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Some Common Coupled Fixed Point Results in Cone Metric Spaces

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Abstract

We introduce the concept of a common coupled fixed point of the mappings $F, G : X \times X \to X$, and we obtain some results for contractive mappings in cone metric space with a cone having nonempty interior. Our results generalize well known results in the literature.

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1 Introduction.

The concept of cone metric space has been investigated initially by Huang and Zhang [6]. They proved some fixed point theorems of contractive type mappings over cone metric spaces. Later, many authors generalized their fixed point theorems in different type. For a survey of coincidence point theory over cone metric spaces, we refer the reader (as examples) to [1-4,6-14]. While V. Bhaskar and Lakshmikantham [5] introduced the concept of a coupled fixed point of a mapping F from $X \times X$ into X and studied fixed point theorems in partially ordered metric spaces. Recently, Sabetghadam et al [13], studied some results of the coupled fixed point for mappings satisfying different contractive conditions on complete metric spaces. in this paper, we introduce the concept of a common coupled fixed point of the mappings $F, G : X \times X \to X$, and we obtain some results for nonlinear contractive mappings in cone metric space with a cone having nonempty interior.

2 Basic Concepts.

In the present paper, E stands for a real Banach space. Let P be a subset of E with $Int(P) \neq \emptyset$. Then P is called a cone if the following conditions are satisfied:

1. P is closed and $P \neq \{\theta\}$.
2. $a, b \in \mathbf{R}^+$, $x, y \in P$ implies $ax + by \in P$.
3. $x \in P \cap -P$ implies $x = \theta$.

For a cone P, define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in IntP$. It can be easily shown that $\lambda Int(P) \subseteq Int(P)$ for all positive scalar λ.

Definition 2.1 [6] *Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies*

1. *$\theta < d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$.*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$.*
3. *$d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y \in X$.*

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 2.2 [6] *Let (X, d) be a cone metric space. Let (x_n) be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$, there is an $N \in \mathbf{N}$ such that $d(x_n, x) \ll c$ for all $n \geq N$, then (x_n) is said to be convergent and (x_n) converges to x and x is the limit of (x_n). We denote this by $\lim_{n \to +\infty} x_n = x$ or $x_n \to x$ as $n \to +\infty$. If for every $c \in E$ with $\theta \ll c$ there is an $N \in \mathbf{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$, then (x_n) is called a Cauchy sequence in X. The space (X, d) is called a complete cone metric space if every Cauchy sequence is convergent.*

The cone P in a real Banach space E is called normal if there is a number $k > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \text{ implies } ||x|| \leq k||y||.$$

Rezapour and Hamlbarani[12] proved that there are no normal cones with normal constant $k < 1$ and that for each $h > 1$ there are cones with normal constant $K > h$. Also, they omitting the assumption of normality they obtain generalizations of some results of [6].

Let (X, d) be a cone metric space with cone P not necessary to be normal. Then the following properties are useful in our subsequent arguments:

1. If $a \leq ha$ and $h \in [0, 1)$, then $a = 0$

2. If $\theta \leq u \ll c$ for each $\theta \ll c$, then $u = \theta$.

3. If $u \leq v$ and $v \ll w$, then $u \ll w$.

Definition 2.3 [5] *An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

3 Main Results.

In order to proceeds in our work and achieve our results we introduce the following definition.

Definition 3.1 *Let X be a nonempty set. Then the point (x, y) in $X \times X$ is called a common coupled fixed point of the mapping $F, G : X \times X \to X$ if*

$$F(x, y) = G(x, y) = x \text{ and } F(y, x) = G(y, x) = y.$$

Theorem 3.1 *Let (X, d) be a complete cone metric space with a cone P having nonempty interior. Let $F, G : X \times X \to X$ be functions such that*

$$d(F(x, y), G(u, v)) \leq hw(x, y, u, v)$$

for all $x, y, u, v \in X$, where

$$w(x,y,u,v) \in \left\{ d(x,u), d(y,v), \frac{1}{2}(d(F(x,y),x) + d(G(u,v),u)), \frac{1}{2}(d(F(x,y),u) + d(G(u,v),x)) \right\}.$$

If $0 \leq h < 1$, then F, G have a unique common coupled fixed point.

Proof. Let x_0, y_0 be two arbitrary elements in X. Choose $x_1, y_1 \in X$ such that $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Again choose $x_2, y_2 \in X$ such that $x_2 = G(x_1, y_1)$ and $y_2 = G(y_1, x_1)$. Continuing this process, we can construct two sequences (x_n) and (y_n) in X such that $x_{2n+1} = F(x_{2n}, y_{2n})$, $y_{2n+1} = F(y_{2n}, x_{2n})$, $x_{2n+2} = G(x_{2n+1}, y_{2n+1})$, and $y_{2n+2} = G(y_{2n+1}, x_{2n+1})$. Let $n \in \mathbf{N} \cup \{\mathbf{0}\}$.
Case 1: $u(x, y, u, v) = d(x, u)$. From

$$d(x_{2n+1}, x_{2n+2}) = d(F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1})) \leq hd(x_{2n}, x_{2n+1}),$$

and

$$d(y_{2n+1}, y_{2n+2}) = d(F(y_{2n}, x_{2n}), G(y_{2n+1}, x_{2n+1})) \leq hd(y_{2n}, y_{2n+1}),$$

we have

$$d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \leq h(d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})).$$

Case 2: $u(x,y,u,v) = d(y,v)$. Similar arguments to Case 1, we have

$$d(x_{2n+1},x_{2n+2}) + d(y_{2n+1},y_{2n+2}) \leq h(d(x_{2n},x_{2n+1}) + d(y_{2n},y_{2n+1})).$$

Case 3: $u(x,y,u,v) = \frac{1}{2}(d(F(x,y),x) + d(G(u,v),u)$. From

$$d(x_{2n+1},x_{2n+2}) \leq \frac{h}{2}((d(x_{2n+1},x_{2n}) + d(x_{2n+2},x_{2n+1})),$$

we get

$$d(x_{2n+1},x_{2n+2}) \leq \frac{h}{2-h}(d(x_{2n+1},x_{2n}). \tag{1}$$

Similarly, we have

$$d(y_{2n+1},y_{2n+2}) \leq \frac{h}{2-h}(d(y_{2n+1},y_{2n}). \tag{2}$$

From Equation (1) and Equation (2), we get

$$d(x_{2n+1},x_{2n+2}) + d(y_{2n+1},y_{2n+2}) \leq \frac{h}{2-h}(d(x_{2n},x_{2n+1}) + d(y_{2n},y_{2n+1})).$$

Case 4: $u(x,y,u,v) = \frac{1}{2}(d(F(x,y),u) + d(G(u,v),x)$. As in Case 3, we get

$$d(x_{2n+1},x_{2n+2}) + d(y_{2n+1},y_{2n+2}) \leq \frac{h}{2-h}(d(x_{2n},x_{2n+1}) + d(y_{2n},y_{2n+1})).$$

Let $r = \max\{h, \frac{h}{2-h}\}$. Then in all case, we get

$$d(x_{2n+1},x_{2n+2}) + d(y_{2n+1},y_{2n+2}) \leq r(d(x_{2n},x_{2n+1}) + d(y_{2n},y_{2n+1})).$$

If we repeat all above arguments for the four cases, we get

$$d(x_{2n+1},x_{2n}) + d(y_{2n+1},y_{2n}) \leq r(d(x_{2n},x_{2n-1}) + d(y_{2n},y_{2n-1})).$$

Hence

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) + d(y_{2n+1},y_{2n+2}) &\leq r(d(x_{2n},x_{2n+1}) + d(y_{2n},y_{2n+1})) \\ &\leq r^2(d(x_{2n},x_{2n-1}) + d(y_{2n},y_{2n-1})) \\ &\vdots \\ &\leq r^{2n+1}(d(x_0,x_1) + d(y_0,y_1)). \end{aligned}$$

So for each $n \in \mathbf{N}$, we have

$$d(x_n,x_{n+1}) + d(y_n,y_{n+1}) \leq r^n(d(x_0,x_1) + d(y_0,y_1)). \tag{3}$$

If $d(x_0, x_1) + d(y_0, y_1) = \theta$, then $y_0 = y_1$ and $x_0 = x_1$. By inequality (3), we get that $x_0 = x_n$ and $y_0 = y_n$ for each $n \in \mathbf{N}$. Hence $x_0 = x_1 = F(x_0, y_0)$ and $y_0 = y_1 = F(y_0, x_0)$. Now, we show that $G(x_0, y_0) = x_0$ and $G(y_0, x_0) = y_0$. Since

$$w(x_0, y_0, x_0, y_0) \in \{d(x_0, x_0), d(y_0, y_0), \frac{1}{2}(d(x_0, x_0) + d(G(x_0, y_0), x_0)), \frac{1}{2}((x_0, x_0) + d(G(x_0, y_0), x_0))\},$$

we have

$$d(x_0, G(x_0, y_0)) = d(F(x_0, y_0), G(x_0, y_0)) \leq \frac{h}{2} d(x_0, G(x_0, y_0)).$$

From the last inequality and the fact that $h < 1$, we get $d(x_0, G(x_0, y_0)) = \theta$, and hence $x_0 = G(x_0, y_0)$. Similarly, we may show that $y_0 = G(y_0, x_0)$. Therefore, (x_0, y_0) is a common coupled fixed point of F and G. Thus we may assume that $d(x_0, x_1) + d(y_0, y_1) \neq \theta$. For $m > n$ we get

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + \ldots + d(x_{m-1}, x_m) \text{ and } d(y_n, y_m) \leq d(y_n, y_{n+1}) + \ldots + d(y_{m-1}, y_m).$$

By Inequality (3) and the fact that $r < 1$, we have

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq \frac{r^n}{1-r}(d(x_0, x_1) + d(y_0, y_1)) \to \theta \text{ as } n \to +\infty.$$

Thus for $c \gg \theta$, we can find $k \in \mathbf{N}$ such that

$$\frac{r^n}{1-r}(d(x_0, x_1) + d(y_0, y_1)) \ll c$$

for all $n \geq k$. Hence $d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \ll c$ for all $n \geq k$. Since $d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) + d(y_n, y_{n+1})$ and $d(y_n, y_{n+1}) \leq d(x_n, x_{n+1}) + d(y_n, y_{n+1})$, we conclude that (x_n) and (y_n) are Cauchy's sequences in (X, d). Since X is complete, we find x, y in X such that $x_n \to x$ and $y_n \to y$. Now, we prove that $F(x, y) = G(x, y) = x$ and $F(y, x) = G(y, x) = x$. For that

$$d(F(x, y), x) \leq d(F(x, y), x_{2n+2}) + d(x_{2n+2}, x). \tag{4}$$

But $d(F(x, y), x_{2n+2}) = d(F(x, y), G(x_{2n+1}, y_{2n+1}))$.

Case 1: If $w(x, y, u, v) = d(x, u)$, then $d(F(x, y), x_{2n+2}) \leq hd(x, x_{2n+1})$. By Inequality (4), we have $d(F(x, y), x) \leq hd(x, x_{2n+1}) + d(x_{2n+2}, x)$. Since (x_{2n+1}) and (x_{2n+2}) are subsequences of (x_n), we get (x_{2n+1}) and (x_{2n+2}) converge to x. Let $c \gg \theta$. Then there are $k_1, k_2 \in \mathbf{N}$ such that $d(x, x_{2n+1}) \ll \frac{c}{2h}$ for all $n \geq k_1$ and $d(x_{2n+2}, x) \leq \frac{c}{2}$ for all $n \geq k_2$. Let $k_0 = \max\{k_1, k_2\}$. Then

$d(F(x,y),x_{2n+2})+d(x_{2n+2},x)\ll c$ for all $n\geq k_0$. Hence $d(F(x,y),x)\ll c$. Therefore $F(x,y)=x$.

Case 2: If $w(x,y,u,v)=d(y,v)$, then $d(F(x,y),x_{2n+2})\leq hd(y,y_{2n+1})$. By inequality (4), we have $d(F(x,y),x)\leq hd(y,y_{2n+1})+d(x_{2n+2},x)$. Noting that the sequences (y_{2n+1}) and (x_{2n+2}) converge to y and x respectively. By similar argument to Case 1, we conclude that $F(x,y)=x$
Case 3: If $w(x,y,u,v)=\frac{1}{2}(d(F(x,y),x)+d(G(u,v),u)$, then

$$d(F(x,y),x_{2n+2})\leq\frac{h}{2}(d(F(x,y),x)+d(x_{2n+2},x_{2n+1})).$$

Since $d(x_{2n+1},x_{2n+2})\leq d(x_{2n+1},x)+d(x,x_{2n+2})$, we have

$$d(F(x,y),x_{2n+2})\leq\frac{h}{2}(d(F(x,y),x)+d(x_{2n+2},x))+d(x,x_{2n+1})).$$

By Inequality 4, we have

$$d(F(x,y),x)\leq\frac{2+h}{2-h}d(x_{2n+2},x)+\frac{h}{2-h}d(x,x_{2n+1}).$$

As similar arguments to Case 1, we get $F(x,y)=x$.

Case 3: If $w(x,y,u,v)=\frac{1}{2}(d(F(x,y),u)+d(F(u,v),x)$,then

$$d(F(x,y),x_{2n+2})\leq\frac{h}{2}(d(F(x,y),x_{2n+1})+d(x_{2n+2},x)).$$

Since $d(F(x,y),x_{2n+1})\leq d(F(x,y),x)+d(x,x_{2n+1})$, by Inequality 4, we have

$$d(F(x,y),x)\leq\frac{2+h}{2-h}d(x_{2n+2},x)+\frac{h}{2-h}d(x,x_{2n+1}).$$

As similar arguments to Case 1, we get $F(x,y)=x$. By the aid of following inequality:

$$d(x,G(x,y))\leq d(x,x_{2n+1})+d(x_{2n+1},G(x,y))=d(x,x_{2n+1})+d(F(x_{2n},y_{2n}),G(x,y))$$

and repeat the above arguments for the four cases, we can show that $G(x,y)=x$. Hence $F(x,y)=G(x,y)=x$. Similarly, we get $F(y,x)=G(y,x)=y$. Therefore (x,y) is a common coupled fixed point of the mappings F and G. Moreover, we show that $x=y$. For that $d(x,y)=d(F(x,y),G(y,x))\leq hw(x,y,y,x)$. Since

$$w(x,y,y,x)\in\{d(x,y),d(y,x),\frac{1}{2}(d(x,x)+d(y,y)),\frac{1}{2}(d(x,y)+d(y,x)),$$

and $h<1$, we conclude that $d(x,y)=\theta$ and hence $x=y$. ■

Our result is an improvement of the following results:

Corollary 3.1 [Theorem 2.5, 13] *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies*

$$d(F(x, y), F(u, v)) \leq ad(F(x, y), x) + bd(F(u, v), u))$$

for all $x, y, u, v \in X$. If a, b are nonnegative real numbers and if $a + b \in [0, 1)$, then F has a unique coupled fixed point.

Proof. Note that if

$$d(F(x, y), F(u, v) \leq ad(F(x, y), x) + bd(F(u, v), u),$$

then

$$d(F(x, y), F(u, v) \leq \frac{a+b}{2}(d(F(x, y), x) + d(F(u, v), u)).$$

Thus the result follows from Theorem 3.1 by taking $G = F$. ■

Corollary 3.2 [Theorem 2.6, 13] *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies*

$$d(F(x, y), F(u, v)) \leq ad(F(x, y), u) + bd(F(u, v), x)$$

for all $x, y, u, v \in X$. If a, b are nonnegative real numbers $a + b < 1$, then F has a unique coupled fixed point.

Proof. Note that if

$$d(F(x, y), F(u, v) \leq ad(F(x, y), u) + bd(F(u, v), x),$$

then

$$d(F(x, y), F(u, v) \leq \frac{a+b}{2}(d(F(x, y), u) + d(F(u, v), x)).$$

Thus the result follows from Theorem 3.1 by taking $G = F$. ■

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