On Cubic Derivations

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Abstract. We say a functional equation \((\xi)\) is stable if any function \(g\) satisfying the equation \((\xi)\) \(approximately\) is near to true solution of \((\xi)\). Also, we say that a functional equation is superstable if every approximately solution is an exact solution of it. In this paper, we investigate the stability and superstability of the system of functional equations

\[
\begin{align*}
 f(xy) &= x^3f(y) + f(x)y^3, \\
 f(2x + y) + f(2x - y) &= 2f(x + y) + 2f(x - y) + 12f(x)
\end{align*}
\]

on Banach algebras.

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1. Introduction

Jun and Kim [41] introduced the following functional equation

\[ f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \]  \hspace{1cm} (1.1)

and the established the general solution for this function equation. It is easy to see that the function \( f(x) = cx^3 \) is a solution of the functional equation (1.1). Thus, it is natural that (1.1) is called a cubic functional equation and every solution of the cubic functional equation is said to be a cubic mapping.

Let \( A \) be a normed algebra and let \( X \) be a Banach \( A \)-module. We say that a mapping \( D : A \to X \) is a cubic derivation if \( D \) is cubic function satisfies

\[ D(ab) = D(a)b^3 + a^3D(b), \quad \text{for all} \ a, b \in A. \]

Example. Let \( A \) be a Banach algebra. Then we take

\[
T = \begin{bmatrix}
0 & A & A & A \\
0 & 0 & A & A \\
0 & 0 & 0 & A \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\( T \) is a Banach algebra equipped with the usual matrix-like operations and the following norm:

\[
\| \begin{bmatrix}
0 & a_1 & a_2 & a_3 \\
0 & 0 & a_4 & a_5 \\
0 & 0 & 0 & a_6 \\
0 & 0 & 0 & 0
\end{bmatrix} \| = \sum_{i=1}^{6} \| a_i \|. \quad (a_i \in A, 1 \leq i \leq 6)
\]

It is known that

\[
T^* = \begin{bmatrix}
o & A^* & A^* & A^* \\
0 & 0 & A^* & A^* \\
0 & 0 & 0 & A^* \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

is the dual of \( T \) under the following norm:

\[
\| \begin{bmatrix}
0 & f_1 & f_2 & f_3 \\
0 & 0 & f_4 & f_5 \\
0 & 0 & 0 & f_6 \\
0 & 0 & 0 & 0
\end{bmatrix} \| = \max \{ \| f_i \| , 1 \leq i \leq 6 \} . \quad (f_i \in A^*, 1 \leq i \leq 6)
\]

Let the left module action of \( T \) on \( T^* \) be trivial and let the right module action of \( T \) on \( T^* \) is defined as follows.

\[
\langle \begin{bmatrix}
0 & f_1 & f_2 & f_3 \\
0 & 0 & f_4 & f_5 \\
0 & 0 & 0 & f_6 \\
0 & 0 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & a_1 & a_2 & a_3 \\
0 & 0 & a_4 & a_5 \\
0 & 0 & 0 & a_6 \\
0 & 0 & 0 & 0
\end{bmatrix} \rangle = \sum_{i=1}^{6} f_i(a_i x_i) , \quad f_i \in A^*, \ a_i, x_i \in A \ (1 \leq i \leq 6),
\]

then \( T^* \) is a Banach \( T \)-module. Let
\[
\begin{bmatrix}
0 & f_1 & f_2 & f_3 \\
0 & f_4 & f_5 \\
0 & 0 & f_6 \\
0 & 0 & 0 & 0
\end{bmatrix} \in T^*. \text{ We define } D : T \rightarrow T^* \text{ by }
\]
\[
D(\begin{bmatrix}
0 & a_1 & a_2 & a_3 \\
0 & 0 & a_4 & a_5 \\
0 & 0 & 0 & a_6 \\
0 & 0 & 0 & 0
\end{bmatrix}) = \begin{bmatrix}
0 & f_1 & f_2 & f_3 \\
0 & 0 & f_4 & f_5 \\
0 & 0 & 0 & f_6 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
0 & 0 & 0 & a_1 a_4 a_6 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]
with \( a_i \in A \). Let now \( a_i, b_i, x_i \in A \ (1 \leq i \leq 6) \), then we have
\[
\langle \langle D(2 \begin{bmatrix}
0 & a_1 & a_2 & a_3 \\
0 & 0 & a_4 & a_5 \\
0 & 0 & 0 & a_6 \\
0 & 0 & 0 & 0
\end{bmatrix}) + D(2 \begin{bmatrix}
0 & a_1 & a_2 & a_3 \\
0 & 0 & a_4 & a_5 \\
0 & 0 & 0 & a_6 \\
0 & 0 & 0 & 0
\end{bmatrix}) \rangle, x \rangle
\]
\[
= \langle f_3, (2a_1 + b_1)(2a_4 + b_4)(2a_6 + b_6) \rangle + \langle f_3, (2a_1 - b_1)(2a_4 - b_4)(2a_6 - b_6) \rangle
\]
\[
= 2\langle f_3, (a_1 + b_1)(a_4 + b_4)(a_6 + b_6) \rangle + 2\langle f_3, (a_1 - b_1)(a_4 - b_4)(a_6 - b_6) \rangle
\]
\[
= 2\langle f_3, a_1 a_4 a_6 x_3 \rangle + 2.2 \langle f_3, a_1 a_4 a_6 x_3 \rangle
\]
\[
= 2\langle D(\begin{bmatrix}
0 & a_1 & a_2 & a_3 \\
0 & 0 & a_4 & a_5 \\
0 & 0 & 0 & a_6 \\
0 & 0 & 0 & 0
\end{bmatrix}) + D(\begin{bmatrix}
0 & a_1 & a_2 & a_3 \\
0 & 0 & a_4 & a_5 \\
0 & 0 & 0 & a_6 \\
0 & 0 & 0 & 0
\end{bmatrix}) \rangle, x \rangle
\]
This means that \( D \) is a cubic function. On the other hand it is easy to check that
The stability of functional equations had been first raised by S. M. Ulam [57] for what metric group \( G \) is it true that a \( \epsilon \)-automorphism of \( G \) is necessarily near to a strict automorphism? In 1941, D. H. Hyers [38] gave a positive answer to the question of Ulam for Banach spaces. Let \( f : E_1 \to E_2 \) be a mapping between Banach spaces such that

\[
\|f(x + y) - f(x) - f(y)\| \leq \delta
\]

for all \( x, y \in E_1 \) and for some \( \delta \geq 0 \). Then there exists a unique additive mapping \( T : E_1 \to E_2 \) satisfying

\[
\|f(x) - T(x)\| \leq \delta
\]

for all \( x \in E_1 \). Moreover, if \( f(tx) \) is continuous in \( t \) for each fixed \( x \in E_1 \), then the mapping \( T \) is linear. In 1950, T. Aoki [2] was the second author to treat this problem for additive mappings. Th. M. Rassias [55] succeeded in extending the result of Hyers’ Theorem by weakening the condition for the Cauchy difference controlled by \((\|x\|^p + \|y\|^p), p \in [0, 1)\) to be unbounded.

Taking into consideration a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Th. M. Rassias is called Hyers-Ulam-Rassias stability (see also [1, 4], [5]–[25], [26, 27, 39, 43], [50]–[52], [54] and [56]).

Jun and Kim [41] generalized Hyers-Ulam-Rassias stability problem for functional equation (1.1). It seems that approximate derivations was first investigated by K. W. Jun and D. W. Park [42]. Recently, the stability of derivations have been investigated by some authors; see [3, 42, 47] and references therein (see also [26, 39, 40, 44, 45, 46, 47, 49, 53, 54] and [56]).

In this paper we study the stability of cubic functional equation that satisfies

\[
f(xy) = x^3f(y) + f(x)y^3
\]

on Banach algebras. Indeed we establish the superstability of equation (1.3) by suitable control functions.
2. Main results

In the following we suppose that $A$ is a commutative Banach algebra and $X$ is a Banach $A$-module. For convenience, we use the following abbreviation for a given function $f : A \rightarrow X$

$$\Delta_f(x, y) := f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x)$$

for all $x, y \in A$.

**Theorem 2.1.** Let $f : A \rightarrow X$ be a mapping and let $\varphi_1 : A^2 \rightarrow \mathbb{R}^+$, $\varphi_2 : A^2 \rightarrow \mathbb{R}^+$ be maps such that

$$\|f(xy) - f(x)y^3 - x^3f(y)\| \leq \varphi_1(x, y), \quad (2.1)$$

and

$$\|\Delta_f(x, y)\| \leq \varphi_2(x, y) \quad (2.2)$$

for all $x, y \in A$. Assume that the series

$$\Psi(x, 0) = \sum_{i=0}^{\infty} \frac{\varphi_2(2^ix, 0)}{2^{3i}}$$

converges and that

$$\lim_{n \rightarrow \infty} \frac{\varphi_1(2^nx, 2^ny)}{2^{6n}} = \lim_{n \rightarrow \infty} \frac{\varphi_2(2^nx, 2^ny)}{2^{3n}} = 0,$$

and for all $x, y \in A$. Then there exists a unique cubic derivation $D : A \rightarrow X$ such that

$$\|D(x) - f(x)\| \leq \frac{1}{16} \Psi(x, 0) \quad (2.3)$$

for all $x \in A$.

**Proof.** Setting $y = 0$ in (2.2) yields

$$\|2f(2x) - 16f(x)\| \leq \varphi_2(x, 0), \quad (2.4)$$

and then result dividing by $2^4$ in (2.4) to obtain

$$\|\frac{f(2x)}{2^3} - f(x)\| \leq \frac{\varphi_2(x, 0)}{2^4} \quad (2.5)$$

for all $x \in A$. In (2.5), replacing $x$ by $2x$ and then result dividing by $2^3$, we have

$$\|\frac{f(2^2x)}{2^6} - \frac{f(2x)}{2^3}\| \leq \frac{\varphi_2(2x, 0)}{2^7}. \quad (2.6)$$

(2.5) combining (2.6) by use of the triangle inequality, we get

$$\|\frac{f(2^2x)}{2^6} - f(x)\| \leq \frac{\varphi_2(2x, 0)}{2^7} + \frac{\varphi_2(x, 0)}{2^4}. \quad (2.7)$$

Now proceed this way to prove by induction that

$$\|\frac{f(2^nx)}{2^{3n}} - f(x)\| \leq \frac{1}{16} \sum_{i=0}^{n-1} \frac{\varphi_2(2^ix, 0)}{2^{3i}}. \quad (2.8)$$
In order to show that the functions $D_n(x) = \frac{f(2^n x)}{2^{3n}}$ is a convergent sequence, we use from the Cauchy convergence criterion. Indeed, replace $x$ by $2^m x$ and divide by $2^{6m}$ in (2.8), where $m$ is an arbitrary positive integer. We find that
\[
\frac{\| f(2^{n+m} x) - f(2^n x) \|}{2^{3(n+m)}} \leq \frac{1}{16} \sum_{i=0}^{n-1} \frac{\varphi_2(2^{i+m} x, 0)}{2^{3(i+m)}} = \frac{1}{16} \sum_{i=m}^{n+m-1} \varphi_2(2^i x, 0)
\]
for all positive integers $m$ and $n$ with $n \geq m$ and all $x \in X$. Hence by the Cauchy criterion the limit $D(x) = \lim_{n \to \infty} D_n(x)$ exists for each $x \in A$. By taking the limit as $n \to \infty$ in (2.8), we see that $\|D(x) - f(x)\| \leq \frac{1}{16} \sum_{i=0}^{\infty} \frac{\varphi_2(2^i x, 0)}{2^{3i}} = \frac{1}{16} \Psi(x, 0)$ and (2.3) holds for all $x \in A$. In order to show that $D$ satisfies (1.2), replace $x$ by $2^n x$ and $y$ by $2^n y$ in (2.1) and divide by $2^{6n}$, to get
\[
\frac{\| f(2^n x^2 y) - f(2^n x) f(2^n y) - (2^n x)^3 f(2^n y) \|}{2^{6n}} \leq \frac{\varphi_1(2^n x, 2^n y)}{2^{6n}}.
\]
Taking the limit as $n \to \infty$, we find that $D$ satisfies (1.2). Now, suppose there is another such function $\hat{D} : A \to X$ satisfies $\Delta_{\hat{D}}(x, y) = 0$ and $\|D(x) - f(x)\| \leq \frac{1}{16} \Psi(x, 0)$. Then for all $x \in A$, we have
\[
\|D(x) - \hat{D}(x)\| = \frac{1}{2^{3n}} \|D(2^n x) - \hat{D}(2^n x)\|
\leq \frac{1}{2^{3n}} (\|D(2^n x) - f(2^n x)\| + \|\hat{D}(2^n x) - f(2^n x)\|)
\leq \frac{1}{2^{3n}} \left( \frac{1}{16} \Psi(2^n x, 0) + \frac{1}{16} \Psi(2^n x, 0) \right)
= \frac{1}{2^{3(n+1)}} \Psi(2^n x, 0) = \frac{1}{2^{3(n+1)}} \sum_{i=0}^{\infty} \frac{1}{2^{3i}} \varphi(2^{n+i} x, 0)
= \frac{1}{2^3} \sum_{i=0}^{\infty} \frac{1}{2^{3(n+i)}} \varphi(2^{n+i} x, 0) = \frac{1}{23} \sum_{i=n}^{\infty} \frac{1}{2^{3i}} \varphi(2^i x, 0).
\]
By $n \to \infty$ we get, $D(x) = \hat{D}(x)$. If we replace $x$ by $2^n x$ and $y$ by $2^n y$ in (2.2) and divide by $2^{6n}$
\[
\frac{\| f(2^{2n} x + 2^n y) - f(2^n x - 2^n y) \|}{2^{3n}} + \frac{f(2^{2n} x - 2^n y)}{2^{3n}} - 2 \frac{f(2^n x + 2^n y)}{2^{3n}} - 2 \frac{f(2^n x - 2^n y)}{2^{3n}}
\leq \frac{\varphi_2(2^n x, 2^n y)}{2^{3n}}.
\]
Taking the limit as $n \to \infty$, we find that $D$ satisfies (1.1).

**Corollary 2.2.** Let $\theta_1$ and $\theta_2$ be nonnegative real numbers, and let $p$ be real number such that $0 < p < 3$. Suppose that a mapping $f : A \to X$ satisfies
\[
\| f(xy) - f(x) y^3 - x^3 f(y) \| \leq \theta_1,
\]
Now, we have the following Hyers-Ulam-Rassias stability of cubic derivations.
On cubic derivations

\[ \|Δ_f(x, y)\| \leq θ_2(\|x\|^p + \|y\|^p) \]

for all \( x, y \in A \). Then there exists a unique cubic derivation \( D : A → X \) such that

\[ \|D(x) - f(x)\| \leq \frac{1}{16} \frac{θ_2 \|x\|^p}{1 - 2^{p-3}} \]

holds for all \( x \in A \).

**Proof.** In theorem 2.1, let \( ϕ_1(x, y) = θ_1 \) and \( ϕ_2(x, y) = θ_2(\|x\|^p + \|y\|^p) \) for all \( x, y \in A \).

**Corollary 2.3.** Let \( θ_1 \) and \( θ_2 \) be nonnegative real numbers. Suppose that a mapping \( f : A → X \) satisfies

\[ \|f(xy) - f(x)y^3 - x^3f(y)\| \leq θ_1 , \]

and

\[ \|Δ_f(x, y)\| \leq θ_2 \]

for all \( x, y \in A \). Then there exists a unique cubic derivation \( D : A → X \) such that

\[ \|D(x) - f(x)\| \leq \frac{θ_2}{14} \]

holds for all \( x \in A \).

**Proof.** In theorem 2.1, let \( ϕ_1(x, y) = θ_1 \) and \( ϕ_2(x, y) = θ_2 \) for all \( x, y \in A \).

**Corollary 2.4.** Let \( 0 < p < 3 \) and \( θ \) be a positive real number. Suppose \( f : A → X, \varphi : A^2 → \mathbb{R}^+ \) be maps such that

\[ \|f(xy) - f(x)y^3 - x^3f(y)\| \leq ϕ(x, y) , \]

and

\[ \|Δ_f(x, y)\| \leq θ\|y\|^p \]

(2.9)

for all \( x, y \in A \). Then \( f \) is a cubic derivation.

**Proof.** Letting \( x = y = 0 \) in (2.9), we get that \( f(0) = 0 \). So by \( y = 0 \), in (2.9), we get \( f(2x) = 2^3f(x) \) for all \( x \in A \). By using induction we have

\[ f(2^n x) = 2^{3n} f(x) \]

(2.10)

for all \( x \in A \) and \( n \in \mathbb{N} \). On the other hand by Theorem 2.1, the mapping \( D : A → X \) defined by

\[ D(x) = \lim_{n→∞} \frac{f(2^n x)}{2^{3n}} , \]

is a unique cubic derivation. Therefore it follows from (2.10) that \( f = D \). So the mapping \( f : A → X \) is a cubic derivation.
Corollary 2.5. Let $A$ be a unital Banach algebra with unit $e$, and $X$ be an $A$-module. Let $\theta$ be a nonnegative real number. Suppose that a mapping $f : A \to X$ satisfies
\[
\|f(2xy + z) + f(2xy - z) - 2[f(xy + z) + f(xy - z)] - 12[f(x)y^3 + x^3f(y)]\| < \theta \tag{2.11}
\]
for all $x, y, z \in A$, and also $f(e) = f(0) = 0$. Then there exists a unique cubic derivation $D : A \to X$ such that
\[
\|D(x) - f(x)\| \leq \frac{\theta}{14}
\]
holds for all $x \in A$.

Proof. By setting $x = e$ in (2.11), we obtain
\[
\|f(2y + z) + f(2y - z) - 2f(y + z) - 2f(y - z) - 12f(y)\| < \theta ,
\]
replacing $y$ by $xy$ and putting $z = 0$, we get
\[
\|f(2xy) - 2^3f(xy)\| < \theta . \tag{2.12}
\]
By setting $z = 0$ in (2.11), we have
\[
\|f(2xy) - 2^3[f(x)y^3 + x^3f(y)]\| < \theta \tag{2.13}
\]
So by (2.12) and (2.13) we get
\[
\|f(xy) - [f(x)y^3 + x^3f(y)]\| < \theta .
\]
Now by applying Corollary 2.3, we obtain the result. \qed

Theorem 2.6. Let $f : A \to X$ be a mapping and let $\varphi_1 : A^2 \to \mathbb{R}^+$, $\varphi_2 : A^2 \to \mathbb{R}^+$ be maps such that
\[
\|f(xy) - f(x)y^3 - x^3f(y)\| \leq \varphi_1(x, y) , \tag{2.14}
\]
and
\[
\|\Delta f(x, y)\| \leq \varphi_2(x, y) \tag{2.15}
\]
for all $x, y \in A$. Assume that the series
\[
\Psi(x, 0) = \sum_{i=0}^{\infty} 2^{3i}\varphi_2\left(\frac{x}{2^i}, 0\right)
\]
converges and that
\[
\lim_{n \to \infty} 2^{6n}\varphi_1\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = \lim_{n \to \infty} 2^{3n}\varphi_2\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 ,
\]
for all $x, y \in A$. Then there exists a unique cubic derivation $D : A \to X$ such that
\[
\|f(x) - D(x)\| \leq \frac{1}{16}\Psi(x, 0) \tag{2.16}
\]
for all $x \in A$. 

**Proof.** Setting $y = 0$ in (2.15) yields
\[
\|2f(2x) - 16f(x)\| \leq \varphi_2(x, 0). \tag{2.17}
\]
Replacing $x$ by $\frac{x}{2}$ in (2.17) and result divide by 2
\[
\|f(x) - 8f(\frac{x}{2})\| \leq \frac{1}{2}\varphi_2(\frac{x}{2}, 0) \tag{2.18}
\]
for all $x \in A$. Now proceed this way to prove by induction that
\[
\|f(x) - 2^{3n}f(\frac{x}{2^n})\| \leq \frac{1}{16}\sum_{i=1}^{n}2^{3i}\varphi_2(\frac{x}{2^i}, 0). \tag{2.19}
\]
In order to show that the functions $D_n(x) = 2^{3n}f(\frac{x}{2^n})$ is a convergent sequence, we use from the Cauchy convergence criterion. Indeed, replace $x$ by $\frac{x}{2^n}$ and multiplier by $2^{3m}$ in (2.19), where $m$ is an arbitrary positive integer. We find that
\[
\|2^{3m}f(\frac{x}{2^m}) - 2^{3(m+m)}f(\frac{x}{2^{m+m}})\| \leq \frac{1}{16}\sum_{i=1}^{n+m}2^{3i}\varphi_2(\frac{x}{2^i}, 0)
= \frac{1}{16}\sum_{i=1+m}^{n+m}2^{3i}\varphi_2(\frac{x}{2^i}, 0)
\]
for all positive integers $m$ and $n$ with $n \geq m$ and all $x \in X$. Hence by the Cauchy criterion the limit $D(x) = \lim_{n \to \infty} D_n(x)$ exists for each $x \in A$. By taking the limit as $n \to \infty$ in (2.19), we see that $\|f(x) - D(x)\| \leq \frac{1}{16}\sum_{i=1}^{\infty}2^{3i}\varphi_2(\frac{x}{2^i}, 0) = \frac{1}{16}\psi(x, 0)$ and (2.16) holds for all $x \in A$. The rest of proof is similar to the proof of Theorem 2.1.

Now, we investigate the superstability of cubic derivations as follows:

**Corollary 2.7.** Let $p > 3$ and $\theta$ be a positive real number. Let $f : A \to X$, $\varphi : A^2 \to \mathbb{R}^+$ be maps such that
\[
\|f(xy) - f(x)y^3 - x^3f(y)\| \leq \varphi(x, y),
\]
and
\[
\|\Delta f(x, y)\| \leq \theta\|y\|^p \tag{2.20}
\]
for all $x, y \in A$. Then $f$ is a cubic derivation.

**Proof.** Letting $x = y = 0$ in (2.20), we get that $f(0) = 0$. So by $y = 0$, in (2.20), we get $f(2x) = 2^3f(x)$ for all $x \in A$. By using induction we have
\[
f(x) = 2^{3n}f(\frac{x}{2^n}) \tag{2.21}
\]
for all $x \in A$ and $n \in \mathbb{N}$. On the other hand by Theorem 2.8, the mapping $D : A \to X$ defined by
\[
D(x) = \lim_{n \to \infty} 2^{3n}f(\frac{x}{2^n})
\]
is a unique cubic derivation. Therefore it follows from (2.21) that $f = D$. So the mapping $f : A \to X$ is a cubic derivation. \qed
**Corollary 2.8.** Let \( p, q, \theta \) be a positive real numbers such that \( p + q > 3 \). Let \( f : A \rightarrow X, \varphi : A^2 \rightarrow X \) be maps such that
\[
\|f(xy) - f(x)y^3 - x^3f(y)\| \leq \varphi(x,y),
\]
and
\[
\|\Delta f(x)\| \leq \theta\|x\|^p\|y\|^p
\]
for all \( x, y \in A \). Then \( f \) is a cubic derivation.

*Proof.* If \( q = 0 \), then by Corollary 2.9 we get the result. Else by the same reasoning as in the proof of (2.15), the mapping \( f : A \rightarrow X \), is cubic derivation. \( \square \)

**Corollary 2.9.** Let \( p > 3 \) and \( \theta \) be a positive real number. Suppose mapping \( f : A \rightarrow X \) satisfies
\[
\|f(xy) - f(x)y^3 - x^3f(y)\| \leq \theta\|y\|^p,
\]
and
\[
\|\Delta f(x,y)\| \leq \theta\|y\|^p
\]
for all \( x, y \in A \). Then \( f \) is a cubic derivation.

*Proof.* Let \( \varphi(x,y) = \theta\|y\|^p \). Then by Corollary 2.9 and 2.10, we get the result. \( \square \)

**Example 2.10.** Let \( x \in X \) be fixed and \( x \neq 0 \). We define \( f : A \rightarrow X \) by \( f(a) := a^3x - xa^3 + x \) and
\[
\varphi_1(a,b) := \|f(ab) - a^3f(b) - f(a)b^3\| = \|x - a^3x - xb^3\|,
\]
and
\[
\varphi_2(a,b) := \|f(2a + b) + f(2a - b) - 2f(a + b) - 2f(a - b) - 12f(a)\| = 14\|x\|.
\]
Then we have
\[
\sum_{i=0}^{\infty} \frac{\varphi_2(2i^2a,2i^2b)}{2^{3i}} = \sum_{i=0}^{\infty} \frac{14\|x\|}{2^{3i}} = 16\|x\|,
\]
and
\[
\lim_{n \rightarrow \infty} \varphi_1(2^n a, 2^n b) = \lim_{n \rightarrow \infty} \frac{\|x - x2^n b^3 - 2^n a^3 x\|}{2^{6n}} = 0.
\]
Thus the limit \( D(a) = \lim_{n \rightarrow \infty} \frac{f(2^n a)}{2^n} = a^3x - xa^3 \) exists. Also,
\[
D(ab) = (ab)^3x - x(ab)^3 = a^3b^3 x - x a^3 b^3,
\]
and
\[
a^3 D(b) + D(a)b^3 = a^3(b^3x - xb^3) + (a^3x - xa^3)b^3 = a^3b^3x - xa^3b^3.
\]
Thus (1.3) holds. Furthermore,
\[
D(2a + b) + D(2a - b) = [(2a + b)^3x - x(2a + b)^3] + [(2a - b)^3x - x(2a - b)^3]
\]
\[
= 2.2^3a^3x + 6.2ab^2x - 2.2^3xa^3 - 6.2xab^2,
\]
on the other hand
\[2[D(a+b) + D(a-b)] + 12D(a)\]
\[= 2[((a+b)^3x - x(a+b)^3) + ((a-b)^3x - x(a-b)^3)] + 12[a^3x - xa^3]\]
\[= 2[2a^3x + 6ab^2x - 2xa^3 - 6xab^2] + 12[a^3x - xa^3]\]
\[= 2.2^3a^3x - 2.2^3xa^3 + 6.2ab^2x - 6.2xab^2.\]

Then \(D\) is cubic.

Also from this example it is clear that the superstability of the functional equation
\[f(2xy + z) + f(2xy - z) = 2[f(xy + z) + f(xy - z)] + 12(x^3f(y) + f(x)y^3)\]
with the control functions in Theorem 2.6, does not hold.

REFERENCES


On cubic derivations


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