F})$ is closed. An $L$-topological space $(L^X, \mathcal{F})$ is said to be regular, if for every $G \in \mathcal{F}$ and $x_\alpha \subseteq G$, there is $A \in \mathcal{F}$ such that $x_\alpha \subseteq A \subseteq \bar{A} \subseteq G$.

For further fuzzy theoretic definitions and results refer to [3].

2 \textbf{L-locally Uniform spaces}

We recall the following definitions and results:

\textbf{Definition 2.1.} [2] Let $U^*$ be the collection of all maps $U : L^X \rightarrow L^X$ which satisfy:

(s1) $\Delta \subseteq U$.
(s2) $U(\bigcup \lambda \{V_\lambda\}) = \bigcup \lambda U(V_\lambda), \ V_\lambda \in L^X$.

Where $\Delta : L^X \rightarrow L^X$ is a mapping such that $\Delta(A) = A, \forall A \in L^X$.

For any $U, V \in U^*, U \circ V$ is the composition of functions. Obviously, $\Delta \circ U = U = U \circ \Delta$.

\textbf{Definition 2.2.} [2] For any $U \in U^*, U^r(x_\alpha) = \bigcap \{y_\beta \mid U(y_\beta') \subseteq x_\alpha'\}$. Then $U^r \in U^*$ and $(U^r)^r = U$.

If $U = U^r$, then $U$ is said to be symmetric.

\textbf{Proposition 2.3.} [2] Let $X$ be a nonempty ordinary set and $L$ a fuzzy lattice. Let $i : L^X \rightarrow L^X$ be a map satisfies the following axioms:

(IO1) $i(\mathbb{1}) = \mathbb{1}$.
(IO2) $i(A) \subseteq A, \forall A \in L^X$.
(IO3) $i(A \cap B) = i(A) \cap i(B), \forall A, B \in L^X$.
(IO4) $i(i(A)) = A, \forall A \in L^X$.

Then $\mathcal{F} = \{A \in L^X \mid i(A) = A\}$ is an $L$-topology and $i(A) = A^\circ$, where $A^\circ$ is the interior of $A$ in the $L$-topological space $(L^X, \mathcal{F})$.

We recall the following definitions and results from [5] for the sake of self containment:

\textbf{Definition 2.4.} An $L$-local uniformity $U$ on $L^X$ is a non empty subfamily of $U^*$ which satisfy the following:
(L1) \( U \cap V \in U, \ \forall U, V \in U \).
(L2) If \( V \in U^* \) such that \( U \subseteq V \), for some \( U \in U \), then \( V \in U \).
(L3) \( U \in U \) implies \( U^r \in U \).
(L4) For any \( U \in U \) and \( x_\alpha \in L^X \) there exists \( V \in U \) such that \( V \circ V(x_\alpha) \subseteq U(x_\alpha) \).

The pair \((L^X, U)\) shall be called an \( L \)-locally uniform space.

**Lemma 2.5.** Let \((L^X, U)\) be an \( L \)-locally uniform space. Let \( \text{int} : L^X \to L^X \) be a mapping defined by \( \text{int}(G) = \bigcup \{ y_\beta \mid \exists U \in U \text{ s.t. } U(y_\beta) \subseteq G \} \). Then \( \text{int} \) satisfies the axioms (IO1), (IO2), (IO3) and (IO4).

**Proof.** (IO1) \( \text{int}(\mathbf{1}) = \mathbf{1} \) and (IO2) \( \text{int}(A) \subseteq A \) for all \( A \in L^X \) are trivially satisfied.

Also if \( G \) is an \( L \)–fuzzy set and \( U \in U \) is such that \( U(x_\alpha) \subseteq G \). Then there is \( V \in U \) such that \( V \circ V(x_\alpha) \subseteq U(x_\alpha) \). This implies that \( V(V(x_\alpha)) \subseteq G \).

Thus \( V(x_\alpha) \subseteq \text{int}(G) \).

This implies that \( x_\alpha \subseteq \text{int}(\text{int}(G)) \) and since the other inclusion follows by (IO2).

So we have (IO3) \( \text{int}(G) = \text{int}(\text{int}(G)) \).

(IO4) Finally, \( \text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B) \) is fulfilled due to (L1).

**Remark 2.6.** In the above lemma the \( L \)-local uniformity \( U \) can be replaced by any of its base.

Now, by proposition 2.3 we have the following:

**Theorem 2.7.** Let \((L^X, U)\) be an \( L \)-local uniform space. Let \( \mathbb{F}(U) = \{ G \in L^X \mid G = \text{int}(G) \} \). Then \( \mathbb{F}(U) \) is an \( L \)-topology on \( L^X \).

The \( L \)-topology generated by an \( L \)-local uniformity \( U \) is the \( L \)-topology generated by \( \text{int} \). Hence, we note that, in particular, for any \( x_\alpha \in L^X \), \( \{ U(x_\alpha) \mid U \in U \} \) is the neighborhood system at \( x_\alpha \) in \((L^X, U)\).

**Definition 2.8.** For any \( V \in U^r \), let \( V^2 = V \circ V \); \( V^{2r} = V^r \circ V^r \) and for each \( n \in \mathbb{N} \), let \( V^{n+1} = V^n \circ V \); \( V^{(n+1)r} = V^{nr} \circ V^r \).

**Theorem 2.9.** If \( U \) is an \( L \)-local uniformity. Then for each \( n \in \mathbb{N} \),
\[
U^n = \{ U : L^X \to L^X \mid \exists V \in U \text{ s.t. } V^n \subseteq U \}
\]
is an \( L \)-local uniformity with the same \( L \)-topology generated by \( U \).

**Proof.** It suffices to prove the theorem for \( U^2 \).

Clearly, \( U^2 \) satisfies the axioms (L1), (L2) and (L3). Now, for \( W \in U^2 \) and \( x_\alpha \in L^X, \exists U, V \in U \) so that \( U^4(x_\alpha) \subseteq V(x_\alpha) \) and \( V^2 \subseteq W \). But \( U^2 \in U^2 \) as \( U \subseteq U^2 \) for any \( U \in U \). Further, \((U^2 \circ U^2)(x_\alpha) \subseteq V(x_\alpha) \subseteq V^2(x_\alpha) \subseteq W(x_\alpha) \).

Hence, \( U^2 \) is an \( L \)-local uniformity.

Now by the definition of \( U^2 \), the relative \( L \)-topology of \( U^2 \) is weaker than that of \( U \). Again as \( U \subseteq U \) implies \( U^2 \subseteq U^2 \), it is also stronger.
Theorem 2.10. Let \((L^X, U)\) be an \(L\)-locally uniform space. Then for any \(A \in L^X\),
\[
\overline{A} = \bigcap\{V(A) \mid V \in U\},
\]
where \(\overline{A}\) is the closure of \(A\) in \((L^X, \mathbb{F}(U))\).

Proof. Let \(A\) be any \(L\)-fuzzy set. Then \(\overline{A} = \text{int}(A')'\).
Let \(B\) be a base for \(U\) consisting of symmetric members of \(U\).
Now, \(\text{int}(A')' = \bigcup\{x_\alpha \mid \exists U \in B \text{ s.t. } U(x_\alpha) \subseteq A'\}\).
\[
= \bigcup\{\bigcup\{x_\alpha \mid U(x_\alpha) \subseteq A', U \in B\}\}'\).
\[
= \bigcap\{U(A) \mid U \in B\}. \quad \text{(as } U \in B \Rightarrow U' = U)\)
\[
= \bigcap\{V(A) \mid V \in U\}. \quad \text{(as } B\text{ is a base for } U)\)
Hence, \(\overline{A} = \bigcap\{V(A) \mid V \in U\}\). \qed

Definition 2.11. An \(L\)-topology is called \(L\)-locally uniformizable if there is an \(L\)-local uniformity \(U\) which generates the \(L\)-topology.

Theorem 2.12. The \(L\)-topology of an \(L\)-locally uniform space is regular. Conversely, for any regular \(L\)-topological space, the set of all neighborhoods of \(\Delta\) is an \(L\)-local uniformity which generates the \(L\)-topology.

Proof. Let \((L^X, U)\) be an \(L\)-locally uniform space.
Now for any \(x_\alpha \in L^X\) and neighborhood \(U(x_\alpha), U \in U\), let \(V \in U\) is such that \(V^2(x_\alpha) \subseteq U(x_\alpha)\). Then, by theorem 2.10, \(V(x_\alpha) \subseteq V(V(x_\alpha)) \subseteq U(x_\alpha)\).
Hence the space is regular.
Conversely, let the \(L\)-topological space \((L^X, \mathbb{F})\) be regular.
Let \(U\) be the collection of all neighborhoods of \(\Delta\). Then, \(U\) satisfies the axioms (L1), (L2), (L3) and generating the \(L\)-topology \(\mathbb{F}\). Also, by regularity, for any \(U \in U\) and \(x_\alpha \in L^X\), there exists an \(L\)-fuzzy set \(B\) s.t.
\[
x_\alpha \subseteq \text{int}B \subseteq \overline{B} \subseteq U(x_\alpha).
\]
Define \(W_B : L^X \to L^X\) as follows:
\[
W_B(A) = \begin{cases} 
1, & \text{if } A \not\subseteq B, \\
B, & \text{otherwise.}
\end{cases}
\]
Then, \(W_B \in U\).
Also, as \(x_\alpha \subseteq B\), we have \(W_B(W_B(x_\alpha)) = W_B(B) = B \subseteq \overline{B} \subseteq U(x_\alpha)\). \qed

3 Stratified \(L\)-locally uniform spaces

Definition 3.1. Let \(U\) be an \(L\)-local uniformity on \(L^X\). Then we shall call \(U\) to be stratified if for any \(A \in L^X\) there is \(U \in U\) such that \(U(A)(x) \leq \bigvee_{y \in X} A(y), \quad \forall x \in \text{supp}(A)\).
We now state the following lemma:

**Lemma 3.2.** Let \((L^X, U)\) be an \(L\)-locally uniform space. Then for any \(A \in L^X\),

\[
\text{int}(A) = \bigcup \{C \in L^X \mid \exists U \in U \text{ s.t. } U(C) \subseteq A\}.
\]

**Proof.** Straightforward.

**Theorem 3.3.** Let \((L^X, U)\) be an \(L\)-locally uniform space. Then \(U\) is stratified iff \(\mathcal{F}(U)\) is stratified.

**Proof.** Let \(U\) be stratified and \(\alpha \in L \setminus \{0_L\}\).
Let \(A = \alpha\). Then \(A(x) = \alpha, \ \forall \ x \in X\).
This implies that \(\bigvee_{y \in X} A(y) = \alpha\) and \(\text{supp}(A) = X\).
Since \(U\) is stratified, therefore there is \(U \in U\) such that
\(U(A)(x) \leq \alpha, \ \forall \ x \in X\). (as \(\text{supp}(A) = X\))
This implies that \(U(A) \subseteq A\) and hence \(U(A) = A\).
Now by lemma 3.2, we get \(\text{int}(A) = \bigcup \{C \mid \exists U \in U \text{ s.t. } U(C) = A\}\).
Therefore, \(A \in \mathcal{F}(U)\) and hence \(\mathcal{F}(U)\) is stratified.

Conversely, let \(\mathcal{F}(U)\) be stratified and \(A\) be any \(L\)-fuzzy set.
Let \(\bigvee_{y \in X} A(y) = \beta\) and \(B : X \to L\) be a mapping defined as follows:
\(B(x) = \beta, \ \forall \ x \in X\).
Then \(B = \beta\). So, \(B \in \mathcal{F}(U)\).
Now, \(B \in \mathcal{F}(U) \Rightarrow B = \text{int}(B) = \bigcup \{C \mid \exists U \in U \text{ s.t. } U(C) \subseteq B\}\).
(by lemma 3.2)
Therefore, there is \(U \in U\) such that
\(U(B) = B \Rightarrow U(B)(x) = B(x) = \beta, \ \forall \ x \in X\).
Further, \(A \subseteq B \Rightarrow U(A) \subseteq U(B) \Rightarrow U(A)(x) \leq U(B)(x), \ \forall \ x \in X\).
\(\Rightarrow U(A)(x) \leq \beta, \ \forall \ x \in X\).
Since \(\text{supp}(A) \subseteq X, \ \forall \ A \in L^X\).
Therefore \(U(A)(x) \leq \beta, \ \forall \ x \in \text{supp}(A)\).
This implies that \(U(A)(x) \leq \bigvee_{y \in X} A(y), \ \forall \ x \in \text{supp}(A)\).
Hence, \(U\) is stratified.

**Definition 3.4.** An \(L\)-locally uniform space \((L^X, U)\) is said to be separated if \(\Delta = \bigcap_{U \in U} U\).

We shall require the following lemma for our subsequent results.

**Lemma 3.5.** Let \((L^X, U)\) be an \(L\)-locally uniform space and \(\{A_\lambda \mid \lambda \in \Lambda\}\) be a sub collection of \(L^X\), where \(\Lambda\) is the index set. Then \(\text{cl}(\bigcup_{\lambda \in \Lambda} A_\lambda) = \bigcup_{\lambda \in \Lambda} \text{cl}(A_\lambda)\).
Proof. \( \text{cl}(\bigcup_{\lambda \in \Lambda} A_\lambda) = \bigcap\{U(\bigcup_{\lambda \in \Lambda} A_\lambda) \mid U \in U\} \) \\
= \bigcap\{U(A_\lambda) \mid U \in U\} \\
= \bigcup_{\lambda \in \Lambda} \bigcap\{U(A_\lambda) \mid U \in U\}. \) (as \( L^X \) is completely distributive) \\
= \bigcup_{\lambda \in \Lambda} \text{cl}(A_\lambda). \]

**Theorem 3.6.** Let \((L^X, U)\) be an \(L\)-locally uniform space such that each \(L\)-fuzzy point \(x_\alpha\) is closed in \(F(U)\). Then \(F(U)\) is stratified.

**Proof.** Let \(x_\alpha\) be any member of \(L^X\). Then \(x_\alpha\) is closed in \(F(U)\).

Now \(x_\alpha' = A \bigcup x_\alpha^*\), where \(A = \bigcup\{y_1 \mid y \in X \{x\}\}\) and \(x_\alpha^*\) is an \(L\)-fuzzy point such that

\[
x_\alpha^*(y) = \begin{cases} 
\alpha' & \text{if } y = x \\
0_L & \text{if } y \neq x.
\end{cases}
\]

Again, \(\text{cl}(A) = \text{cl}(\bigcup\{y_1 \mid y \in X \{x\}\})\).

\[
= \bigcup\{\text{cl}(y_1) \mid y \in X \{x\}\}. \quad \text{(By lemma 3.5)}
\]

\[
= \bigcup\{y_1 \mid y \in X \{x\}\}. \quad \text{(as each } y_1 \text{ is closed in } F(U))
\]

\[
= A.
\]

Hence \(A\) is closed.

Also, \(x_\alpha^*\) is closed in \(F(U)\), being an \(L\)-fuzzy point.

Therefore, \(x_\alpha'\) is closed in \(F(U)\).

This implies that \(x_\alpha\) is open in \(F(U)\).

Again as \(\forall \alpha \in L, \alpha = \bigcup_{x \in X} x_\alpha\), therefore \(\alpha \in F(U)\), \(\forall \alpha \in L\).

Thus, \(F(U)\) is stratified. \(\square\)

Now by theorem 2.12, we may conclude the following:

**Corollary 3.7.** Every typically \(T_1\) regular \(L\)-topology is stratified.

**Theorem 3.8.** An \(L\)-locally uniform space \((L^X, U)\) is separated iff \(F(U)\) is typically \(T_1\).

**Proof.** Let \((L^X, U)\) be separated and \(x_\alpha\) be any \(L\)-fuzzy point.

Then by theorem 2.10,

\[
\overline{x_\alpha} = \bigcap\{U(x_\alpha) \mid U \in U\}.
\]

\[
= \Delta(x_\alpha). \quad \text{(as } L^X, U \text{ is separated)}
\]

\[
= x_\alpha.
\]

This implies that every \(L\)-fuzzy point \(x_\alpha\) is a closed subset in \((L^X, F(U))\).

Conversely, it can be shown that if every \(L\)-fuzzy point is closed in \(F(U)\). Then \((L^X, U)\) is separated. \(\square\)

**Theorem 3.9.** Every separated \(L\)-locally uniform space is stratified.

**Proof.** Follows from theorems 3.8 and 3.6. \(\square\)
References


Received: July, 2010