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On the Nonlinear Product of Laplacian Related to the Biharmonic Equation

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Abstract

In this paper, we study the solution of nonlinear equation of the form

$$\triangle^k(\triangle + m^2)^k u(x) = f(x, \triangle^{k-1}(\triangle + m^2)^k u(x)),$$

where the operator $\triangle^k$ is the Laplacian defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^k,$$

n is the dimension of the Euclidean space $\mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, k is a positive integer, $u(x)$ is an unknown and f is a given function. It is found that the existence of the solution $u(x)$ of such equation depending on the condition of f and $\triangle^{k-1}(\triangle + m^2)^k u(x)$ and moreover such solution $u(x)$ related to the Laplacian depending on the conditions of k

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1 Introduction

Gelfand and Shilov [4] have shown that the iterated Laplace equation $\triangle^k u(x) = f(x)$ will be solved when we have obtained an elementary solution $E(x)$. Kananthai [6], [7] has shown that $u(x) = (-1)^k R^e_{2k}(x)$ be the elementary solution of the equation $\triangle^k u(x) = \delta(x)$, where $R^e_{2k}(x)$ defined by (5) and $u(x) = ((-1)^{k-1} R^e_{2(k-1)}(x))^{(l)}$ be a solution of $\triangle^k u(x) = 0$.

R.Courant and D.Hilbert [2] have studied the nonlinear equation of the form $\triangle u(x) = f(x, u(x))$ with f defined and continuous function for all $x \in \Omega \cup \partial\Omega$ where Ω is an open set in $\mathbb{R}^n$, $\partial\Omega$ denotes the boundary of Ω and $\triangle$ is the Laplace operator, defined by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \tag{1}$$

They found that the solution $u(x)$ of such equation is unique under the condition $|f(x, u(x))| \leq N$ where N is a constant for all $x \in \Omega$ and the boundary condition $u(x) = 0$ for all $x \in \partial\Omega$.

In this paper, we study the nonlinear equation of the form

$$\triangle^k (\triangle + m^2)^k u(x) = f(x, \triangle^{k-1}(\triangle + m^2)^k u(x)), \tag{2}$$

where $\triangle + m^2$ defined by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} + m^2, \tag{3}$$

k is a positive integer, f defined and continuous for all $x \in \Omega \cup \partial\Omega$ where Ω is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ denotes the boundary of Ω. We can find the solution $u(x)$ of (2) which is unique under the condition $|f(x, \triangle^{k-1}(\triangle + m^2)^k u(x))| \leq N$ where N is a constant for all $x \in \Omega$ and the boundary condition $\triangle^{k-1}(\triangle + m^2)^k u(x) = 0$ for $x \in \partial\Omega$. Moreover the solution $u(x)$ related to the nonhomogeneous biharmonic equation depend on the conditions of k.

2 Preliminaries

Definition 2.1. *Let $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and write*

$$v = x_1^2 + x_2^2 + ... + x_n^2. \tag{4}$$

For any complex number β, define

$$R^e_\beta(v) = 2^{-\beta} \pi^{\frac{-n}{2}} \Gamma(\frac{n-\beta}{2}) \frac{v^{\frac{\beta-n}{2}}}{\Gamma(\frac{\beta}{2})}. \tag{5}$$

The function $R^e_\beta(v)$ is called the elliptic kernel of Marcel Riesz and is ordinary function for $Re(\beta) \geq n$ and is a distribution of β for $Re(\beta) < n$.

Definition 2.2. *For any complex number β, define*

$$W_{2k}^{e}(v,m)=\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(k+r)}{r!\Gamma(k)}(m^2)^r(-1)^{k+r}R_{2k+2r}^{e}(v), \tag{6}$$

where $R_{2k+2r}^{e}(v)$ is defined by (5) with $\beta=2k+2r$, m is a nonnegative real number.

Lemma 2.3. *Given the equation*

$$\triangle^k u(x)=0, \tag{7}$$

where $\triangle^k$ is the Laplacian iterated k-times defined by equation (1) we obtain $u(x)=((-1)^{k-1}R_{2(k-1)}^{e}(x))^{(l)}$ as a solutions of (7) where $l=(n-4)/2, n\geq 4$ is nonnegative integer and n is even and $R_{2(k-1)}^{e}(x)$ defined by equation (5) with m derivatives and $\beta=2(k-1)$.

Proof. see [7].

Lemma 2.4. *Given the equation $\triangle^k u(x)=\delta(x)$ for $x\in\mathbb{R}^n$, where $\triangle^k$ is the Laplace operator iterated k-times defined by (1). Then $u(x)=(-1)^kR_{2k}^{e}(x)$ is an elementary solution of the operator $\triangle^k$.*

Proof. See [6].

Lemma 2.5. *The function $W_{2k}^{e}(v,m)$ is an elementary solutions of the operator $(\triangle+m^2)^k$ where $(\triangle+m^2)^k$ is the Helmholtz operator iterated k-times, $\triangle$ is the Laplacian, and $W_{2k}^{e}(v,m)$ defined by equation (6)*

Proof. At first, the following formula is valid ([1] p.3)

$$\Gamma\left(\frac{\eta}{2}+r\right)=\frac{\eta}{2}\left(\frac{\eta}{2}+1\right)\cdots\left(\frac{\eta}{2}+r-1\right)\Gamma\left(\frac{\eta}{2}\right).$$

Equivalently,

$$\begin{aligned}\frac{(-1)^r}{r!}\Gamma\left(\frac{\eta}{2}+r\right)&=\frac{(-1)^r}{r!}\frac{\eta}{2}\left(\frac{\eta}{2}+1\right)\cdots\left(\frac{\eta}{2}+r-1\right)\Gamma\left(\frac{\eta}{2}\right)\\&=\frac{1}{r!}\left(\frac{-\eta}{2}\right)\left(\frac{-\eta}{2}-1\right)\cdots\left(\frac{-\eta}{2}-r+1\right)\Gamma\left(\frac{\eta}{2}\right).\end{aligned}$$

We have,

$$\frac{(-1)^r}{r!}\Gamma\left(\frac{\eta}{2}+r\right)=\binom{-\frac{\eta}{2}}{r}\Gamma\left(\frac{\eta}{2}\right).$$

Then, we obtain the function $W^e_{2k}(v,m)$ is defined by (6) become

$$W^e_{2k}(v,m) = \sum_{r=0}^{\infty} \begin{pmatrix} -k \\ r \end{pmatrix} (m^2)^r(-1)^{k+r}R^e_{2k+2r}(v).$$

Since the operator $\triangle$ is a linearly continuous and have $1-1$ mapping, it has an inverse. By Lemma 2.4, we obtain

$$W^e_{2k}(v,m) = \sum_{r=0}^{\infty} \begin{pmatrix} -k \\ r \end{pmatrix} (m^2)^r\triangle^{-k-r}\delta(x) = (\triangle + m^2)^{-k}\delta(x),$$

where $(\triangle + m^2)^{-k}$ is the inverse operator of the operator $(\triangle + m^2)^k$. By applying the operator $(\triangle + m^2)^k$ to both sides of the above equation, we have

$$(\triangle + m^2)^k W^e_{2k}(v,m) = (\triangle + m^2)^k(\triangle + m^2)^{-k}\delta(x) = \delta(x).$$

Lemma 2.6. *Given the equation*

$$\triangle u(x) = f(x, u(x)), \tag{8}$$

where f is defined and has continuous first derivatives for all $x \in \Omega \cup \partial\Omega, \Omega$ is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ denotes the boundary of Ω. Assume f is a bounded, that is $|f(x,u)| \leq N$ and the boundary condition $u(x) = 0$ for $x \in \partial\Omega$. Then we obtain $u(x)$ as a unique solution of (8).

Proof. We can prove this lemma by the method of iterations and the Schauder's estimates, see [2].

3 Main results

Theorem 3.1. *Given the nonlinear equation*

$$\triangle^k(\triangle + m^2)^k u(x) = f(x, \triangle^{k-1}(\triangle + m^2)^k u(x)) \tag{9}$$

where $\triangle^k$ is the Laplacian iterated k times, defined by (1) and $(\triangle + m^2)^k$ is the Helmholtz operator iterated k times, defined by (2). Let f be defined and have continuous first derivatives for all $x \in \Omega \cup \partial\Omega, \Omega$ is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ denotes the boundary of Ω and n is even with $n \geq 4$. Let f be a bounded function, that is

$$|f(x, \triangle^{k-1}(\triangle + m^2)^k u(x))| \leq N \tag{10}$$

and the boundary condition

$$\triangle^{k-1}(\triangle + m^2)^k u(x) = 0, \quad \textit{for } x \in \partial\Omega \tag{11}$$

then we obtain

$$u(x) = (-1)^{k-1} R^e_{2(k-1)}(x) * W^e_{2k}(v, m) * W(x) \tag{12}$$

as a solution of (9) *with the boundary condition*

$$u(x) = (-1)^{k-2} (R^e_{2(k-2)}(x))^{(l)} * W^e_{2k}(v, m)$$

for $x \in \partial\Omega, l = (n-4)/2,\ k = 2, 3, 4, \ldots$ *and* v *is given by* (4), $W(x)$ *is a continuous function for* $x \in \Omega \cup \partial\Omega$. $R^e_{2(k-2)}(x)$, *with* $\beta = 2(k-2)$, *and* $W^e_{2k}(v, m)$ *are given by* (5) *and* (6), *respectively. Moreover, for* $k = 1$ *then* (9) *becomes*

$$\triangle(\triangle + m^2) u(x) = f(x, (\triangle + m^2) u(x)) \tag{13}$$

with boundary condition

$$(\triangle + m^2) u(x) = 0, \quad for\ x \in \partial\Omega; \tag{14}$$

we have

$$u(x) = W^e_2(v, m) * W(x) \tag{15}$$

as a solution of (13) *and we can write* (13) *as*

$$\triangle^2 u(x) = g(x, \triangle u(x)),$$

which is called the nonhomogeneous biharmonic equation, where $g(x, \triangle u(x)) = f(x, (\triangle + m^2) u(x)) - m^2 \triangle u(x)$

Proof. From equation (9), we have

$$\begin{aligned} \triangle^k (\triangle + m^2)^k u(x) &= \triangle(\triangle^{k-1} (\triangle + m^2)^k u(x)) \\ &= f(x, \triangle^{k-1} (\triangle + m^2)^k u(x)). \end{aligned} \tag{16}$$

Since $u(x)$ has continuous derivatives up to order $2k$ for $k = 1, 2, 3, \ldots$ we can assume

$$\triangle^{k-1} (\triangle + m^2)^k u(x) = W(x), \quad for\ x \in \partial\Omega. \tag{17}$$

Thus, (16) can be written in the form

$$\triangle^k u(x) = \triangle W(x) = f(x, W(x)). \tag{18}$$

by (10)

$$|f(x, W(x))| \leq N. \tag{19}$$

and by (11), $W(x)$=0 or

$$\triangle^{k-1}(\triangle+m^2)^k u(x)=0, \quad for\ x\in\partial\Omega. \tag{20}$$

Thus by Lemma (2.6) there exist a unique solution $W(x)$ of (18) which satisfies (19).
Now consider the Equation (17); we have $(-1)^{k-1}R^e_{2(k-1)}(x)$ and $W^e_{2k}(v,m)$ are the elementary solutions of the operators $\triangle^{k-1}$ and $(\triangle+m^2)^k$, respectively. Thus, convolving both sides of (17) by $(-1)^{k-1}R^e_{2(k-1)}(x)*W^e_{2k}(v,m)$ we obtain

$$\begin{aligned}[(-1)^{k-1}R^e_{2(k-1)}(x)*W^e_{2k}(v,m)]*\triangle^{k-1}(\triangle+m^2)^k u(x)\\ =(-1)^{k-1}R^e_{2(k-1)}(x)*W^e_{2k}(v,m)*W(x).\end{aligned}$$

By properties of convolution, we obtain

$$\begin{aligned}[\triangle^{k-1}(-1)^{k-1}R^e_{2(k-1)}(x)][(\triangle+m^2)^k W^e_{2k}(v,m)]*u(x)\\ =(-1)^{k-1}R^e_{2(k-1)}(x)*W^e_{2k}(v,m)*W(x)\\ \delta*\delta*u(x)=(-1)^{k-1}R^e_{2(k-1)}(x)*W^e_{2k}(v,m)*W(x).\end{aligned}$$

Thus

$$u(x)=(-1)^{k-1}R^e_{2(k-1)}(x)*W^e_{2k}(v,m)*W(x) \tag{21}$$

as required. Consider $\triangle^{k-1}(\triangle+m^2)^k u(x)=0,\ for\ x\in\partial\Omega$. By Lemma (2.3), we have

$$\begin{aligned}(\triangle+m^2)^k u(x)=(-1)^{k-2}(R^e_{2(k-2)}(x))^{(l)}\\ u(x)=(-1)^{k-2}(R^e_{2(k-2)}(x))^{(l)}*W^e_{2k}(v,m)\end{aligned}$$

$for\ x\in\partial\Omega$ and $k=2,3,4,\ldots$.

Moreover, if we put $k=1$ in (9), then

$$\triangle(\triangle+m^2)u(x)=f(x,(\triangle+m^2)u(x)) \tag{22}$$

with boundary condition

$$(\triangle+m^2)u(x)=0, \quad for\ x\in\partial\Omega,$$

respectively, we obtain

$$u(x)=W^e_{2k}(v,m)*W(x).$$

From (22) we can write

$$\triangle^2 u(x)=g(x,(\triangle)u(x)), \tag{23}$$

where $g(x, (\triangle)u(x)) = f(x, (\triangle + m^2)u(x)) - m^2\triangle u(x)$ and (23) is called the nonhomogeneous biharmonic equation.

This completes the proof. □

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