Some Common Coupled Fixed Point Results in Cone Metric Spaces

W. Shatanawi

Dept. of Mathematics, Hashemite University
P.O. Box 150459, Zarqa 13115, Jordan
swasfi@hu.edu.jo

Abstract
We introduce the concept of a common coupled fixed point of the mappings $F, G : X \times X \to X$, and we obtain some results for contractive mappings in cone metric space with a cone having nonempty interior. Our results generalize well known results in the literature.

Mathematics Subject Classifications: 54H25, 47H10, 54E50

Keywords: Common fixed point, Coupled fixed point, Cone metric space

1 Introduction.

The concept of cone metric space has been investigated initially by Huang and Zhang [6]. They proved some fixed point theorems of contractive type mappings over cone metric spaces. Later, many authors generalized their fixed point theorems in different type. For a survey of coincidence point theory over cone metric spaces, we refer the reader (as examples) to [1-4,6-14]. While V. Bhaskar and Lakshmikantham [5] introduced the concept of a coupled fixed point of a mapping $F$ from $X \times X$ into $X$ and studied fixed point theorems in partially ordered metric spaces. Recently, Sabetghadam et al [13], studied some results of the coupled fixed point for mappings satisfying different contractive conditions on complete metric spaces. in this paper, we introduce the concept of a common coupled fixed point of the mappings $F, G : X \times X \to X$, and we obtain some results for nonlinear contractive mappings in cone metric space with a cone having nonempty interior.

2 Basic Concepts.

In the present paper, $E$ stands for a real Banach space. Let $P$ be a subset of $E$ with $\text{Int}(P) \neq \emptyset$. Then $P$ is called a cone if the following conditions are
satisfied:

1. $P$ is closed and $P \neq \{\theta\}$.
2. $a, b \in \mathbb{R}^+, x, y \in P$ implies $ax + by \in P$.
3. $x \in P \cap -P$ implies $x = \theta$.

For a cone $P$, define a partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{Int}P$. It can be easily shown that $\lambda \text{Int}(P) \subseteq \text{Int}(P)$ for all positive scalar $\lambda$.

**Definition 2.1** [6] Let $X$ be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

1. $\theta < d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y \in X$.

Then $d$ is called a cone metric on $X$, and $(X, d)$ is called a cone metric space.

**Definition 2.2** [6] Let $(X, d)$ be a cone metric space. Let $(x_n)$ be a sequence in $X$ and $x \in X$. If for every $c \in E$ with $\theta \ll c$, there is an $N \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq N$, then $(x_n)$ is said to be convergent and $(x_n)$ converges to $x$ and $x$ is the limit of $(x_n)$. We denote this by $\lim_{n \to +\infty} x_n = x$ or $x_n \to x$ as $n \to +\infty$. If for every $c \in E$ with $\theta \ll c$ there is an $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$, then $(x_n)$ is called a Cauchy sequence in $X$. The space $(X, d)$ is called a complete cone metric space if every Cauchy sequence is convergent.

The cone $P$ in a real Banach space $E$ is called normal if there is a number $k > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \text{ implies } ||x|| \leq k||y||.$$ 

Rezapour and Hamlbarani[12] proved that there are no normal cones with normal constant $k < 1$ and that for each $h > 1$ there are cones with normal constant $K > h$. Also, they omitting the assumption of normality they obtain generalizations of some results of [6].

Let $(X, d)$ be a cone metric space with cone $P$ not necessary to be normal. Then the following properties are useful in our subsequent arguments:

1. If $a \leq ha$ and $h \in [0, 1)$, then $a = 0$
2. If \( \theta \leq u \ll c \) for each \( \theta \ll c \), then \( u = \theta \).

3. If \( u \leq v \) and \( v \ll w \), then \( u \ll w \).

**Definition 2.3** [5] An element \((x, y) \in X \times X\) is called a coupled fixed point of a mapping \(F : X \times X \rightarrow X\) if \(F(x, y) = x\) and \(F(y, x) = y\).

### 3 Main Results.

In order to proceed in our work and achieve our results we introduce the following definition.

**Definition 3.1** Let \(X\) be a nonempty set. Then the point \((x, y)\) in \(X \times X\) is called a common coupled fixed point of the mapping \(F, G : X \times X \rightarrow X\) if

\[
F(x, y) = G(x, y) = x \quad \text{and} \quad F(y, x) = G(y, x) = y.
\]

**Theorem 3.1** Let \((X, d)\) be a complete cone metric space with a cone \(P\) having nonempty interior. Let \(F, G : X \times X \rightarrow X\) be functions such that

\[
d(F(x, y), G(u, v)) \leq hw(x, y, u, v)
\]

for all \(x, y, u, v \in X\), where

\[
w(x, y, u, v) \in \left\{d(x, u), d(y, v), \frac{1}{2}(d(F(x, y), x) + d(G(u, v), u)), \frac{1}{2}(d(F(x, y), u) + d(G(u, v), x))\right\}.
\]

If \(0 \leq h < 1\), then \(F, G\) have a unique common coupled fixed point.

**Proof.** Let \(x_0, y_0\) be two arbitrary elements in \(X\). Choose \(x_1, y_1 \in X\) such that \(x_1 = F(x_0, y_0)\) and \(y_1 = F(y_0, x_0)\). Again choose \(x_2, y_2 \in X\) such that \(x_2 = G(x_1, y_1)\) and \(y_2 = G(y_1, x_1)\). Continuing this process, we can construct two sequences \((x_n)\) and \((y_n)\) in \(X\) such that \(x_{2n+1} = F(x_{2n}, y_{2n})\), \(y_{2n+1} = F(y_{2n}, x_{2n})\), \(x_{2n+2} = G(x_{2n+1}, y_{2n+1})\), and \(y_{2n+2} = G(y_{2n+1}, x_{2n+1})\). Let \(n \in \mathbb{N} \cup \{0\}\).

**Case 1:** \(u(x, y, u, v) = d(x, u)\). From

\[
d(x_{2n+1}, x_{2n+2}) = d(F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1})) \leq hd(x_{2n}, x_{2n+1}),
\]

and

\[
d(y_{2n+1}, y_{2n+2}) = d(F(y_{2n}, x_{2n}), G(y_{2n+1}, x_{2n+1})) \leq hd(y_{2n}, y_{2n+1}),
\]

we have

\[
d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \leq h(d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})).
\]
Case 2: \( u(x, y, u, v) = d(y, v) \). Similar arguments to Case 1, we have
\[
d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \leq h(d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})).
\]

Case 3: \( u(x, y, u, v) = \frac{1}{2}(d(F(x, y), x) + d(G(u, v), u) \). From
\[
d(x_{2n+1}, x_{2n+2}) \leq \frac{h}{2}((d(x_{2n+1}, x_{2n}) + d(x_{2n+2}, x_{2n+1})),
\]
we get
\[
d(x_{2n+1}, x_{2n+2}) \leq \frac{h}{2-h}(d(x_{2n+1}, x_{2n})).
\]
(1)

Similarly, we have
\[
d(y_{2n+1}, y_{2n+2}) \leq \frac{h}{2-h}(d(y_{2n+1}, y_{2n})).
\]
(2)

From Equation (1) and Equation (2), we get
\[
d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \leq \frac{h}{2-h}(d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})).
\]
Case 4: \( u(x, y, u, v) = \frac{1}{2}(d(F(x, y), u) + d(G(u, v), x) \). As in Case 3, we get
\[
d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \leq \frac{h}{2-h}(d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})).
\]

Let \( r = \max\{h, \frac{h}{2-h}\} \). Then in all case, we get
\[
d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \leq r(d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})).
\]

If we repeat all above arguments for the four cases, we get
\[
d(x_{2n+1}, x_{2n}) + d(y_{2n+1}, y_{2n}) \leq r(d(x_{2n}, x_{2n-1}) + d(y_{2n}, y_{2n-1})).
\]

Hence
\[
d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \leq r(d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1}))
\]
\[
\leq r^2(d(x_{2n}, x_{2n-1}) + d(y_{2n}, y_{2n-1}))
\]
\[
\vdots
\]
\[
\leq r^{2n+1}(d(x_0, x_1) + d(y_0, y_1)).
\]

So for each \( n \in \mathbb{N} \), we have
\[
d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq r^n(d(x_0, x_1) + d(y_0, y_1)).
\]
(3)
If \( d(x_0, x_1) + d(y_0, y_1) = \theta \), then \( y_0 = y_1 \) and \( x_0 = x_1 \). By inequality (3), we get that \( x_0 = x_n \) and \( y_0 = y_n \) for each \( n \in \mathbb{N} \). Hence \( x_0 = x_1 = F(x_0, y_0) \) and \( y_0 = y_1 = F(y_0, x_0) \). Now, we show that \( G(x_0, y_0) = x_0 \) and \( G(y_0, x_0) = y_0 \). Since
\[
 w(x_0, y_0, x_0, y_0) \in \{ d(x_0, x_0), d(y_0, y_0), \frac{1}{2}(d(x_0, x_0) + d(G(x_0, y_0), x_0)), \\
 \frac{1}{2}((x_0, x_0) + d(G(x_0, y_0), x_0)) \},
\]
we have
\[
 d(x_0, G(x_0, y_0)) = d(F(x_0, y_0), G(x_0, y_0)) \leq \frac{h}{2} d(x_0, G(x_0, y_0)).
\]
From the last inequality and the fact that \( h < 1 \), we get \( d(x_0, G(x_0, y_0)) = \theta \), and hence \( x_0 = G(x_0, y_0) \). Similarly, we may show that \( y_0 = G(y_0, x_0) \).

Therefore, \((x_0, y_0)\) is a common coupled fixed point of \( F \) and \( G \). Thus we may assume that \( d(x_0, x_1) + d(y_0, y_1) \neq \theta \). For \( m > n \) we get
\[
 d(x_n, x_m) \leq d(x_n, x_{n+1}) + \ldots + d(x_{m-1}, x_m) \text{ and } d(y_n, y_m) \leq d(y_n, y_{n+1}) + \ldots + d(y_{m-1}, y_m).
\]
By Inequality (3) and the fact that \( r < 1 \), we have
\[
 d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq \frac{r^n}{1-r} (d(x_0, x_1) + d(y_0, y_1)) \rightarrow \theta \text{ as } n \rightarrow +\infty.
\]
Thus for \( c \gg \theta \), we can find \( k \in \mathbb{N} \) such that
\[
 \frac{r^n}{1-r} (d(x_0, x_1) + d(y_0, y_1)) \ll c
\]
for all \( n \geq k \). Hence \( d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \ll c \) for all \( n \geq k \). Since
\[
 d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \text{ and } d(y_n, y_{n+1}) \leq d(x_n, x_{n+1}) + d(y_n, y_{n+1}),
\]
we conclude that \((x_n)\) and \((y_n)\) are Cauchy’s sequences in \((X, d)\). Since \( X \) is complete, we find \( x, y \) in \( X \) such that \( x_n \rightarrow x \) and \( y_n \rightarrow y \). Now, we prove that \( F(x, y) = G(x, y) = x \) and \( F(y, x) = G(y, x) = x \). For that
\[
 d(F(x, y), x) \leq d(F(x, y), x_{2n+2}) + d(x_{2n+2}, x). \quad (4)
\]
But \( d(F(x, y), x_{2n+2}) = d(F(x, y), G(x_{2n+1}, y_{2n+1})) \).

Case 1: If \( w(x, y, u, v) = d(x, u) \), then \( d(F(x, y), x_{2n+2}) \leq \theta d(x, x_{2n+1}) \). By Inequality (4), we have \( d(F(x, y), x) \leq \theta d(x, x_{2n+1}) + d(x_{2n+2}, x) \). Since \((x_{2n+1})\) and \((x_{2n+2})\) are subsequences of \((x_n)\), we get \((x_{2n+1})\) and \((x_{2n+2})\) converge to \( x \). Let \( c \gg \theta \). Then there are \( k_1, k_2 \in \mathbb{N} \) such that \( d(x, x_{2n+1}) \ll \frac{c}{2} \) for all \( n \geq k_1 \) and \( d(x_{2n+2}, x) \leq \frac{c}{2} \) for all \( n \geq k_2 \). Let \( k_0 = \max\{k_1, k_2\} \). Then
\[ d(F(x, y), x_{2n+2}) + d(x_{2n+2}, x) \ll c \text{ for all } n \geq k_0. \text{ Hence } d(F(x, y), x) \ll c. \]

Therefore \( F(x, y) = x \).

Case 2: If \( w(x, y, u, v) = d(y, v) \), then \( d(F(x, y), x_{2n+2}) \leq hd(y, y_{2n+1}) \). By inequality (4), we have \( d(F(x, y), x) \leq hd(y, y_{2n+1}) + d(x_{2n+2}, x) \). Noting that the sequences \((y_{2n+1}) \) and \((x_{2n+2}) \) converge to \( y \) and \( x \) respectively. By similar argument to Case 1, we conclude that \( F(x, y) = x \).

Case 3: If \( w(x, y, u, v) = \frac{1}{2}(d(F(x, y), x) + d(G(u, v), u)) \), then

\[ d(F(x, y), x_{2n+2}) \leq \frac{h}{2}(d(F(x, y), x) + d(x_{2n+2}, x_{2n+1})). \]

Since \( d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n+1}, x) + d(x, x_{2n+2}) \), we have

\[ d(F(x, y), x_{2n+2}) \leq \frac{h}{2}(d(F(x, y), x) + d(x_{2n+2}, x)) + d(x, x_{2n+1}). \]

By Inequality 4, we have

\[ d(F(x, y), x) \leq \frac{2 + h}{2 - h}d(x_{2n+2}, x) + \frac{h}{2 - h}d(x, x_{2n+1}). \]

As similar arguments to Case 1, we get \( F(x, y) = x \).

Case 3: If \( w(x, y, u, v) = \frac{1}{2}(d(F(x, y), u) + d(F(u, v), x)) \), then

\[ d(F(x, y), x_{2n+2}) \leq \frac{h}{2}(d(F(x, y), x_{2n+1}) + d(x_{2n+2}, x)). \]

Since \( d(F(x, y), x_{2n+1}) \leq d(F(x, y), x) + d(x, x_{2n+1}) \), by Inequality 4, we have

\[ d(F(x, y), x) \leq \frac{2 + h}{2 - h}d(x_{2n+2}, x) + \frac{h}{2 - h}d(x, x_{2n+1}). \]

As similar arguments to Case 1, we get \( F(x, y) = x \). By the aid of following inequality:

\[ d(x, G(x, y)) \leq d(x, x_{2n+1}) + d(x_{2n+1}, G(x, y)) = d(x, x_{2n+1}) + d(F(x_{2n}, y_{2n}), G(x, y)) \]

and repeat the above arguments for the four cases, we can show that \( G(x, y) = x \). Hence \( F(x, y) = G(x, y) = x \). Similarly, we get \( F(y, x) = G(y, x) = y \). Therefore \((x, y) \) is a common coupled fixed point of the mappings \( F \) and \( G \). Moreover, we show that \( x = y \). For that \( d(x, y) = d(F(x, y), G(y, x)) \leq hw(x, y, y, x) \). Since

\[ w(x, y, y, x) \in \{d(x, y), d(y, x), \frac{1}{2}(d(x, x) + d(y, y)), \frac{1}{2}(d(x, y) + d(y, x))\}, \]

and \( h < 1 \), we conclude that \( d(x, y) = 0 \) and hence \( x = y \).

Our result is an improvement of the following results:
Corollary 3.1 [Theorem 2.5, 13] Let \((X, d)\) be a complete cone metric space. Suppose that the mapping \(F : X \times X \rightarrow X\) satisfies
\[
d(F(x, y), F(u, v)) \leq a d(F(x, y), x) + b d(F(u, v), u)
\]
for all \(x, y, u, v \in X\). If \(a, b\) are nonnegative real numbers and if \(a + b \in [0, 1)\), then \(F\) has a unique coupled fixed point.

Proof. Note that if
\[
d(F(x, y), F(u, v)) \leq a d(F(x, y), x) + b d(F(u, v), u),
\]
then
\[
d(F(x, y), F(u, v)) \leq a + b \frac{d(F(x, y), x) + d(F(u, v), u)}{2}.
\]
Thus the result follows from Theorem 3.1 by taking \(G = F\).

Corollary 3.2 [Theorem 2.6, 13] Let \((X, d)\) be a complete cone metric space. Suppose that the mapping \(F : X \times X \rightarrow X\) satisfies
\[
d(F(x, y), F(u, v)) \leq a d(F(x, y), u) + b d(F(u, v), x)
\]
for all \(x, y, u, v \in X\). If \(a, b\) are nonnegative real numbers \(a + b < 1\), then \(F\) has a unique coupled fixed point.

Proof. Note that if
\[
d(F(x, y), F(u, v)) \leq a d(F(x, y), u) + b d(F(u, v), x),
\]
then
\[
d(F(x, y), F(u, v)) \leq a + b \frac{d(F(x, y), u) + d(F(u, v), x)}{2}.
\]
Thus the result follows from Theorem 3.1 by taking \(G = F\).

References


Received: June, 2010