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Properties of a Class of p-Valent Analytic Functions Defined by Using Convolution

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Abstract

In this paper we have introduced a new class $S^*(\alpha, p, \lambda, \beta)$ of analytic and p-valent functions in the unit disc $U = \{z : |z| < 1\}$ and obtain some sharp results including coefficient estimates, closure theorems, distortion theorem, integral operators and some results for convolution of functions in the class $S^*(\alpha, p, \lambda, \beta)$. Also we define some operators of fractional calculus and we obtain several sharp results of growth and distortion properties of the functions belonging to the class $S^*(\alpha, p, \lambda, \beta)$.

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1 Introduction

Let A denote the class of functions of the form,

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, a_n \geq 0, p \in N \tag{1.1}$$

which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$.
Let S denote the subclass of A consisting of analytic and p-valent functions $f(z)$ in U

Definition 1.1 *A function $f(z) \in S$ is said to be p-starlike of order $\alpha, 0 \leq \alpha < p$ if and only if*

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U)$$

we denote the class of all p-valent starlike functions of order α by $S^*(\alpha)$

Definition 1.2 *A function $f(z) \in S$ is said to be convex of order $\alpha, 0 \leq \alpha < p$ if and only if*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in U)$$

we denote the class of all p-valent convex functions of order α by $K(\alpha)$

We note that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$
The classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Owa [3] and studied by Srivastava and Owa [9], Patil and Thakare [5] and A.Shakor S.Teim [1]
Now, the function $S(z) = z^p(1 - z)^{-2(1-\alpha)}, 0 \leq \alpha < p, p = 1, 2, 3\cdot$
is well known extreme function for the class $S^*(\alpha)$ setting

$$G(\alpha, n) = \frac{\prod_{i=2}^{n+1}(i - 2\alpha)}{n!}, n \geq 1$$

then $S(z)$ can be written in the form

$$S(z) = z^p + \sum_{n=p+1}^{\infty} G(\alpha, n)z^n$$

We note that $G(\alpha, n)$ is a decreasing function in α and that

$$\lim_{n\to\infty} G(\alpha, n) \begin{cases} \infty & , \quad \alpha < \dfrac{1}{2} \\ 1 & , \quad \alpha = \dfrac{1}{2} \\ 0 & , \quad \alpha > \dfrac{1}{2} \end{cases}$$

Let $(f * g)(z)$ denote the Hadamard product (Convolution) of two analytic functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n,$$

then $(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n,$

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, \quad a_n \geq 0 \tag{1.2}$$

A function $f(z)$ defined by (1.1) belong to the class $S(\alpha, p, \lambda, \beta)$ if and only if $f(z)$ satisfies the condition

$$\left| \frac{z(f * S(z))' - p(f * S(z))}{(1-\lambda)(f * S(z)) + \frac{\lambda}{p} z(f * S(z))'} \right| < \beta$$

for some $\alpha, 0 \leq \alpha < p, 0 \leq \lambda < p, 0 \leq \beta < p$ and all $z \in U$.
Further we denote
$S^*(\alpha, p, \lambda, \beta) = S(\alpha, p, \lambda, \beta) \cap T$

2 Coefficient Estimates

Theorem 1 *Let the function $f(z)$ be defined by (1.2).*
Then $f(z)$ is in the class $S(\alpha, p, \lambda, \beta)$ if and only if

$$\sum_{n=p+1}^{\infty} [p(n-p) + \beta(\lambda n - p\lambda + p)] \, a_n G(\alpha, n) \leq p\beta \tag{2.1}$$

and the result is sharp

Proof:- Assume that inequality (2.1) holds true and $|z| = 1$. Then we obtain

$$|z(f * S(z))' - p(f * S(z))| - \beta \left| (1-\lambda)(f * S(z)) + \frac{\lambda}{p} z(f * S(z)') \right|$$

$$\leq \left| \sum_{n=p+1}^{\infty} p(n-p) a_n G(\alpha, n) z^n \right| - \beta \left| pz^p - \sum_{n=p+1}^{\infty} [p(1-\lambda) + \lambda n] a_n G(\alpha, n) z^n \right|$$

$$\leq \sum_{n=p+1}^{\infty} [p(n-p) + \beta(\lambda n - p\lambda + p)] a_n G(\alpha, n) \leq p\beta$$

by hypothesis
Hence by maximum modulus principle, we have
$f(z) \in S^*(\alpha, p, \lambda, \beta)$
Conversly, let $f(z) \in S^*(\alpha, p, \lambda, \beta)$

Then $\left|\dfrac{z(f*S(z))'-p(f*S(z))}{(1-\lambda)(f*S(z))+\dfrac{\lambda z}{p}(f*S(z))'}\right|<\beta, \quad z\in U$

That is

$$=\frac{\left|\sum_{n=p+1}^{\infty}p(n-p)a_nG(\alpha,n)z^n\right|}{\left|pz^p-p(1-\lambda)\sum_{n=p+1}^{\infty}a_nG(\alpha,n)z^n-\lambda\sum_{n=p+1}^{\infty}na_nG(\alpha,n)z^n\right|}<\beta, \tag{2.2}$$

Since $|Ref(z)|\leq|f(z)|$ for all z, we have

$$\left|Re\left\{\frac{\sum_{n=p+1}^{\infty}p(n-p)a_nG(\alpha,n)z^n}{pz^p-p(1-\lambda)\sum_{n=p+1}^{\infty}a_nG(\alpha,n)z^n-\lambda\sum_{n=p+1}^{\infty}na_nG(\alpha,n)z^n}\right\}\right|<\beta \tag{2.3}$$

Choosing z on real axis and allowing $z\rightarrow 1$

$$\frac{\sum_{n=p+1}^{\infty}p(n-p)a_nG(\alpha,n)}{p-p(1-\lambda)\sum_{n=p+1}^{\infty}a_nG(\alpha,n)-\lambda\sum_{n=p+1}^{\infty}na_nG(\alpha,n)}\leq\beta$$

$$\therefore\sum_{n=p+1}^{\infty}[p(n-p)+\beta(\lambda n-p\lambda+p)]a_nG(\alpha,n)\leq p\beta$$

which gives (2.1)

Finally, the result is sharp with extremal function $f(z)$ given by

$$f(z)=z^p-\frac{p\beta}{[p(n-p)+\beta(\lambda n-p\lambda+p)]G(\alpha,n)}z^n, \quad n\geq p+1 \tag{2.4}$$

Corollary 2.1 : *Let the function $f(z)$ defined by (1.2) be in the class $S^*(\alpha,p,\lambda,\beta)$ Then we have*

$$a_n\leq\frac{p\beta}{[p(n-p)+\beta(\lambda n-p\lambda+p)]G(\alpha,n)}, \quad n\geq p+1 \tag{2.5}$$

The equality in (2.5) is attained for the function $f(z)$ given by (2.4)

3 Closure Theorem

Theorem 2 *Let the function $f_j(z), j = 1, 2, \dots m$ defined by*

$$f_j(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,j} z^n, a_{n,j} \geq 0 \quad for \quad z \in U \tag{3.1}$$

be in the class $S^*(\alpha, p, \lambda, \beta)$. Then the function $h(z)$ defined by $h(z) = z^p - \sum_{n=p+1}^{\infty} b_n z^n$ also belongs to the class $S^*(\alpha, p, \lambda, \beta)$ where $b_n = \frac{1}{m} \sum_{j=p+1}^{\infty} a_{n,j}$

Proof:- Since $f_j(z) \in S^*(\alpha, p, \lambda, \beta)$ it follows fron Theorem 1 that $\sum_{n=p+1}^{\infty} [p(n-p) + \beta(\lambda n - p\lambda + p)]G(\alpha, n)a_{n,j} \leq p\beta, \quad j = 1, 2, \dots m$
Therefore

$$\begin{aligned}
& \sum_{n=p+1}^{\infty} [p(n-p) + \beta(\lambda n - p\lambda + p)]G(\alpha, n)b_n \\
= & \sum_{m=p+1}^{\infty} [p(n-p) + \beta(\lambda n - p\lambda + p)]G(\alpha, n) \left(\frac{1}{m} \sum_{j=p}^{\infty} a_{n,j} \right) \\
= & \frac{1}{m} \sum_{j=1}^{\infty} \sum_{n=p+1}^{\infty} [p(n-p) + \beta(\lambda n - p\lambda + p)]G(\alpha, n)a_{n,j} \\
\leq & \; p\beta
\end{aligned}$$

Hence by Theorem 1, $h(z) \in S^*(\alpha, p, \lambda, \beta)$
Thus we have the result

Theorem 3 *The class $S^*(\alpha, p, \lambda, \beta)$ is closed under convex linear combinations. As a consequence of Theorem 5, there exists extreme points of the class $S^*(\alpha, p, \lambda, \beta)$*

Theorem 4 *Let $f_p(z) = z$ and*

$$f_n(z) = z^p - \frac{p\beta}{[p(n-p) + \beta(\lambda n - p\lambda + p)]G(\alpha, n)} z^n, \quad n \geq p + 1 \tag{3.2}$$

for $0 \leq \alpha < p, \; 0 \leq \lambda < p$ and $0 \leq \beta < p$.
Then $f(z)$ is in the class $S^*(\alpha, p, \lambda, \beta)$ if and only if it can be expressed in the form

$f(z) = \sum_{n=p+1}^{\infty} \mu_n f_n(z),\ where\ \mu_n \geq 0,\ n = p+1$

$and\ \sum_{n=p+1}^{\infty} \mu_n = p$

4 Distortion Theorem

Using Theorem 1, we may find bounds of the modulus of $f(z)$ and $f'(z)$ for $f(z) \in S^*(\alpha, p, \lambda, \beta)$

Theorem 5 *If the function $f(z)$ defined by (1.2) is in the class $S^*(\alpha, p, \lambda, \beta)$,$0 \leq \lambda < p$ and $0 \leq \beta < p$, and either $0 \leq \alpha \leq \frac{5}{6}$ or $|z| \leq \frac{3}{4}$ then*

$$|f(z)| \geq max\{0, 1 - \frac{p\beta}{2[p(n-p) + \beta(\lambda n - p\lambda + p)](1-\alpha)} |z|^{n-p}\}$$

and

$$|f(z)| \leq 1 + \frac{p\beta}{2[p(n-p) + \beta(\lambda n - p\lambda + p)](1-\alpha)} |z|^{n-p}$$

The bounds are sharp

Proof:- By virtue of Theorem 1, we note that

$$|f(z)| \geq max\left\{0, |z|^p - {}^{max}_{n\in N\setminus\{1\}} \frac{p\beta}{2[p(n-p) + \beta(\lambda n - p\lambda + p)](1-\alpha)} |z|^n\right\}$$

$$|f(z)| \leq |z|^p + {}^{max}_{n\in N\setminus\{1\}} \frac{p\beta}{2[p(n-p) + \beta(\lambda n - p\lambda + p)](1-\alpha)} |z|^n$$

for $z \in U$. Hence it suffices to deduce that

$$H(\alpha, \lambda, \beta, |z|, n) \frac{p\beta}{2[p(n-p) + \beta(\lambda n - p\lambda + p)](1-\alpha)} |z|^n$$

in decreasing function of $n (n \geq p+1)$

Since $G(\alpha, n+1) = \frac{n+1-2\alpha}{n} G(\alpha, n)$, we can see that,

for $|z| \neq 0, H(\alpha, \lambda, \beta, |z|, n) \geq H(\alpha, \lambda, \beta, |z|, n+1)$ if and only if

$I(\alpha, |z|, n) = (n+1)(n+1-2\alpha) - n^2 |z| \geq 0$

It is easy to show that $I(\alpha, |z|, n)$ is a decreasing function for α for fixed $|z|$.

Consequently it follows that

$I(\alpha, |z|, n) \geq I(5/6, |z|, n) = n^2(1-|z|) + \frac{1}{3}(n-2) \geq 0$

for $0 \leq \alpha \leq 5/6, \quad z \in U, n \geq 2$

Further, since $I(\alpha, |z|, n)$ is decreasing in $|z|$ and decreasing in n, we obtain that

$I(\alpha, |z|, n) > I(1, |z|, n) \geq I(1, 3/4, 2) = 0$

for $0 \leq \alpha \leq 1, |z| < 3/4$ and $n \geq 2$. Thus $max_{n \in N \setminus \{1\}}(\alpha, \lambda, \beta, |z|, n)$ is attained at $n = 2$

Finally, since the function $f_n(z)(n \geq 2)$ defined in Theorem 6 are extreme points of the class $S^*(\alpha, p, \lambda, \beta)$, we can see that the bounds of Theorem 7 are attained by the $f_{p+1}(z)$, that is

$f_{p+1}(z) = z^p - \frac{p\beta}{2[p(n-p) + \beta(\lambda n - p\lambda + p)](1-\alpha)} |z|^n$

Theorem 6 *If the function $f(z)$ defined by (1.2) is in the class $S^*(\alpha, p, \lambda, \beta)$, $0 \leq \lambda < p$ and $0 \leq \beta < p$ and either $0 \leq \alpha \leq \frac{p}{2}$ or $|z| < \frac{p}{2}$ then*

$$\begin{aligned} 1 &- \frac{p\beta}{2[p(n-p) + \beta(\lambda n - p\lambda + p)](1-\alpha)} z^{n-p} \\ \leq |f'(z)| \leq 1 &+ \frac{p\beta}{2[p(n-p) + \beta(\lambda n - p\lambda + p)](1-\alpha)} z^{n-p} \end{aligned}$$

The bounds are sharp

Proof:-

5 Integral Operators

Theorem 7 *Let the function $f(z)$ defined by (1.2) is in the class $S^*(\alpha, p, \lambda, \beta)$ and let d be real number such that $d > -p$ then the function $F(z)$ defined by*

$$F(z) = \frac{d+p}{z^d} \int_0^z t^{d-1} f(t) dt \quad \textit{belongs to} \quad S^*(\alpha, p, \lambda, \beta) \tag{5.1}$$

Proof:- From the representation of $F(z)$, it follows that

$$F(z) = z^p - \sum_{n=p+1}^{\infty} b_n z^n, \quad \text{where} \quad b_n = \frac{d+p}{d+n} a_n$$

Therefore

$$\begin{aligned}&\sum_{n=p+1}^{\infty}[p(n-p)+\beta(\lambda n-p\lambda+p)]G(\alpha,n)b_n\\ =\ &\sum_{n=p+1}^{\infty}[p(n-p)+\beta(\lambda n-p\lambda+p)]G(\alpha,n)\left(\frac{d+p}{d+n}\right)a_n\\ \leq\ &\sum_{n=p+1}^{\infty}[p(n-p)+\beta(\lambda n-p\lambda+p)]G(\alpha,n)a_n\\ \leq\ &p\beta\end{aligned}$$

Hence by Theorem 1, $F(z)\in S^*(\alpha,p,\lambda,\beta)$

Theorem 8 *Let the function* $F(z)=z^p-\sum_{n=p+1}^{\infty}a_nz^n \quad (a_n\geq 0)$ *be in the class* $S^*(\alpha,p,\lambda,\beta)$ *and let* d *be real number such that* $d>-p$. *Then the function* $F(z)$ *defined by (5.1) is p-valent in* $|z|<R$

Where $R={}_{n}^{inf}\left\{\frac{p(d+p)[p(n-p)+\beta(\lambda n-p\lambda+p)]G(\alpha,n)}{n^2\beta(d+n)}\right\}^{\frac{1}{n-p}}, \quad n\geq p+1$

The result is sharp

Proof:- From (5.1), we have $f(z)=\frac{z^{1-d}(z^dF(z))'}{d+p}=z^p-\sum_{n=p+1}^{\infty}\frac{d+n}{d+p}a_nz^n$

In order to obtain the required result it is sufficient to show that

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right|\leq p$$

$$\begin{aligned}Re\left(1+\frac{zf''(z)}{f'(z)}-p\right)\ &\leq\ \left|\frac{n\left(\frac{d+p}{d+n}\right)(n-p)\sum_{n=p+1}^{\infty}a_nz^{n-1}}{pz^{p-1}-n\left(\frac{d+p}{d+n}\right)\sum_{n=p+1}^{\infty}a_nz^{n-1}}\right|\\ &\leq\ \frac{n\left(\frac{d+p}{d+n}\right)(n-p)\sum_{n=p+1}^{\infty}a_nz^{n-1}}{pz^{p-1}-n\left(\frac{d+p}{d+n}\right)\sum_{n=p+1}^{\infty}a_nz^{n-1}}\\ &\leq\ p\end{aligned}$$

which is equivalent to show that

$$\left(\frac{n}{p}\right)^2 \frac{d+n}{d+p} \sum_{n=p+1}^{\infty} a_n \left|z\right|^{n-p} \leq 1$$

As $f(z) \in S^*(\alpha, p, \lambda, \beta)$, we have from Theorem 1,

$$\left(\frac{n}{p}\right)^2 \frac{d+n}{d+p} \left|z\right|^{n-p} \leq \frac{[p(n-p)+\beta(\lambda n - p\lambda + p)]G(\alpha, n)}{p\beta}$$

$$\text{or } |z| \leq \left\{ \frac{[p(n-p)+\beta(\lambda n - p\lambda + p)]G(\alpha, n)}{\left(\frac{n}{p}\right)^2 \left(\frac{d+n}{d+p}\right) p\beta} \right\}^{\frac{1}{n-p}}$$

Setting $|z| = R$, we get the result

$$R =_{n}^{inf} \left\{ \frac{p(d+p)[p(n-p)+\beta(\lambda n - p\lambda + p)]G(\alpha, n)}{n^2 \beta (d+n)} \right\}^{\frac{1}{n-p}}, \quad n \geq p+1$$

6 Fractional Calculus Operators

In this section our aim is to obtain several growth and distortion properties of functions in the class $S^*(\alpha, p, \lambda, \beta)$ involving a family of operators of Fractional Calculus (fractional integral and fractional derivative operators)

First of all, in terms of Gauss hyper geometric function

$${}_2F_1(\delta, \tau; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\delta)_k (\tau)_k z^k}{(\gamma)_k k!} (z \in U; \delta, \tau; \gamma \in C; \gamma; \neq 0, -1, -2 \ldots)$$

where $(m)_k = \dfrac{\lceil\overline{(m+k)}}{\lceil\overline{m}}$ denotes the Pochhamer symbol,

we recall the definitions of fractional integral operator $J_{o,z}^{\mu,\xi,\eta}$ and the fractional derivative operator $k_{o,z}^{\mu,\xi,\eta}$ as follows (cf., eg., [4] and [11] see also [9])

Definition 6.1 *The fractional integral of order μ is defined for a function $f(z)$, by*

$$J_o^{\mu,\xi,\eta} f(z) = \frac{z^{-\mu-\xi}}{\lceil\overline{\mu}} \int_0^z (z-t)^{\mu-1} {}_2F_1(\mu+\xi, -\eta; \mu; 1-\frac{t}{z}) f(t) dt, \mu > 0$$

where $f(z)$ is analytic function is a simple connected region of z-plane containing the origin, and the multiplicity of $(z-t)^{\mu-1}$ is removed by requiring $log(z-t)$ to be real when $(z-t)>0$, provided further that

$$f(z) = O(|z|^6)(z \in 0; \epsilon > max\{0, \xi - \eta\} - 1) \tag{6.1}$$

Definition 6.2 *The fractional derivative of order μ is defined for a function $f(z)$, by*

$$\begin{aligned} K_{o,z}^{\mu,\xi,\eta} f(z) &= \frac{1}{\Gamma 1-\mu} \frac{d}{dz} \left\{ z^{\mu-\xi} \int_0^z (z-t)^{-\mu} {}_2F_1(\xi-\mu, 1-\eta; 1-\mu; 1-\frac{t}{z}) f(t)dt, \right\} \\ &= \frac{d^n}{dz^n} K_{0,z}^{\mu-\eta,\xi,\eta} f(z), (\eta \leq \mu < n+1; \eta \in N) \end{aligned}$$

where the function $f(z)$ is constrained and the multiplicity of $(z-t)^{-\mu}$ is removed, as in definition 6.1, and $\in$ is given by the order estimate (6.1). If following from the definition 1.1 *and* 1.2 *that*

$$J_{0,2}^{\mu,-\mu,\eta} f(z) = D_z^{-\mu} f(z), \quad (\mu > 0) \tag{6.2}$$

$$K_{0,z}^{\mu,\mu,\eta} f(z) = D_z^{\mu} f(z), \quad (0 \leq \mu < 1) \tag{6.3}$$

where $D_z^\mu f(z)(\mu \in R)$ is the fractional calculus operator consider by Owa [2] and subsequently by Srivastava and Owa [9]

Furthermore, in terms of Gamma functions Definition 6.1 and 6.2

Lemma 6.1 *(cf. Srivastava et al. [11]). The (generalized) fractional integral and the (generalized) fractional of a power functions are given by*

$$J_{0,z}^{\mu,-\xi,\eta} z^p = \frac{\Gamma(p+1)\Gamma(p-\xi+\eta+1)}{\Gamma(p-\xi+1)\Gamma(p+\mu+\xi+1)} \cdot z^{p-\xi} \tag{6.4}$$

where $(\mu > 0; p > max\{0, \xi - \eta\} - 1)$

and

$$K_{0,z}^{\mu,\xi,\eta} z^p = \frac{\Gamma(p+1)\Gamma(p-\xi+\eta+1)}{\Gamma(p-\xi+1)\Gamma(p-\mu+\eta+1)} z^{p-\xi} \tag{6.5}$$

where $(\mu \leq 0; p > max\{0, \xi - \eta\} - 1)$

Theorem 9 *Let the function $f(z)$ defined by (2.1) be in the class $S^*(\alpha, p, \lambda, \beta)$, with $0 \leq \alpha < \frac{p}{2}$, $0 \leq \alpha < \frac{p}{2}$ and $0 \leq \beta < p$*

Then

$$\frac{\Gamma(p+1)\Gamma(p-\xi+\eta+1)}{\Gamma(p+1-\xi)\Gamma(p+1+\mu+\eta)}|z|^{p-\xi}\times$$
$$\left[1-\frac{\beta(p-\xi+\eta+1)}{(p+1)(p+1-\xi)(p+1+\mu+\eta)[p(n-p)+\beta(\lambda n-p\lambda+p)]\cdot(1-\alpha)}|z|\right]$$
$$\leq\left|J_{0,z}^{\mu,\xi,\eta}f(z)\right|\leq$$

$$\frac{\Gamma(p+1)\Gamma(p-\xi+\eta+1)}{\Gamma(p+1-\xi)\Gamma(p+1+\mu+\eta)}|z|^{p-\xi}\times$$
$$\left[1+\frac{\beta(p-\xi+\eta+1)}{(p+1)(p+1-\xi)(p+1+\mu+\eta)[p(n-p)+\beta(\lambda n-p\lambda+p)]\cdot(1-\alpha)}|z|\right] \tag{6.6}$$

$(z\in U_0;\mu>0, max\{\xi,\xi-\eta,-\mu-\eta\}<2;\xi(\mu+\eta)\leq 3\mu)$ and

$$\frac{\Gamma(p+1)\Gamma(p-\xi+\eta+1)}{\Gamma(p+1-\xi)\Gamma(p+1-\mu+\eta)}|z|^{p-\xi}\times$$
$$\left[1-\frac{\beta(p-\xi+\eta+1)}{(p+1)(p+1-\xi)(p+1-\mu+\eta)[p(n-p)+\beta(\lambda n-p\lambda+p)]\cdot(1-\alpha)}|z|\right]$$
$$\leq\left|K_{0,z}^{\mu,\xi,\eta}f(z)\right|\leq$$

$$\frac{\Gamma(p+1)\Gamma(p-\xi+\eta+1)}{\Gamma(p+1-\xi)\Gamma(p+1-\mu+\eta)}|z|^{p-\xi}\times$$
$$\left[1+\frac{\beta(p-\xi+\eta+1)}{(p+1)(p+1-\xi)(p+1-\mu+\eta)[p(n-p)+\beta(\lambda n-p\lambda+p)]\cdot(1-\alpha)}|z|\right] \tag{6.7}$$

$(z\in U_0;0\leq\mu<p, max\{\xi,\xi-\eta,-\mu-\eta\}<2;\xi(\mu-\eta)\geq 3\mu)$,

$$\text{where}\quad U_0=\begin{cases} U & ,\quad \xi\leq p\\ U/\{0\} & ,\quad \xi>p\end{cases}$$

Each of these results is sharp for the function $f(z)$ given by

$$f(z)=z^p-\frac{p\beta}{2[p(n-p)+\beta(\lambda n-p\lambda+p)](1-\alpha)}z^n \tag{6.8}$$

Proof : Applying Theorem 1, we have

$$\sum_{n=p+1}^{\infty} a_n \leq \frac{p\beta}{[p(n-p)+\beta(\lambda n - p\lambda + p)]G(\alpha, n)} \tag{6.9}$$

Next, using the assertion (6.4) of Lemma 1, we find from (1.2) that

$$F(z) = \frac{\Gamma(p-\xi+1)\Gamma(p+\mu+\xi+1)}{\Gamma(p+1)\Gamma(p-\xi+\eta+1)} z^{\xi} J_{0,z}^{\mu,\xi,\eta} f(z) = z^p - \sum_{n=p+1}^{\infty} \phi(n) a_n z^n \tag{6.10}$$

$$\text{where} \quad \phi(n) = \frac{(p+1)_n (p-\xi+\eta+1)_n}{(p-\xi+1)_n (p+\mu+\xi+1)_n}, \quad \eta \geq p+1 \tag{6.11}$$

The function $\phi(n)$ defined by (6.11) can easily be seen to be non-increasing under the parameter constraints stated in (6.6) and thus we have

$$0 < \phi(n) \leq \phi(p+1) = \frac{(p+1)(p-\xi+\eta+1)}{(p-\xi+1)(p+\mu+\xi+1)} \tag{6.12}$$

The assertation (6.6) of the theorem follows from (6.9), (6.10) and (6.12)
The assertation (6.7) of the theorem can be proved similarly by noting from (6.5) that

$$G(z) = \frac{\Gamma(p+1-\xi)\Gamma(p+1-\mu+\eta)}{\Gamma(p+1)\Gamma(p-\xi+\eta+1)} z^{\xi} K_{0,z}^{\mu,\xi,\eta} f(z) = z^p - \sum_{n=p+1}^{\infty} \psi(n) a_n z^n \tag{6.13}$$

where

$$\psi(n) = \frac{(p+1)_n (p-\xi+\eta+1)_n}{(p+1-\xi)_n (p+1-\mu+\eta)_n}, \quad n \geq p+1 \tag{6.14}$$

Thus similarly,

$$0 < \psi(n) \leq \psi(p+1) = \frac{(p+1)(p-\xi+\eta+1)}{(p+1-\xi)(p+1-\mu+\eta)}, \quad n \in N \backslash \{1\} \tag{6.15}$$

The assertion (6.6) and (6.7) are attained

Corollary 6.1 *Let the function $f(z)$ defined by (1.2) be in the class $S^*(\alpha, p, \lambda, \beta)$, with $0 \leq \alpha < \frac{p}{2}$, $0 \leq \alpha < p$ and $0 \leq \beta < p$*

Then

$$\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)}|z|^{p+\mu}[1-\frac{p(p+1)\beta}{2(p+1)(p+1+\mu)[p(n-p)+\beta(\lambda n-p\lambda+p)](1-\alpha)}|z|]$$

$$\leq \left|D_z^{-\mu}f(z)\right| \leq \frac{\Gamma(p+1)}{\Gamma(p+1+\mu)}|z|^{p+\mu}\left[1+\frac{p(p+1)\beta}{2(p+1)(p+1+\mu)[p(n-p)+\beta(\lambda n-p\lambda+p)](1-\alpha)}|z|\right] \tag{6.16}$$

and

$$\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)}|z|^{p-\mu}\left[1-\frac{p(p+1)\beta}{2(p+1)(p+1-\mu)[p(n-p)+\beta(\lambda n-p\lambda+p)](1-\alpha)}|z|\right]$$

$$\leq |D_z^{\mu}| \leq \frac{\Gamma(p+1)}{\Gamma(p+1-\mu)}|z|^{p-\mu}\left[1+\frac{p(p+1)\beta}{2(p+1)(p+1-\mu)[p(n-p)+\beta(\lambda n-p\lambda+p)](1-\alpha)}|z|\right] \tag{6.17}$$

$$(z\in U, 0\leq\mu<p)$$

Each of these result are sharp for the function $f(z)$ given by (6.8)

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