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A General Iterative Method for Solving the Generalized Equilibrium Problems for Strictly Pseudocontractive Mappings in Hilbert Spaces

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Abstract. In this paper, the researcher found a general iterative method for solving the generalized equilibrium problems for strictly pseudocontractive mappings in Hilbert spaces. The results presented in this paper improve and extend the corresponding results announced by many others.

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Keywords: Nonexpansive mappings, Generalized equilibrium problems, k-strictly pseudocontractive, α-inverse strongly monotone operator, Strong convergence, Hilbert spaces

1. **Introduction**

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $T : H \longrightarrow H$ be a nonlinear mapping. In this paper, we use $F(T)$ to denote the fixed point set of T.

Recall the following definitions.

(1) The mappings T is said to be contractive with the coefficient $\beta \in (0, 1)$ if

$$\|Tx - Ty\| \leq \beta \|x - y\|, \quad \forall x, y \in H. \tag{1.1.1}$$

(2) The mappings T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H. \tag{1.1.2}$$

(3) The mappings T is said to be strictly pseudocontractive with the coefficient $k \in [0, 1)$ if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H. \tag{1.1.3}$$

(4) The mappings T is said to be pseudocontractive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H. \tag{1.1.4}$$

Next, let F be a bifunction from $C \times C$ into $\mathbb{R}$, the set of real and $A : C \longrightarrow H$ be a nonlinear mapping. The generalized equilibrium problem is to find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x \rangle \geq 0, \forall y \in C. \tag{1.1.5}$$

The set of solutions of the generalized equilibrium problem is denoted by EP. Let H be a real Hilbert space and C be a closed convex subset of H. Let $B : C \longrightarrow H$ be a mapping. The classical variational inequality, denoted by $VI(B, C)$, is to find $x^* \in C$ such that

$$\langle Bx^*, v - x^* \rangle \geq 0$$

for all $v \in C$. The variational inequality has been extensively studied in the literature. See, e.g. [17, 19] and the references therein. We recall that a mapping $B : C \longrightarrow H$ is said to be: The variational inequality has been extensively studied in the literature. See, e.g. [17, 19] and the references therein. We recall that a mapping $B : C \longrightarrow H$ is said to be:

(i) monotone if $\langle Bu - Bv, u - v \rangle \geq 0, \ \forall u, v \in C$.
(ii) L-Lipschitz if there exists a constant $L > 0$ such that $\|Bu - Bv\| \leq L\|u - v\|, \ \forall u, v \in C$.
(iii) α-inverse-strongly monotone [3, 7] if there exists a positive real number α such that

$$\langle Bu - Bv, u - v \rangle \geq \alpha\|Bu - Bv\|^2 \quad \forall u, v \in C.$$

It is obvious that any α-inverse-strongly monotone mapping B is monotone and Lipschitz continuous. A mapping S of C into itself is called nonexpansive if

$$\|Su - Sv\| \leq \|u - v\|$$

for all $u, v \in C$. We denote by $F(S)$ the set of fixed points of S. An operator A is strongly positive on H if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.1.6}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, B)$; see [12]. For finding an element of $F(S) \cap VI(B, C)$, Takahashi and Toyoda [14] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n Bx_n) \tag{1.1.7}$$

for every $n = 0, 1, 2, ...,$ where $x_0 = x \in C, \{\alpha_n\}$ is a sequence in (0, 1), and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that, if $F(S) \cap VI(B, C)$ is nonempty, then the sequence $\{x_n\}$ generated by (2.2.5) converges weakly to some $z \in F(S) \cap VI(B,C)$. On the other hand, for solving the variational inequality problem in the finite-dimensional Euclidean space $\mathbb{R}^n$ under the assumption that a set $C \subset \mathbb{R}^n$ is closed and convex, a mapping B of C into $\mathbb{R}^n$ is monotone and k-Lipschitz continuous and $VI(B,C)$ is nonempty, Korpelevich [6] introduced the following so-called extragradient method:

$$(1.1.8) \qquad \begin{cases} x_1 = u \in C \\ y_n = P_C(x_n - \lambda B x_n) \\ x_{n+1} = P_C(x_n - \lambda B y_n), \ n \geq 1, \end{cases}$$

where $\lambda \in (0, \frac{1}{k})$. He proved that the sequences $\{x_n\}$ and $\{y_n\}$ generated by this iterative process converge to the same point $z \in VI(B,C)$. Recently, Nadezhkina and Takahashi [10], Zeng and Yao [20] proposed some new iterative schemes for finding elements in $F(S) \cap VI(B,C)$. Recently, Iiduka and Takahashi [5] proposed another iterative scheme as following

$$(1.1.9) \qquad \begin{cases} x_1 = x \in C \text{ chosen arbitrary,} \\ x_{n+1} = \alpha_n x + (1-\alpha_n) S P_C(x_n - \lambda_n B x_n), \ \ n \geq 1 \end{cases}$$

where B is an α-cocoerceive map, $\{\alpha_n\} \subseteq (0,1)$ and $\{\lambda_n\} \subseteq (0, 2\alpha)$ satisfy some parameters controlling conditions. They showed that, if $F(S) \cap VI(B,C)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.1.9) converges strongly to some $z \in F(S) \cap VI(B,C)$. By using this idea, Yao and Yao [18] gave the iterative scheme (1.1.10) below for finding an element of $F(S) \cap VI(B,C)$ under the assumption that a set $C \subseteq H$ is nonempty, closed and convex, a mapping $S : C \longrightarrow C$ is nonexpansive and a mapping $B : C \longrightarrow H$ is α-inverse-strongly-monotone:

$$(1.1.10) \qquad \begin{cases} x_1 = u \in C \\ y_n = P_C(x_n - \lambda_n B x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda_n B y_n), \ n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. They proved that if $F(S) \cap VI(B,C) \neq \emptyset$ and and the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ of parameters satisfy appropriate conditions, then the sequence $\{x_n\}$ defined by (1.1.10) converges strongly to $q \in F(S) \cap VI(B,C)$.

On the other hand, Moudafi [8] introduced the viscosity approximation method for nonexpansive mappings (see [16] for further developments in both Hilbert and Banach spaces). Let f be a contraction on C. Starting with an arbitrary initial $x_1 \in C$, define a sequence $\{x_n\}$ recursively by

$$(1.1.11) \qquad x_{n+1} = (1-\sigma_n) T x_n + \sigma_n f(x_n), \ n \geq 0,$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$. It is proved [8, 16] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.1.11) strongly converges to the unique solution q in C of the variational inequality

$$\langle (I-f)q, p-q\rangle \geq 0, p \in C.$$

Recently, Marino and Xu [9] introduced the following general iterative method:

$$x_{n+1} = (I-\alpha_n A)Tx_n + \alpha_n \gamma f(x_n), n \geq 0. \tag{1.1.12}$$

where A is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.1.12) converges strongly to the unique solution of the variational inequality

$$\langle (A-\gamma f)x^*, x-x^*\rangle \geq 0, x \in C \tag{1.1.13}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for $\gamma f (i.e., h'(x) = \gamma f(x)$ for $x \in H)$.

Very recently, to find a common fixed point of a countable family of nonexpansive mappings in Banach spaces , Aoyama et al.[1] introduced the following iterative sequence:

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = \alpha_n x + (1-\alpha_n)S_n x_n, \ n \geq 1, \end{cases} \tag{1.1.14}$$

where C is a nonempty closed convex subset of a Banach space, $\{\alpha_n\}$ is a sequence of $[0,1]$, and $\{S_n\}$ is a sequence of nonexpansive mappings with some conditions. Then they proved that $\{x_n\}$ defined by (1.1.14) converges strongly to a common fixed point of $\{S_n\}$.

Inspired and motivated by the above research, we suggest and analyze a new iterative scheme for finding a common element of the fixed point set of common fixed points of a countable family of nonexpansive mappings and the solution set of the variational inequality problem for an α-inverse-strongly monotone mapping in a real Hilbert space. Under some appropriate conditions imposed on the parameters, we obtain a strong convergence theorem for the sequence generated by the proposed method. The results of this paper extend and improve the results of Yao and Yao [18] and many others.

2. **Preliminaries**

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\longrightarrow$, respectively.

A space X is said to satisfy Opial's condition [11] if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\longrightarrow\infty} \|x_n - x\| < \liminf_{n\longrightarrow\infty} \|x_n - y\|, \ \ \forall y \in X, y \neq x.$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by P_Cx, such that

$$\|x - P_Cx\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_Cx - P_Cy\rangle \geq \|P_Cx - P_Cy\|^2 \tag{2.2.1}$$

for every $x, y \in H$. Moreover, P_Cx is characterized by the following properties: $P_Cx \in C$ and

$$\langle x - P_Cx, y - P_Cx\rangle \leq 0, \tag{2.2.2}$$

$$\|x - y\|^2 \geq \|x - P_Cx\|^2 + \|y - P_Cx\|^2 \tag{2.2.3}$$

for all $x \in H, y \in C$. It is easy to see that the following is true:

$$u \in VI(A, C) \Leftrightarrow u = P_C(u - \lambda Au), \lambda > 0. \tag{2.2.4}$$

If A an $\alpha-$inverse-strongly monotone mapping of C into H, then it is obvious that A is $\frac{1}{\alpha} - Lipschitz$ continuous. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|x - y - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle Ax - Ay, x - y\rangle + \lambda^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2 \qquad (2.2.5) \end{aligned}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of C into H.

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1. *Let H be a real Hilbert space. Then for all $x, y \in H$,*

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$
(2) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$.

Lemma 2.2. ([13]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\longrightarrow\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\longrightarrow\infty} \|y_n - x_n\| = 0$.*

For solving the equilibrium problems for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F such that satisfies the following conditions:

$(A1)$ $F(x, x) = 0$ for all $x \in C$;
$(A2)$ F is monotone, i.e, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
$(A3)$ for each $x, y \in C$, $\lim_{t\to\ 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
$(A4)$ for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.3. ([2]) *Let C be a nonempty closed convex subset of H and F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C.$$

Lemma 2.4. ([4]) *Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \ \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

1. *T_r is single-valued;*
2. *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

3. *$F(T_r) = EP(F)$;*
4. *EP(F) is closed and convex.*

Lemma 2.5. ([11]) *Let H be a Hilbert space, C a closed convex subset of H, and $S : C \longrightarrow C$ a nonexpansive mapping with $F(S) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{(I - S)x_n\}$ converges strongly to y, then $(I - S)x = y$.*

Lemma 2.6. ([16]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \sigma_n, \ n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$

(2) $\limsup_{n\longrightarrow\infty} \frac{\sigma_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.7. [1, Lemma 3.2] *Let C be a nonempty closed subset of a Banach space and let $\{S_n\}$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in C\} < \infty$. Then, for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a mapping of C into itself defined by*

$$Sy = \lim_{n\longrightarrow\infty} S_n y, \ \ \forall y \in C.$$

Then $\lim_{n\longrightarrow\infty} \sup\{\|Sz - S_n z\| : z \in C\} = 0$.

Lemma 2.8. ([9]) *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.9. ([?]) *Let C be a closed convex subset of a real Hilbert space H and $T : C \longrightarrow C$ a k-strict pseudocontraction. Define $S : C \longrightarrow H$ by*

$$Sx = \alpha x + (1-\alpha)Tx, \text{ for all } x \in C.$$

Then, as $\alpha \in [k, 1)$, S is nonexpansive such that $F(S) = F(T)$.

3. Main Results

In this section, we prove the strong convergence theorem for solving the generalized equilibrium problems for strictly pseudocontractive mappings in a real Hilbert spaces.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $f : C \longrightarrow C$ be a contraction with coefficient $\beta \in (0, 1)$, A an $\alpha-$inverse-strongly monotone mapping of C into H and let T be a k-strict pseudocontraction of C into itself such that $\Omega = F(T) \cap EP \neq \emptyset$. Define $T_k : C \longrightarrow C$ by*

$$T_k x = kx + (1-k)Tx, \text{ for all } x \in C.$$

Let B be a strongly positive bounded linear operator on C with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\beta}$. Suppose the sequences $\{x_n\}, \{y_n\}$ are given by

(3.3.1)
$$\begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ F(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)[\mu T_k x_n + (1-\mu)u_n], \ n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $[0, 1]$ and $\{r_n\}$ is a sequence in $(0, 2\alpha]$. Suppose that $\{\alpha_n\}, \{\beta_n\}$ and $\{r_n\}$ are chosen so that $r_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$ satisfying

(i) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,
(iii) $\lim_{n\longrightarrow\infty} |r_{n+1} - r_n| = 0$.

Then $\{x_n\}$ converges strongly to a point $z \in \Omega$ which is the unique solution of the variational inequality

$$\langle (B - \gamma f)z, z - x\rangle \leq 0, \ \ x \in \Omega. \tag{3.3.2}$$

Equivalently, we have $z = P_\Omega(I - B + \gamma f)(z)$.

Proof. Note that from the condition (i), we may assume, without loss of generality, that $\alpha_n \leq (1-\beta_n)\|B\|^{-1}$ for all $n \in \mathbb{N}$. From Lemma 2.8, we know that if $0 \leq \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$. We will assume that $\|I - B\| \leq 1 - \bar{\gamma}$. since B is a strongly positive bounded linear operator on C, we have

$$\|B\| = \sup\{|\langle Bx, x\rangle| : x \in C, \|x\| = 1\}.$$

For any x such that $\|x\| = 1$, we have

$$\langle ((1-\beta_n)I - \alpha_n B)x, x\rangle \ \ = \ \ 1 - \beta_n - \alpha_n \langle Bx, x\rangle$$

$$\begin{array}{rcl} & \geq & 1-\beta_n-\alpha_n\|B\| \\ & \geq & 0 \end{array}$$

this show that $(1-\beta_n)I-\alpha_n B$ is positive. It follows that

$$\begin{array}{rcl} \|(1-\beta_n)I-\alpha_n B\| & = & \sup\{|\langle((1-\beta_n)I-\alpha_n B)x,x\rangle| : x\in C, \|x\|=1\} \\ & = & \sup\{1-\beta_n-\alpha_n\langle Bx,x\rangle : x\in C, \|x\|=1\} \\ & \leq & 1-\beta_n-\alpha_n\bar{\gamma}. \end{array}$$

We note that by hypothesis $F(T)\cap EP\neq\emptyset$. Put $u_n := T_{r_n}(x_n-r_nAx_n)$, $n\geq 1$. Let $p\in\Omega$ and $\{T_{r_n}\}$ be a sequence of mappings defined as Lemma 2.4. Since $I-\lambda_nA$ is a nonexpansive and $p=T_{r_n}(p-r_nAp)$.
By (2.2.5) , we have

$$\begin{array}{rcl} \|u_n-p\|^2 & = & \|T_{r_n}(x_n-r_nAx_n)-p\|^2 \\ & = & \|T_{r_n}(x_n-r_nAx_n)-T_{r_n}(p-r_nAp)\|^2 \\ & \leq & \|(I-r_nA)x_n-(I-r_nA)p\|^2 \\ & \leq & \|x_n-p\|^2+r_n(r_n-2\alpha)\|Ax_n-Ap\|^2 \\ & \leq & \|x_n-p\|^2. \end{array}$$

Set $t_n=\mu T_kx_n+(1-\mu)u_n$. From Lemma 2.9 we see that T_k is nonexpansive. Thus

$$\begin{array}{rcl} \|t_n-p\| & = & \|\mu T_kx_n+(1-\mu)u_n-p\| \\ & = & \|\mu T_kx_n+(1-\mu)u_n-\mu p-(1-\mu)p\| \\ & = & \|\mu(T_kx_n-p)+(1-\mu)(u_n-p)\| \\ & \leq & \|\mu(T_kx_n-p)\|+\|(1-\mu)(u_n-p)\| \\ & = & \mu\|(T_kx_n-p)\|+(1-\mu)\|u_n-p\| \\ & = & \mu\|(T_kx_n-Tp)\|+(1-\mu)\|u_n-p\| \\ & \leq & \mu\|x_n-p\|+(1-\mu)\|x_n-p\| \\ & = & \|x_n-p\| \end{array}$$

It then follows that

$$\begin{array}{rcl} \|x_{n+1}-p\| & = & \|\alpha_n(\gamma f(x_n)-Bp)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_nB)(t_n-p)\| \\ & \leq & (1-\beta_n-\alpha_n\bar{\gamma})\|t_n-p\|+\beta_n\|x_n-p\|+\alpha_n\|\gamma f(x_n)-Bp\| \\ & \leq & (1-\beta_n-\alpha_n\bar{\gamma})\|x_n-p\|+\beta_n\|x_n-p\|+\alpha_n\|\gamma f(x_n)-Bp\| \\ & \leq & (1-\alpha_n\bar{\gamma})\|x_n-p\|+\alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-Bp\| \\ & \leq & (1-\alpha_n\bar{\gamma})\|x_n-p\|+\alpha_n\gamma\beta\|x_n-p\|+\alpha_n\|\gamma f(p)-Bp\| \\ & = & (1-(\bar{\gamma}-\gamma\beta)\alpha_n)\|x_n-p\|+(\bar{\gamma}-\gamma\beta)\alpha_n\dfrac{\|\gamma f(p)-Bp\|}{\bar{\gamma}-\gamma\beta}. \end{array}$$

It follows from induction that

$$\|x_n - p\| \le \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - Bp\|}{\bar{\gamma} - \gamma\beta}\right\}, n \ge 1. \tag{3.3.3}$$

Hence $\{x_n\}$ is bounded, so are $\{u_n\}$ and $\{t_n\}$. Now,

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \|T_{r_{n+1}}(x_{n+1} - r_{n+1}Ax_{n+1}) - T_{r_n}(x_n - r_nAx_n)\| \\
&\le \|(x_{n+1} - r_{n+1}Ax_{n+1}) - (x_n - r_nAx_n)\| \\
&= \|(x_{n+1} - r_{n+1}Ax_{n+1}) - (x_n - r_{n+1}Ax_n) + (r_nAx_n - r_{n+1}Ax_n)\| \\
&\le \|x_{n+1} - x_n\| + |r_n - r_{n+1}|\|Ax_n\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|t_{n+1} - t_n\| &= \|\mu T_k x_{n+1} + (1-\mu)u_{n+1} - [\mu T_k x_n + (1-\mu)u_n]\| \\
&= \|(\mu T_k x_{n+1} - \mu T_k x_n) + ((1-\mu)u_{n+1} - (1-\mu)u_n)\| \\
&\le \mu\|T_k x_{n+1} - T_k x_n\| + (1-\mu)\|u_{n+1} - u_n\| \\
&\le \mu\|x_{n+1} - x_n\| + (1-\mu)\|u_{n+1} - u_n\| \\
&\le \mu\|x_{n+1} - x_n\| + (1-\mu)\|x_{n+1} - x_n\| + (1-\mu)|r_n - r_{n+1}|\|Ax_n\| \\
&= \|x_{n+1} - x_n\| + (1-\mu)|r_n - r_{n+1}|\|Ax_n\|. \qquad (3.3.4)
\end{aligned}$$

Setting

$$x_{n+1} = (1-\beta_n)e_n + \beta_n x_n, \forall n \ge 1, \tag{3.3.5}$$

we see that

$$\begin{aligned}
e_{n+1} - e_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + [(1-\beta_{n+1})I - \alpha_{n+1}B]t_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n\gamma f(x_n) + [(1-\beta_n)I - \alpha_n B]t_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}[\gamma f(x_{n+1}) - Bt_{n+1}] + t_{n+1} \\
&\quad - \frac{\alpha_n}{1-\beta_n}[\gamma f(x_n) - Bt_n] - t_n. \qquad (3.3.6)
\end{aligned}$$

It follows that

$$\begin{aligned}
\|e_{n+1} - e_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Bt_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - Bt_n\| \\
&\quad + \|t_{n+1} - t_n\|, \qquad (3.3.7)
\end{aligned}$$

which combines with (3.3.4) yields that

$$\begin{aligned}
\|e_{n+1} - e_n\| - \|x_{n+1} - x_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Bt_{n+1}\| \\
&\quad + \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - Bt_n\|
\end{aligned}$$

$$+(1-\mu)|r_n - r_{n+1}|\|Ax_n\|. \tag{3.3.8}$$

It follows from the conditions (i), (iii) and that

$$\limsup_{n\longrightarrow\infty}(\|e_{n+1}-e_n\| - \|x_{n+1}-x_n\|) \le 0.$$

Hence, from Lemma 2.2, one obtains

$$\lim_{n\longrightarrow\infty} \|e_n - x_n\| = 0. \tag{3.3.9}$$

From (3.3.5), one has $\|x_{n+1}-x_n\| = (1-\beta_n)\|e_n - x_n\|$. From (3.3.9), we see that

$$\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0. \tag{3.3.10}$$

On the other hand, we have

$$\begin{aligned} x_{n+1} - x_n &= \alpha_n \gamma f(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n B]t_n - x_n \\ &= \alpha_n(\gamma f(x_n) - Bt_n) + (1-\beta_n)(t_n - x_n). \end{aligned} \tag{3.3.11}$$

It follows that

$$(1-\beta_n)\|t_n - x_n\| \le \|x_{n+1}-x_n\| + \alpha_n\|\gamma f(x_n) - Bt_n\|. \tag{3.3.12}$$

From the conditions (i) and (3.3.10), we see that

$$\lim_{n\longrightarrow\infty} \|t_n - x_n\| = 0. \tag{3.3.13}$$

Next, we prove that

$$\limsup_{n\longrightarrow\infty}\langle(\gamma f - B)z, x_n - z\rangle \le 0, \text{where } z = P_\Omega(I - B + \gamma f)(z).$$

To see this, we choose a subsequence $\{x_{n_i}\}$ of x_n such that

$$\limsup_{n\longrightarrow\infty}\langle(\gamma f - B)z, x_n - z\rangle = \lim_{i\longrightarrow\infty}\langle(\gamma f - B)z, x_{n_i} - z\rangle. \tag{3.3.14}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$. Next, we show that $w \in F(T)\cap EP$
First, observe that

$$\begin{aligned} \|x_{n+1}-p\|^2 &= \|((1-\beta_n)I - \alpha_n B)(t_n - p) + \beta_n(x_n - p) + \alpha_n(rf(x_n) - Bp)\|^2 \\ &= \|((1-\beta_n)I - \alpha_n B)(t_n - p) + \beta_n(x_n - p)\|^2 + \alpha_n^2\|rf(x_n) - Bp\|^2 \\ &\quad + 2\beta_n\alpha_n\langle x_n - p, rf(x_n) - Bp\rangle \\ &\le \left[(1-\beta_n - \alpha_n\bar{\gamma})\|t_n - p\| + \beta_n\|x_n - p\|\right]^2 + c_n \\ &= (1-\beta_n-\alpha_n\bar{\gamma})^2\|t_n - p\|^2 + \beta_n^2\|x_n - p\|^2 \\ &\quad + 2\beta_n(1-\beta_n-\alpha_n\bar{\gamma})\|t_n - p\|\|x_n - p\| + c_n \end{aligned}$$

$$
\begin{aligned}
&= \big[(1-\alpha_n\bar{\gamma})^2 - 2(1-\alpha_n\bar{\gamma})\beta_n + \beta_n^2\big]\|t_n - p\|^2 + \beta_n^2\|x_n - p\|^2 \\
&\quad +(1-\beta_n-\alpha_n\bar{\gamma})\beta_n\big[\|t_n - p\|^2 + \|x_n - p\|^2\big] + c_n \\
&= \big[(1-\alpha_n\bar{\gamma})^2 - 2(1-\alpha_n\bar{\gamma})\beta_n + \beta_n^2\big]\|t_n - p\|^2 + \beta_n^2\|x_n - p\|^2 \\
&\quad + \big[(1-\alpha_n\bar{\gamma})\beta_n - \beta_n^2\big]\big(\|t_n - p\|^2 + \|x_n - p\|^2\big) + c_n \\
&= (1-\alpha_n\bar{\gamma})^2\|t_n - p\|^2 - (1-\alpha_n\bar{\gamma})\beta_n\|t_n - p\|^2 \\
&\quad +(1-\alpha_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\|t_n - p\|^2 + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\le (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\big(\mu\|x_n - p\|^2 \\
&\quad +(1-\mu)\|u_n - p\|^2\big) + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&= (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\mu\|x_n - p\|^2 \\
&\quad +(1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})(1-\mu)\big[\|x_n - p\|^2 \\
&\quad +r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2\big] + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\le \big[(1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\mu + (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})(1-\mu) \\
&\quad +(1-\alpha_n\bar{\gamma})\beta_n\big]\|x_n - p\|^2 + a(b-2\alpha)\|Ax_n - Ap\|^2 + c_n \\
&= \big[(1-\alpha_n\bar{\gamma})\big((1-\beta_n-\alpha_n\bar{\gamma})\mu + (1-\beta_n-\alpha_n\bar{\gamma})(1-\mu) + \beta_n\big)\big]\|x_n - p\|^2 \\
&\quad +a(b-2\alpha)\|Ax_n - Ap\|^2 + c_n \\
&= \big[(1-\alpha_n\bar{\gamma})\big(1-\beta_n-\alpha_n\bar{\gamma}\beta_n\big)\big]\|x_n - p\|^2 + a(b-2\alpha)\|Ax_n - Ap\|^2 + c_n \\
&= (1-\alpha_n\bar{\gamma})^2\|x_n - p\|^2 + a(b-2\alpha)\|Ax_n - Ap\|^2 + c_n \\
&\le \|x_n - p\|^2 + a(b-2\alpha)\|Ax_n - Ap\|^2 + c_n
\end{aligned} \tag{3.3.15}
$$

where

$$
c_n = \alpha_n^2\|rf(x_n) - Bp\|^2 + 2\beta_n\alpha_n\langle x_n - p, rf(x_n) - Bp\rangle. \tag{3.3.16}
$$

This implies that

$$
\begin{aligned}
-a(b-2\alpha)\|Ax_n - Ap\|^2 &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + c_n \\
&\le \|x_n - x_{n+1}\|\big(\|x_n - p\| + \|x_{n+1} - p\|\big) + c_n.
\end{aligned}
$$

Since $\lim_{n\longrightarrow\infty} c_n = 0$ and from (3.3.10), we obtain

$$
\lim_{n\longrightarrow\infty}\|Ax_n - Ap\| = 0. \tag{3.3.17}
$$

Next, we show that $\lim_{n\longrightarrow\infty}\|x_n - u_n\| = 0$. Consider,

$$
\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - p\|^2 \\
&= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp)\|^2 \\
&\le \langle (I - r_nA)x_n - (I - r_nA)p, u_n - p\rangle \\
&= \frac{1}{2}\big[\|(I - r_nA)x_n - (I - r_nA)p\|^2 + \|u_n - p\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad -\|(I-r_nA)x_n-(I-r_nA)p-(u_n-p)\|^2\big] \\
&\leq \frac{1}{2}\big[\|x_n-p\|^2+\|u_n-p\|^2-\|(x_n-u_n)-r_n(Ax_n-Ap)\|^2\big] \\
&= \frac{1}{2}\big[\|x_n-p\|^2+\|u_n-p\|^2-\|(x_n-u_n)\|^2 \\
&\quad +2r_n\langle x_n-u_n,Ax_n-Ap\rangle-r_n^2\|Ax_n-Ap\|^2\big].
\end{aligned}
$$

Thus,

(3.3.18)

$$\|u_n-p\|^2\leq\|x_n-p\|^2-\|x_n-u_n\|^2+2r_n\langle x_n-u_n,Ax_n-Ap\rangle-r_n^2\|Ax_n-Ap\|^2.$$

It follows that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\big(\mu\|x_n-p\|^2 \\
&\quad +(1-\mu)\|u_n-p\|^2\big)+(1-\alpha_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \\
&\leq (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\mu\|x_n-p\|^2 \\
&\quad +(1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})(1-\mu)\big[\|x_n-p\|^2-\|x_n-u_n\|^2 \\
&\quad +2r_n\langle x_n-u_n,Ax_n-Ap\rangle-r_n^2\|Ax_n-Ap\|^2\big] \\
&\quad +(1-\alpha_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n \\
&= \big[(1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\mu+(1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})(1-\mu) \\
&\quad +(1-\alpha_n\bar{\gamma})\beta_n\big]\|x_n-p\|^2-(1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})(1-\mu)\|x_n-u_n\|^2 \\
&\quad +(1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})(1-\mu)2r_n\|x_n-u_n\|\|Ax_n-Ap\| \\
&\quad +(1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})(1-\mu)r_n^2\|Ax_n-Ap\|^2+c_n.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|x_n-u_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\quad +(1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})(1-\mu)2r_n\|x_n-u_n\|\|Ax_n-Ap\| \\
&\quad +(1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})(1-\mu)r_n^2\|Ax_n-Ap\|^2+c_n \\
&\leq \|x_n-x_{n+1}\|\big(\|x_n-p\|+\|x_{n+1}-p\|\big) \\
&\quad +(1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})(1-\mu)2r_n\|x_n-u_n\|\|Ax_n-Ap\| \\
&\quad +(1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})(1-\mu)r_n^2\|Ax_n-Ap\|^2+c_n.
\end{aligned}
$$

Since (3.3.10), (3.3.17) and $\lim_{n\longrightarrow\infty}c_n=0$, we obtain

(3.3.19)
$$\lim_{n\longrightarrow\infty}\|x_n-u_n\|=0.$$

Next, we have

$$t_n-x_n=\mu(T_kx_n-x_n)+(1-\mu)(u_n-x_n).$$

It follows that

$$\mu\|T_k x_n - x_n\| \leq \|t_n - x_n\| + (1-\mu)\|u_n - x_n\|.$$

From (3.3.13) and (3.3.19), we see that

$$\lim_{n\longrightarrow\infty} \|T_k x_n - x_n\| = 0. \tag{3.3.20}$$

Hence

$$\begin{aligned}\|T_k u_n - u_n\| &\leq \|T_k u_n - T_k x_n\| + \|T_k x_n - x_n\| + \|x_n - u_n\| \\ &\leq 2\|u_n - x_n\| + \|T_k x_n - x_n\|.\end{aligned}$$

By (3.3.19) and (3.3.20), we get

$$\lim_{n\longrightarrow\infty} \|T_k u_n - u_n\| = 0. \tag{3.3.21}$$

Observe that $P_\Omega(I - B + \gamma f)$ is a contraction of C into itself. Indeed, for all $x, y \in C$, we have

$$\begin{aligned}\|P_\Omega(I - B + \gamma f)(x) - P_\Omega(I - B + \gamma f)(y)\| &\leq \|(I - B + \gamma f)(x) - (I - B + \gamma f)(y)\| \\ &\leq \|I - B\|\|x - y\| + \gamma\|f(x) - f(y)\| \\ &\leq (1-\bar{\gamma})\|x - y\| + \gamma\beta\|x - y\| \\ &= (1 - (\bar{\gamma} - \gamma\beta))\|x - y\|.\end{aligned}$$

Since H is complete, there exists a unique element $z \in C$ such that $z = P_\Omega(I - B + \gamma f)(z)$. Next, we show that

$$\limsup_{n\longrightarrow\infty} \langle (B - \gamma f)z, z - x_n\rangle \leq 0. \tag{3.3.22}$$

We choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\lim_{i\longrightarrow\infty} \langle (B - \gamma f)z, z - v_{n_i}\rangle = \limsup_{n\longrightarrow\infty} \langle (B - \gamma f)z, z - v_n\rangle.$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to $w \in C$. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. From $\|T_k u_n - u_n\| \longrightarrow 0$, we obtain $T_k u_{n_i} \rightharpoonup w$. Next, we show that $w \in \Omega$. First, we show that

$$w \in F(T_k).$$

Assume $w \notin F(T_k)$. Since $u_{n_i} \rightharpoonup w$ and $w \neq T_k w$, it follows by the Opial's condition that

$$\begin{aligned}\liminf_{i\longrightarrow\infty} \|u_{n_i} - w\| &< \liminf_{i\longrightarrow\infty} \|u_{n_i} - T_k w\| \\ &\leq \liminf_{i\longrightarrow\infty} \{\|u_{n_i} - T_k u_{n_i}\| + \|T_k u_{n_i} - T_k w\|\} \\ &< \liminf_{i\longrightarrow\infty} \|u_{n_i} - w\|\end{aligned}$$

which derives a contradiction. Thus, we have $w \in F(T_k) = F(T)$. Next, we show that $w \in EP$. We have $u_{n_i} \rightharpoonup w$. and $u_n = T_{r_n}(x_n - r_n A x_n)$, for any $y \in C$ we have

$$F(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0.$$

From (A2), we have

$$\langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n).$$

Replacing n by n_i, we get

$$\langle Ax_{n_i}, y - u_{n_i}\rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}). \tag{3.3.23}$$

Put $u_t = ty + (1-t)w$ for all $t \in (0, 1]$ and $y \in C$. Then, we have $u_t \in C$. So, from (3.3.23) we have

$$\begin{aligned}
\langle u_t - u_{n_i}, Au_t\rangle &\geq \langle u_t - u_{n_i}, Au_t\rangle - \langle u_t - u_{n_i}, Ax_{n_i}\rangle - \langle u_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(u_t, u_{n_i}) \\
&= \langle u_t - u_{n_i}, Au_t - Au_{n_i}\rangle + \langle u_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle \\
&\quad - \langle u_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(u_t, u_{n_i}).
\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Au_{n_i} - Ax_{n_i}\| \to 0$. Further, from monotonicity of A, we have $\langle u_t - u_{n_i}, Au_t - Au_{n_i}\rangle \geq 0$. So, from (A4) we have

$$\langle u_t - w, Au_t\rangle \geq F(u_t, w), \tag{3.3.24}$$

as $i \to \infty$. From (A1), (A4) and (3.3.24), we also have

$$\begin{aligned}
0 &= F(u_t, u_t) \leq tF(u_t, y) + (1-t)F(u_t, w) \\
&\leq tF(u_t, y) + (1-t)\langle u_t - w, Au_t\rangle \\
&= tF(u_t, y) + (1-t)\langle y - w, Au_t\rangle
\end{aligned}$$

and hence

$$0 \leq F(u_t, y) + (1-t)\langle y - w, Au_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq F(w, y) + \langle y - w, Aw\rangle. \tag{3.3.25}$$

This implies $w \in EP$. Hence $w \in \Omega$.
Since $z = P_\Omega(I - B + \gamma f)(z)$, it follows that

$$\begin{aligned}
\limsup_{n\longrightarrow\infty}\langle (B - \gamma f)z, z - x_n\rangle &= \limsup_{n\longrightarrow\infty}\langle (B - \gamma f)z, z - v_n\rangle \\
&= \lim_{i\longrightarrow\infty}\langle (B - \gamma f)z, z - v_{n_i}\rangle \\
&= \langle (B - \gamma f)z, z - w\rangle \leq 0. \qquad (3.3.26)
\end{aligned}$$

It follows from the last inequality, (3.3.13), (3.3.17) and (3.3.19) that

$$\limsup_{n\longrightarrow\infty}\langle\gamma f(z)-Bz, t_n-z\rangle \leq 0. \tag{3.3.27}$$

Finally, we prove $x_n \longrightarrow z$ as $n \longrightarrow \infty$. To this end, we calculate

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n B)t_n - z\|^2\\
&= \|((1-\beta_n)I-\alpha_n B)(t_n-z)+\beta_n(x_n-z)+\alpha_n(\gamma f(x_n)-Bz)\|^2\\
&= \|((1-\beta_n)I-\alpha_n B)(t_n-z)+\beta_n(x_n-z)\|^2+\alpha_n^2\|\gamma f(x_n)-Bz\|^2\\
&\quad +2\beta_n\alpha_n\langle x_n-z,\gamma f(x_n)-Bz\rangle\\
&\quad +2\alpha_n\langle((1-\beta_n)I-\alpha_n B)(t_n-z),\gamma f(x_n)-Bz\rangle\\
&\leq ((1-\beta_n-\alpha_n\bar{\gamma})\|t_n-z\|+\beta_n\|x_n-z\|)^2+\alpha_n^2\|\gamma f(x_n)-Bz\|^2\\
&\quad +2\beta_n\alpha_n\gamma\langle x_n-z, f(x_n)-f(z)\rangle+2\beta_n\alpha_n\langle x_n-z,\gamma f(z)-Bz\rangle\\
&\quad +2(1-\beta)\gamma\alpha_n\langle t_n-z, f(x_n)-f(z)\rangle\\
&\quad +2(1-\beta)\alpha_n\langle t_n-z,\gamma f(z)-Bz\rangle-\ 2\alpha_n^2\langle B(t_n-z),\gamma f(z)-Bz\rangle\\
&\leq ((1-\beta_n-\alpha_n\bar{\gamma})\|x_n-z\|+\beta_n\|x_n-z\|)^2+\alpha_n^2\|\gamma f(x_n)-Bz\|^2\\
&\quad +2\beta_n\alpha_n\gamma\beta\|x_n-z\|^2+2\beta_n\alpha_n\langle x_n-z,\gamma f(z)-Bz\rangle\\
&\quad +2(1-\beta_n)\gamma\alpha_n\beta\|x_n-z\|^2+2(1-\beta)\alpha_n\langle t_n-z,\gamma f(z)-Bz\rangle\\
&\quad -2\alpha_n^2\langle A(t_n-z),\gamma f(z)-Bz\rangle\\
&= \left[(1-\alpha_n\bar{\gamma})^2+2\beta_n\alpha_n\gamma\beta+2(1-\beta_n)\gamma\alpha_n\beta\right]\|x_n-z\|^2\\
&\quad +\alpha_n^2\|\gamma f(x_n)-Bz\|^2+\ 2\beta_n\alpha_n\langle x_n-z,\gamma f(z)-Bz\rangle\\
&\quad +2(1-\beta)\alpha_n\langle t_n-z,\gamma f(z)-Bz\rangle-2\alpha_n^2\langle B(t_n-z),\gamma f(z)-Bz\rangle\\
&\leq [1-2(\bar{\gamma}-\alpha\gamma)\alpha_n]\|x_n-z\|^2+\bar{\gamma}^2\alpha_n^2\|x_n-z\|^2+\alpha_n^2\|\gamma f(x_n)-Bz\|^2\\
&\quad +2\beta_n\alpha_n\langle x_n-z,\gamma f(z)-Bz\rangle+2(1-\beta)\alpha_n\langle t_n-z,\gamma f(z)-Bz\rangle\\
&\quad +2\alpha_n^2\|B(t_n-z)\|\|\gamma f(z)-Bz\|\\
&= [1-2(\bar{\gamma}-\alpha\gamma)\alpha_n]\|x_n-z\|^2+\alpha_n\{\alpha_n(\bar{\gamma}^2\|x_n-z\|^2\\
&\quad +\|\gamma f(x_n)-Bz\|^2+2\|B(t_n-z)\|\|\gamma f(z)-Bz\|)\\
&\quad +2\beta_n\langle x_n-z,\gamma f(z)-Bz\rangle+2(1-\beta)\langle t_n-z,\gamma f(z)-Bz\rangle\}. \qquad (3.3.28)
\end{aligned}$$

Since $\{x_n\}$, $\{f(x_n)\}$ and $\{t_n\}$ are bounded, we can take a constant $M > 0$ such that

$$\bar{\gamma}^2\|x_n-z\|^2+\|\gamma f(x_n)-Bz\|^2+2\|B(t_n-z)\|\gamma f(z)-Bz\| \leq M,$$

for all $n \geq 0$. It then follows that

$$\|x_{n+1}-z\|^2 \leq [1-2(\bar{\gamma}-\alpha\gamma)\alpha_n]\|x_n-z\|^2+\alpha_n\sigma_n, \tag{3.3.29}$$

where

$$\sigma_n = 2\beta_n\langle x_n-z,\gamma f(z)-Bz\rangle+2(1-\beta)\langle t_n-z,\gamma f(z)-Bz\rangle+\alpha_n M.$$

Using (i), (3.3.26) and (3.3.27), we get $\limsup_{n\longrightarrow\infty}\sigma_n \leq 0$. Now applying Lemma 2.6 to (3.3.29), we conclude that $x_n \longrightarrow z$. □

Corollary 3.2. *Let C be a closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $f : C \longrightarrow C$ be a contraction with coefficient $\beta \in (0,1)$ and let T be a k-strict pseudocontraction of C into itself such that $\Omega = F(T) \cap EP \neq \emptyset$. Define $T_k : C \longrightarrow C$ by*

$$T_k x = kx + (1-k)Tx, \text{ for all } x \in C.$$

Let B be a strongly positive bounded linear operator on C with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\beta}$. Suppose the sequences $\{x_n\}, \{y_n\}$ are given by

(3.3.30)

$$\begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)[\mu T_k x_n + (1-\mu)u_n], \ n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $[0,1]$ and $\{r_n\}$ is a sequence in $(0, 2\alpha]$. Suppose that $\{\alpha_n\}, \{\beta_n\}$ and $\{r_n\}$ are chosen so that $r_n \in [a,b]$ for some a, b with $0 < a < b < 2\alpha$ satisfying

(i) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,
(iii) $\lim_{n\longrightarrow\infty} |r_{n+1} - r_n| = 0$.

Then $\{x_n\}$ converges strongly to a point $z \in \Omega$ which is the unique solution of the variational inequality

$$\langle (B - \gamma f)z, z - x\rangle \leq 0, \ \ x \in \Omega. \tag{3.3.31}$$

Equivalently, we have $z = P_\Omega(I - B + \gamma f)(z)$.

Proof. Taking $A \equiv 0$ in Theorem 3.1, we obtain the desired result. □

Corollary 3.3. *Let C be a closed convex subset of a real Hilbert space H. Let $f : C \longrightarrow C$ be a contraction with coefficient $\beta \in (0,1)$ and let T be a k-strict pseudocontraction of C into itself such that $\Omega = F(T) \neq \emptyset$. Define $T_k : C \longrightarrow C$ by*

$$T_k x = kx + (1-k)Tx, \text{ for all } x \in C.$$

Let B be a strongly positive bounded linear operator on C with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\beta}$. Suppose the sequences $\{x_n\}, \{y_n\}$ are given by

(3.3.32)

$$\begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)[\mu T_k x_n + (1-\mu)x_n], \ n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $[0,1]$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying

(i) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.

Then $\{x_n\}$ *converges strongly to a point* $z \in \Omega$ *which is the unique solution of the variational inequality*

$$\langle (B - \gamma f)z, z - x\rangle \leq 0, \quad x \in \Omega. \tag{3.3.33}$$

Equivalently, we have $z = P_\Omega(I - B + \gamma f)(z)$.

Proof. Taking $A \equiv 0$, $F(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ in Theorem 3.1, we obtain the desired result. □

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