Properties of a Class of p-Valent Analytic Functions Defined by Using Convolution

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Abstract

In this paper we have introduced a new class \( S^*(\alpha, p, \lambda, \beta) \) of analytic and p-valent functions in the unit disc \( U = \{ z : |z| < 1 \} \) and obtain some sharp results including coefficient estimates, closure theorems, distortion theorem, integral operators and some results for convolution of functions in the class \( S^*(\alpha, p, \lambda, \beta) \). Also we define some operators of fractional calculus and we obtain several sharp results of growth and distortion properties of the functions belonging to the class \( S^*(\alpha, p, \lambda, \beta) \).

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1 Introduction

Let \( A \) denote the class of functions of the form,

\[
f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad a_n \geq 0, \quad p \in N
\]  

(1.1)
which are analytic and p-valent in the unit disc $U = \{ z : |z| < 1 \}$.

Let $S$ denote the subclass of $A$ consisting of analytic and p-valent functions $f(z)$ in $U$

**Definition 1.1** A function $f(z) \in S$ is said to be p-starlike of order $\alpha$, $0 \leq \alpha < p$ if and only if

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U)$$

we denote the class of all p-valent starlike functions of order $\alpha$ by $S^*(\alpha)$

**Definition 1.2** A function $f(z) \in S$ is said to be convex of order $\alpha$, $0 \leq \alpha < p$ if and only if

$$\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in U)$$

we denote the class of all p-valent convex functions of order $\alpha$ by $K(\alpha)$

We note that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$

The classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Owa [3] and studied by Srivastava and Owa [9], Patil and Thakare [5] and A. Shakor S. Teim [1]

Now, the function $S(z) = z^p(1-z)^{-2(1-\alpha)}, 0 \leq \alpha < p, p = 1, 2, 3$ is well known extreme function for the class $S^*(\alpha)$ setting

$G(\alpha, n) = \prod_{i=2}^{n+1} (i-2\alpha) / n!$, $n \geq 1$

then $S(z)$ can be written in the form

$S(z) = z^p + \sum_{n=p+1}^{\infty} G(\alpha, n) z^n$

We note that $G(\alpha, n)$ is a decreasing function in $\alpha$ and that

$$\lim_{n \to \infty} G(\alpha, n) \begin{cases} \infty , \quad \alpha < \frac{1}{2} \\ 1 , \quad \alpha = \frac{1}{2} \\ 0 , \quad \alpha > \frac{1}{2} \end{cases}$$

Let $(f * g)(z)$ denote the Hadamard product (Convolution) of two analytic functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and

$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$,

then $(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n$, 

Let \( T \) denote the subclass of \( S \) consisting of functions of the form
\[
f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, \quad a_n \geq 0
\]

(1.2)

A function \( f(z) \) defined by (1.1) belong to the class \( S(\alpha, p, \lambda, \beta) \) if and only if
\[
\left| \frac{z(f \ast S(z))' - p(f \ast S(z))}{(1 - \lambda)(f \ast S(z)) + \frac{\lambda}{p} z(f \ast S(z))'} \right| < \beta
\]

for some \( \alpha, 0 \leq \alpha < p, 0 \leq \lambda < p, 0 \leq \beta < p \) and all \( z \in U \).

Further we denote
\[
S^*(\alpha, p, \lambda, \beta) = S(\alpha, p, \lambda, \beta) \cap T
\]

### 2 Coefficient Estimates

**Theorem 1** Let the function \( f(z) \) be defined by (1.2).
Then \( f(z) \) is in the class \( S(\alpha, p, \lambda, \beta) \) if and only if
\[
\sum_{n=p+1}^{\infty} \left[ p(n - p) + \beta(\lambda n - p\lambda + p) \right] a_n G(\alpha, n) \leq p\beta
\]

(2.1)

and the result is sharp

**Proof:** Assume that inequality (2.1) holds true and \( |z| = 1 \). Then we obtain
\[
\left| \frac{z(f \ast S(z))' - p(f \ast S(z))}{(1 - \lambda)(f \ast S(z)) + \frac{\lambda}{p} z(f \ast S(z))'} \right| = \beta
\]
\[
\leq \left| \sum_{n=p+1}^{\infty} p(n - p)a_n G(\alpha, n)z^n \right| \leq p\beta - \sum_{n=p+1}^{\infty} \left| \left[ p(1 - \lambda) + \lambda n \right] a_n G(\alpha, n)z^n \right|
\]
\[
\leq \sum_{n=p+1}^{\infty} \left[ p(n - p) + \beta(\lambda n - p\lambda + p) \right] a_n G(\alpha, n) \leq p\beta
\]

by hypothesis.
Hence by maximum modulus principle, we have
\( f(z) \in S^*(\alpha, p, \lambda, \beta) \)
Conversely, let \( f(z) \in S^*(\alpha, p, \lambda, \beta) \)
Then \( \left| \frac{z(f * S(z))' - p(f * S(z))}{(1 - \lambda)(f * S(z)) + \frac{\lambda z}{p}(f * S(z))'} - \frac{\lambda z}{p}(f * S(z))' \right| < \beta, \quad z \in U \)
That is
\[
= \left| \sum_{n=p+1}^{\infty} p(n-p)a_n G(\alpha, n) z^n \right| < \beta, \quad (2.2)
\]

Since \(|Re f(z)| \leq |f(z)|\) for all \(z\), we have
\[
Re \left\{ \sum_{n=p+1}^{\infty} p(n-p)a_n G(\alpha, n) z^n \right\} < \beta \quad (2.3)
\]

Choosing \(z\) on real axis and allowing \(z \to 1\)
\[
\sum_{n=p+1}^{\infty} p(n-p)a_n G(\alpha, n) \leq \beta \quad (2.1)
\]

Finally, the result is sharp with extremal function \(f(z)\) given by
\[
f(z) = z^p - \frac{p\beta}{[p(n-p) + \beta(\lambda n - p\lambda + p)]G(\alpha, n)} z^n, \quad n \geq p + 1 \quad (2.4)
\]

**Corollary 2.1**: Let the function \(f(z)\) defined by (1.2) be in the class \(S^*(\alpha, p, \lambda, \beta)\). Then we have
\[
a_n \leq \frac{p\beta}{[p(n-p) + \beta(\lambda n - p\lambda + p)]G(\alpha, n)}, \quad n \geq p + 1 \quad (2.5)
\]
The equality in (2.5) is attained for the function \(f(z)\) given by (2.4)
3 Closure Theorem

Theorem 2 Let the function $f_j(z), j = 1, 2, \ldots m$ defined by

$$f_j(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,j} z^n, a_{n,j} \geq 0 \text{ for } z \in U \tag{3.1}$$

be in the class $S^*(\alpha, p, \lambda, \beta)$. Then the function $h(z)$ defined by

$$h(z) = z^p - \sum_{n=p+1}^{\infty} b_n z^n$$

also belongs to the class $S^*(\alpha, p, \lambda, \beta)$ where

$$b_n = \frac{1}{m} \sum_{j=p+1}^{\infty} a_{n,j}$$

Proof: Since $f_j(z) \in S^*(\alpha, p, \lambda, \beta)$ it follows from Theorem 1 that

$$\sum_{n=p+1}^{\infty} [p(n-p)+\beta(\lambda n - p\lambda + p)]G(\alpha, n)a_{n,j} \leq p\beta, \quad j = 1, 2, \ldots m$$

Therefore

$$\sum_{n=p+1}^{\infty} [p(n-p)+\beta(\lambda n - p\lambda + p)]G(\alpha, n)b_n$$

$$= \sum_{j=p+1}^{\infty} [p(n-p)+\beta(\lambda n - p\lambda + p)]G(\alpha, n) \left( \frac{1}{m} \sum_{j=p}^{\infty} a_{n,j} \right)$$

$$= \frac{1}{m} \sum_{j=1}^{\infty} \sum_{n=p+1}^{\infty} [p(n-p)+\beta(\lambda n - p\lambda + p)]G(\alpha, n)a_{n,j}$$

$$\leq p\beta$$

Hence by Theorem 1, $h(z) \in S^*(\alpha, p, \lambda, \beta)$

Thus we have the result

Theorem 3 The class $S^*(\alpha, p, \lambda, \beta)$ is closed under convex linear combinations. As a consequence of Theorem 5, there exists extreme points of the class $S^*(\alpha, p, \lambda, \beta)$

Theorem 4 Let $f_p(z) = z$ and

$$f_n(z) = z^p - \frac{p\beta}{[p(n-p)+\beta(\lambda n - p\lambda + p)]G(\alpha, n)}z^n, \quad n \geq p + 1 \tag{3.2}$$

for $0 \leq \alpha < p, \quad 0 \leq \lambda < p$ and $0 \leq \beta < p.

Then $f(z)$ is in the class $S^*(\alpha, p, \lambda, \beta)$ if and only if it can be expressed in the form
\[ f(z) = \sum_{n=p+1}^{\infty} \mu_n f_n(z), \text{ where } \mu_n \geq 0, \quad n = p + 1 \]

and \[ \sum_{n=p+1}^{\infty} \mu_n = p \]

4 \quad \textbf{Distortion Theorem}

Using Theorem 1, we may find bounds of the modulus of \( f(z) \) and \( f'(z) \) for \( f(z) \in S^*(\alpha, p, \lambda, \beta) \)

Theorem 5 \textit{If the function } f(z) \textit{defined by (1.2) is in the class } S^*(\alpha, p, \lambda, \beta) \textit{,} 0 \leq \lambda < p \textit{ and } 0 \leq \beta < p, \textit{ and either } 0 \leq \alpha \leq \frac{5}{6} \textit{ or } |z| \leq \frac{3}{4} \textit{ then}

\[ |f(z)| \geq \max \{0, 1 - \frac{p\beta}{2[p(n-p) + \beta(\lambda n - p\lambda + p)](1-\alpha)}|z|^{n-p}\} \]

and

\[ |f(z)| \leq 1 + \frac{p\beta}{2[p(n-p) + \beta(\lambda n - p\lambda + p)](1-\alpha)}|z|^{n-p} \]

The bounds are sharp

**Proof:**- By virtue of Theorem 1, we note that

\[ |f(z)| \geq \max \left\{0, |z|^p + \max_{\n \in \mathbb{N} \setminus \{1\}} \frac{p\beta}{2[p(n-p) + \beta(\lambda n - p\lambda + p)](1-\alpha)}|z|^n\right\} \]

\[ |f(z)| \leq |z|^p + \max_{\n \in \mathbb{N} \setminus \{1\}} \frac{p\beta}{2[p(n-p) + \beta(\lambda n - p\lambda + p)](1-\alpha)}|z|^n \]

for \( z \in U \). Hence it suffices to deduce that

\[ H(\alpha, \lambda, \beta, |z|, n) \frac{p\beta}{2[p(n-p) + \beta(\lambda n - p\lambda + p)](1-\alpha)}|z|^n \]

in decreasing function of \( n(n \geq p + 1) \)

Since \( G(\alpha, n+1) = \frac{n+1-2\alpha}{n}G(\alpha, n) \), we can see that,

for \( |z| \neq 0, H(\alpha, \lambda, \beta, |z|, n) \geq H(\alpha, \lambda, \beta, |z|, n+1) \) if and only if
$I(\alpha, |z|, n) = (n + 1)(n + 1 - 2\alpha) - n^2 |z| \geq 0$

It is easy to show that $I(\alpha, |z|, n)$ is a decreasing function for $\alpha$ for fixed $|z|$.

Consequently it follows that

$I(\alpha, |z|, n) \geq I(5/6, |z|, n) = n^2(1 - |z|) + \frac{1}{3}(n - 2) \geq 0$

for $0 \leq \alpha \leq 5/6$, $z \in U, n \geq 2$

Further, since $I(\alpha, |z|, n)$ is decreasing in $|z|$ and decreasing in $n$, we obtain that

$I(\alpha, |z|, n) > I(1, |z|, n) \geq I(1, 3/4, 2) = 0$

for $0 \leq \alpha \leq 1$, $|z| < 3/4$ and $n \geq 2$. Thus $\max_{n \in \mathbb{N}\{1\}}(\alpha, \lambda, \beta, |z|, n)$ is attained at $n = 2$

Finally, since the function $f_n(z)(n \geq 2)$ defined in Theorem 6 are extreme points of the class $S^*(\alpha, p, \lambda, \beta)$, we can see that the bounds of Theorem 7 are attained by the $f_{p+1}(z)$, that is

$$f_{p+1}(z) = z^p - \frac{p\beta}{2[p(n - p) + \beta(\lambda n - p\lambda + p)](1 - \alpha)}|z|^n$$

**Theorem 6** If the function $f(z)$ defined by (1.2) is in the class $S^*(\alpha, p, \lambda, \beta)$, $0 \leq \lambda < p$ and $0 \leq \beta < p$ and either $0 \leq \alpha \leq \frac{p}{2}$ or $|z| < \frac{p}{2}$ then

$$1 - \frac{p\beta}{2[p(n - p) + \beta(\lambda n - p\lambda + p)](1 - \alpha)}z^{n-p} \leq |f'(z)| \leq 1 + \frac{p\beta}{2[p(n - p) + \beta(\lambda n - p\lambda + p)](1 - \alpha)}z^{n-p}$$

The bounds are sharp

**Proof:**

5 Integral Operators

**Theorem 7** Let the function $f(z)$ defined by (1.2) is in the class $S^*(\alpha, p, \lambda, \beta)$ and let $d$ be real number such that $d > -p$ then the function $F(z)$ defined by

$$F(z) = \frac{d + p}{z^d} \int_0^z t^{d-1} f(t) dt \text{ belongs to } S^*(\alpha, p, \lambda, \beta) \quad (5.1)$$

**Proof:** From the representation of $F(z)$, it follows that

$$F(z) = z^p - \sum_{n=p+1}^{\infty} b_n z^n, \quad \text{where} \quad b_n = \frac{d + p}{d + n} a_n$$
Therefore
\[
\sum_{n=p+1}^{\infty} \left[ p(n-p) + \beta(\lambda n - p\lambda + p) \right] G(\alpha, n) b_n \\
= \sum_{n=p+1}^{\infty} \left[ p(n-p) + \beta(\lambda n - p\lambda + p) \right] G(\alpha, n) \left( \frac{d+p}{d+n} \right) a_n \\
\leq \sum_{n=p+1}^{\infty} \left[ p(n-p) + \beta(\lambda n - p\lambda + p) \right] G(\alpha, n) a_n \\
\leq p \beta
\]

Hence by Theorem 1, \( F(z) \in S^*(\alpha, p, \lambda, \beta) \)

**Theorem 8** Let the function \( F(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n \) \((a_n \geq 0)\) be in the class \( S^*(\alpha, p, \lambda, \beta) \) and let \( d \) be real number such that \( d > -p \). Then the function \( F(z) \) defined by (5.1) is \( p \)-valent in \(|z| < R\)

Where \( R = \inf_n \left\{ \frac{p(d+p)[p(n-p) + \beta(\lambda n - p\lambda + p)]G(\alpha, n)}{n^2 \beta(d+n)} \right\} \frac{1}{n-p}, \quad n \geq p+1 \n
The result is sharp

**Proof:** From (5.1), we have
\[
f(z) = \frac{z^{1-d}(z^d F(z))'}{d+p} = z^p - \sum_{n=p+1}^{\infty} \frac{d+n}{d+p} a_n z^n
\]

In order to obtain the required result it is sufficient to show that
\[
Re \left( 1 + \frac{zf''(z)}{f'(z)} - p \right) \leq p
\]

\[
\leq \left| \frac{\left| \sum_{n=p+1}^{\infty} n \left( \frac{d+p}{d+n} \right) (n-p) \sum_{n=p+1}^{\infty} a_n z^{n-1} \right|}{p \sum_{n=p+1}^{\infty} a_n z^{n-1}} \right|
\]

\[
= \left| \frac{n \left( \frac{d+p}{d+n} \right) (n-p) \sum_{n=p+1}^{\infty} a_n z^{n-1}}{p \sum_{n=p+1}^{\infty} a_n z^{n-1}} \right|
\]

\[
\leq \left| \frac{p \sum_{n=p+1}^{\infty} a_n z^{n-1}}{p \sum_{n=p+1}^{\infty} a_n z^{n-1}} \right|
\]

\[
\leq p
\]
which is equivalent to show that

\[
\left( \frac{n}{p} \right)^2 \frac{d+n}{d+p} \sum_{n=p+1}^{\infty} a_n |z|^{n-p} \leq 1
\]

As \( f(z) \in S^*(\alpha, p, \lambda, \beta) \), we have from Theorem 1,

\[
\left( \frac{n}{p} \right)^2 \frac{d+n}{d+p} |z|^{n-p} \leq \frac{[p(n-p) + \beta(\lambda n - p\lambda + p)]G(\alpha, n)}{p\beta}
\]

or \(|z| \leq \left\{ \frac{[p(n-p) + \beta(\lambda n - p\lambda + p)]G(\alpha, n)}{\left( \frac{n}{p} \right)^2 \left( \frac{d+n}{d+p} \right) p\beta} \right\} \frac{1}{n-p} \) \( n \geq p+1 \)

Setting \(|z| = R\), we get the result

\[
R = \inf_n \left\{ \frac{p(d+p)[p(n-p) + \beta(\lambda n - p\lambda + p)]G(\alpha, n)}{n^2\beta(d+n)} \right\} \frac{1}{n-p}, \ n \geq p+1
\]

6 Fractional Calculus Operators

In this section our aim is to obtain several growth and distortion properties of functions in the class \( S^*(\alpha, p, \lambda, \beta) \) involving a family of operators of Fractional Calculus (fractional integral and fractional derivative operators)

First of all, in terms of Gauss hyper geometric function

\[
2F_1(\delta, \tau; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\delta)_k(\tau)_k z^k}{(\gamma)_k k!}, \ (z \in U; \delta, \tau; \gamma \in C; \gamma \neq 0, -1, -2 \ldots)
\]

where \((m)_k = \frac{(m+k)!}{m!}\) denotes the Pochhammer symbol,

we recall the definitions of fractional integral operator \( J_{o,z}^{\mu,\xi,\eta} \) and the fractional derivative operator \( k_{o,z}^{\mu,\xi,\eta} \) as follows (cf., eg., [4] and [11] see also [9])

**Definition 6.1** The fractional integral of order \( \mu \) is defined for a function \( f(z) \), by

\[
J_{o,z}^{\mu,\xi,\eta} f(z) = \frac{z^{-\mu-\xi}}{|\gamma|} \int_{0}^{z} (z-t)^{\mu-1} 2F_1(\mu+\xi, -\eta; \mu; 1-\frac{t}{z}) f(t) dt, \mu > 0
\]
where \( f(z) \) is analytic function is a simple connected region of \( z \)-plane containing the origin, and the multiplicity of \((z-t)^{\mu-1}\) is removed by requiring log\((z-t)\) to be real when \((z-t) > 0\), provided further that

\[
f(z) = O(|z|^\delta)(z \in \Omega; \epsilon > \max\{0, \xi - \eta\} - 1)
\] (6.1)

**Definition 6.2** The fractional derivative of order \( \mu \) is defined for a function \( f(z) \), by

\[
K^{\mu,\xi,\eta}_{0,z}f(z) = \frac{1}{|1-\mu|} \frac{d}{dz} \left\{ z^{\mu-\xi} \int_0^z (z-t)^{-\mu} F_1(\xi-\mu,1-\eta;1-\mu,1-\frac{t}{z}) f(t) dt \right\}
\]

\[
= \frac{d^n}{dz^n} K^{\mu-\xi,\eta}_{0,z}f(z), (\eta \leq \mu < n+1; \eta \in \mathbb{N})
\]

where the function \( f(z) \) is constrained and the multiplicity of \((z-t)^{-\mu}\) is removed, as in definition 6.1, and \( \epsilon \) is given by the order estimate (6.1). If following from the definition 1.1 and 1.2 that

\[
J^{\mu,-\xi,\eta}_{0,2}f(z) = D_z^{-\mu} f(z), (\mu > 0)
\] (6.2)

\[
K^{\mu,\xi,\eta}_{0,z}f(z) = D_z^{\mu} f(z), (0 \leq \mu < 1)
\] (6.3)

where \( D_z^{\mu} f(z)(\mu \in \mathbb{R}) \) is the fractional calculus operator consider by Owa [2] and subsequently by Srivastava and Owa [9]

Furthermore, in terms of Gamma functions Definition 6.1 and 6.2

**Lemma 6.1** (cf. Srivastava et al. [11]). The (generalized) fractional integral and the (generalized) fractional of a power functions are given by

\[
J^{\mu,-\xi,\eta}_{0,z}z^p = \frac{(p+1)(p+\xi+1)}{(p-\xi+1)(p-\mu+\xi+1)} z^{p-\xi}
\] (6.4)

where \((\mu > 0; p > \max\{0, \xi - \eta\} - 1)\)

and

\[
K^{\mu,\xi,\eta}_{0,z}z^p = \frac{(p+1)(p-\xi+1)}{(p-\xi+1)(p-\mu+\eta+1)} z^{p-\xi}
\] (6.5)

where \((\mu \leq 0; p > \max\{0, \xi - \eta\} - 1)\)

**Theorem 9** Let the function \( f(z) \) defined by (2.1) be in the class \( S^*(\alpha, p, \lambda, \beta) \), with \( 0 \leq \alpha < \frac{p}{2} \), \( 0 \leq \alpha < \frac{p}{2} \) and \( 0 \leq \beta < p \)
Then
\[
\left(\frac{(p+1)(p+\xi+\eta+1)}{(p+1-\xi)(p+1+\mu+\eta)}\right) |z|^{p-\xi} \times \frac{\beta(p-\xi+\eta+1)}{(p+1)(p+1-\xi)(p+1+\mu+\eta)[p(n-p)+\beta(\lambda n-p\alpha+p)] \cdot (1-\alpha)} |z| \\
\leq \left| J_{0,z}^{\mu,\xi,\eta} f(z) \right| \leq \left(\frac{u+1}{u+\xi+\eta+1}\right) |z|^{p-\xi} \times \frac{\beta(p-\xi+\eta+1)}{(p+1)(p+1-\xi)(p+1+\mu+\eta)[p(n-p)+\beta(\lambda n-p\alpha+p)] \cdot (1-\alpha)} |z| \\
(6.6)\]
\[
(z \in U_0; \mu > 0, \max\{\xi, \xi-\eta, -\mu-\eta\} < 2; \xi(\mu+\eta) \leq 3\mu) \text{ and}
\]
\[
\left(\frac{(p+1)(p+\xi+\eta+1)}{(p+1-\xi)(p+1+\mu+\eta)}\right) |z|^{p-\xi} \times \frac{\beta(p-\xi+\eta+1)}{(p+1)(p+1-\xi)(p+1+\mu+\eta)[p(n-p)+\beta(\lambda n-p\alpha+p)] \cdot (1-\alpha)} |z| \\
\leq \left| K_{0,z}^{\mu,\xi,\eta} f(z) \right| \leq \left(\frac{u+1}{u+\xi+\eta+1}\right) |z|^{p-\xi} \times \frac{\beta(p-\xi+\eta+1)}{(p+1)(p+1-\xi)(p+1+\mu+\eta)[p(n-p)+\beta(\lambda n-p\alpha+p)] \cdot (1-\alpha)} |z| \\
(6.7)\]
\[
(z \in U_0; 0 \leq \mu < p, \max\{\xi, \xi-\eta, -\mu-\eta\} < 2; \xi(\mu-\eta) \geq 3\mu),
\]
\[
\text{where } U_0 = \begin{cases} U & , \xi \leq p \\ U/\{0\} & , \xi > p \end{cases}
\]
Each of these results is sharp for the function \( f(z) \) given by
\[
f(z) = z^p - \frac{\beta p^p}{2[p(n-p) + \beta(\lambda n-p\alpha+p)](1-\alpha)} z^n \quad (6.8)
\]
Proof: Applying Theorem 1, we have
\[ \sum_{n=p+1}^{\infty} a_n \leq \frac{p\beta}{[p(n-p) + \beta(\lambda n - p\lambda + p)]} G(\alpha, n) \]  
(6.9)

Next, using the assertion (6.4) of Lemma 1, we find from (1.2) that
\[ F(z) = \frac{\Gamma(p+\xi+\eta+1)}{\Gamma(p+1)\Gamma(p-\xi+\eta+1)} z^\xi J_{\nu_n}^{\mu,\xi,n} f(z) = z^p - \sum_{n=p+1}^{\infty} \phi(n) a_n z^n \]  
(6.10)

where \( \phi(n) = \frac{(p+1)(p-\xi+\eta+1)}{(p-\xi+1)(p+\mu+\xi+1)} \), \( \eta \geq p+1 \)  
(6.11)

The function \( \phi(n) \) defined by (6.11) can easily be seen to be non-increasing under the parameter constraints stated in (6.6) and thus we have
\[ 0 < \phi(n) \leq \phi(p+1) = \frac{(p+1)(p-\xi+\eta+1)}{(p-\xi+1)(p+\mu+\xi+1)} \]  
(6.12)

The assertion (6.6) of the theorem follows from (6.9), (6.10) and (6.12)

The assertion (6.7) of the theorem can be proved similarly by noting from (6.5) that
\[ G(z) = \frac{(p+1-\xi)(p+1-\mu+\eta)}{(p+1)\Gamma(p-\xi+\eta+1)} z^\xi K_{\nu_n}^{0,\xi,n} f(z) = z^p - \sum_{n=p+1}^{\infty} \psi(n) a_n z^n \]  
(6.13)

where
\[ \psi(n) = \frac{(p+1)(p-\xi+\eta+1)}{(p+1-\xi)(p+1-\mu+\eta)} \], \( n \geq p+1 \)  
(6.14)

Thus similarly,
\[ 0 < \psi(n) \leq \psi(p+1) = \frac{(p+1)(p-\xi+\eta+1)}{(p+1-\xi)(p+1-\mu+\eta)}, \quad n \in \mathbb{N}\backslash\{1\} \]  
(6.15)

The assertion (6.6) and (6.7) are attained

**Corollary 6.1** Let the function \( f(z) \) defined by (1.2) be in the class \( S^*(\alpha, p, \lambda, \beta) \), with \( 0 \leq \alpha < \frac{p}{2} \), \( 0 \leq \alpha < p \) and \( 0 \leq \beta < p \)
Then
\[
\frac{|(p+1)|}{|(p+1+\mu)|} |z|^{p+\mu} \left[ 1 - \frac{p(p+1)\beta}{2(p+1)(p+1+\mu)|p(n-p)+\beta(\lambda n-p\lambda + p)|(1-\alpha)|z|} \right]
\]
\[
\leq |D_z f(z)| \leq \frac{|(p+1)|}{|(p+1+\mu)|} |z|^{p+\mu} \left[ 1 + \frac{p(p+1)\beta}{2(p+1)(p+1+\mu)|p(n-p)+\beta(\lambda n-p\lambda + p)|(1-\alpha)|z|} \right]
\]
and
\[
\frac{|(p+1)|}{|(p+1-\mu)|} |z|^{p-\mu} \left[ 1 - \frac{p(p+1)\beta}{2(p+1)(p+1-\mu)|p(n-p)+\beta(\lambda n-p\lambda + p)|(1-\alpha)|z|} \right]
\]
\[
\leq |D_z^\mu| \leq \frac{|(p+1)|}{|(p+1-\mu)|} |z|^{p-\mu} \left[ 1 + \frac{p(p+1)\beta}{2(p+1)(p+1-\mu)|p(n-p)+\beta(\lambda n-p\lambda + p)|(1-\alpha)|z|} \right]
\]
(6.16)
(6.17)

Each of these result are sharp for the function \( f(z) \) given by (6.8)

References


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