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Boundary Value of Analytic Functions and Boehmians of Analytic Type

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Abstract

In this paper, we shall associate a unique Boehmian of analytic type to every given analytic function defined in the open unit disc of the finite complex plane. We shall show that this Boehmian can be regarded as the boundary value of the given analytic function in a natural way. This answers a question raised in the literature.

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1 Introduction

Let $L^1(T)$ denote the set of all complex-valued Lebesgue integrable functions on the unit circle T. We make no distinction between a function on T and a 2π-periodic function on the real line $\mathbb{R}$.

The convolution of two functions $f, g \in L^1(T)$, denoted by $f * g$, is defined by

$$(f * g)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x-t)g(t)dt \qquad x \in [0, 2\pi].$$

It is easy to see that $f * g \in L^1(T)$ (see [2]). The norm on $L^1(T)$ is defined by

$$\|f\|_1 = \frac{1}{2\pi}\int_0^{2\pi} |f(t)|dt \qquad (f \in L^1(T)).$$

Under this norm $L^1(T)$ becomes a Banach space (see [2]).

The n^{th} Fourier co-efficient of $f \in L^1(T)$ is defined by

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx.$$

Definition 1.1 *A summability kernel is a sequence of continuous* 2π*-periodic functions* $\{\phi_k\}$ *satisfying*

(*i*) $\frac{1}{2\pi}\int_0^{2\pi} \phi_k(t)dt = 1 \quad (k \in \mathbb{N})$

(*ii*) $\frac{1}{2\pi}\int_0^{2\pi} |\phi_k(t)|dt \leq M,$ *a constant* $(k \in \mathbb{N})$

(*iii*) $\lim_{k\to\infty} \frac{1}{2\pi}\int_\delta^{2\pi-\delta} \phi_k(t)dt = 0$ *for each* δ *with* $0 < \delta < \pi$.

The collection of all summability kernels will be denoted by Δ.

The following lemmas can be easily proved (see [2]).

Lemma 1.2 *If* $f \in L^1(T)$ *and* $\{\phi_k\} \in \Delta$ *then* $f * \phi_k \to f$ *in* $L^1(T)$ *as* $k \to \infty$.

Lemma 1.3 *If* $\{\phi_k\}, \{\psi_k\} \in \Delta$ *then* $\{\phi_k * \psi_k\} \in \Delta$.

Lemma 1.4 *If* $f_n \to f$ *in* $L^1(T)$ *as* $n \to \infty$ *and* $\{\phi_k\} \in \Delta$ *then* $f_n * \phi_k \to f * \phi_k$ *in* $L^1(T)$ *as* $n \to \infty$ *for each fixed* k.

A pair of sequences $(\{f_k\}, \{\phi_k\})$ where $f_k \in L^1(T), \{\phi_k\} \in \Delta$, is said to be a quotient if

$$f_k * \phi_m = f_m * \phi_k \text{ for all } k, m \in \mathbb{N}.$$

The collection of all quotients is denoted by A. Two elements $(\{f_k\}, \{\phi_k\})$ and $(\{g_k\}, \{\psi_k\})$ of A are said be equivalent if

$$f_k * \psi_m = g_m * \phi_k \text{ for all } k, m \in \mathbb{N}.$$

It is easy to show that this relation is an equivalence relation on A. The equivalence classes are called periodic Boehmians denoted by β. A typical element of β will be written as $[(\{f_k\}, \{\phi_k\})]$. In a canonical way we can define addition, convolution and scalar multiplication on β as follows:

$$[(\{f_k\},\{\phi_k\})] + [(\{g_k\},\{\psi_k\})] = [(\{f_k * \psi_k + g_k * \phi_k\},\{\phi_k * \psi_k\})]$$
$$[(\{f_k\},\{\phi_k\})] * [(\{g_k\},\{\psi_k\})] = [(\{f_k * g_k\},\{\phi_k * \psi_k\})]$$
$$\alpha\,[(\{f_k\},\{\phi_k\})] = [(\{\alpha f_k\},\{\phi_k\})] \ \ (\alpha \in \mathbb{C}).$$
$$[(\{f_k\},\{\phi_k\})] * \phi = [(\{f_k * \phi\},\{\phi_k\})] \ \ (\phi \in L^1(T)).$$

It is easy to see that β becomes a commutative algebra with identity $[(\{\phi_k\},\{\phi_k\})]$.

Note 1.5 *The space $L^1(T)$ can be identified with a subspace of β using the injective map $f \to [(\{f * \phi_k\},\{\phi_k\})]$ for any delta sequence $\{\phi_k\} \in \Delta$.*

Definition 1.6 *A sequence $\{X_n\}$ in β is said to δ-converge to some $X \in \beta$, denoted by $X_n \xrightarrow{\delta} X$ as $n \to \infty$ if there exists a sequence $\{\phi_k\} \in \Delta$ such that $X_n * \phi_k, X * \phi_k \in L^1(T)$ $(n, k \in \mathbb{N})$ and $X_n * \phi_k \to X * \phi_k$ in $L^1(T)$ as $n \to \infty$ for each fixed k.*

Definition 1.7 *Let $X = [(\{f_k\},\{\phi_k\})] \in \beta$. The n^{th} Fourier co-efficient of X, denoted by $\hat{X}(n)$, is defined by $\hat{X}(n) = \lim_{k\to\infty} \hat{f}_k(n)$.*

The above limit always exists and is independent of the representative of X. This can be easily seen using the quotient property of the Boehmian X.

A Boehmian is said to be of analytic type if $\hat{X}(n) = 0$ for $n = -1, -2, \cdots$.

In [1], the following question is raised:

"What type of function in the unit disc has a Boehmian of analytic type as a boundary value?"

In this paper, we shall prove that to each analytic function (bounded or not) in the open unit disc we can associate a unique Boehmian in β which can be regarded as representing the boundary value of the given analytic function. We can further show that this Boehmian is always of analytic type. This answers affirmatively the question raised above in this more general context.

2 Main Results

Definition 2.1 *Let $H = H(\mathbb{D})$ denote the class of all analytic functions defined in the open unit disc $\mathbb{D}$. We shall equip H with the usual topology of uniform convergence on compact subsets of $\mathbb{D}$.*

Let $A = \{f \in H(\mathbb{D}) / f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\sum_{n=0}^{\infty} |a_n| < \infty\}$.

If $g(z) = \sum_{n=0}^{\infty} a_n z^n \in A$ then the boundary value of g is obviously given by $g^*(\theta) = \sum_{n=0}^{\infty} a_n e^{in\theta} \quad (\theta \in [0, 2\pi])$. Further $g^* \in L^1(T)$.

Theorem 2.2 *Let $f \in H$ then there exists a sequence $\{S_m\}$ in A and $X \in \beta$ such that $S_m \to f$ in H and $S_m^* \xrightarrow{\delta} X$ in β as $m \to \infty$.*

Proof: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H$. For each $m \in \mathbb{N}$, define

$$S_m(z) = \sum_{n=0}^{m} a_n z^n, \quad z \in \mathbb{D}.$$

Then obviously $S_m \in A$ and $S_m \to f$ in H as $m \to \infty$. Also $S_m^*(\theta) = \sum_{n=0}^{m} a_n e^{in\theta}$. Consider Fejer's kernel (see [2]) given by

$$\psi_k(\theta) = \sum_{j=-k}^{k} \left(1 - \frac{|j|}{k+1}\right) e^{ij\theta} \qquad (\theta \in [0, 2\pi]).$$

It is easy to see that $\{\psi_k\} \in \Delta$ and

$$\hat{\psi}_k(n) = \begin{cases} \left(1 - \frac{|n|}{k+1}\right) & \text{for } |n| \le k \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Now

$$\begin{aligned} (S_m^* * \psi_k)(\theta) &= \frac{1}{2\pi} \int_0^{2\pi} S_m^*(\theta - \eta) \psi_k(\eta) d\,\eta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=0}^{m} a_n e^{in(\theta-\eta)} \psi_k(\eta) d\,\eta \\ &= \sum_{n=0}^{m} a_n e^{in\theta} \hat{\psi}_k(n). \end{aligned} \tag{2}$$

Define $g_k(\theta) = \sum_{n=0}^{k} a_n e^{in\theta} \hat{\psi}_k(n) \in L^1(T)$. Then using (1) we can easily prove that for each k, $S_m^* * \psi_k \to g_k$ in $L^1(T)$ as $m \to \infty$ and $(\{g_k\}, \{\psi_k\}) \in$ A. Take $X = [(\{g_k\}, \{\psi_k\})] \in \beta$. By definition $S_m \xrightarrow{\delta} X$ in β as $m \to \infty$. This completes the proof of our theorem.

Theorem 2.3 *Let $f \in H$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $X \in \beta$ be as in Theorem 2.2. Suppose that there exists a sequence $\{h_m\}$ in A such that $h_m \to f$ in H and $h_m^* \xrightarrow{\delta} Y$ in β as $m \to \infty$. Then $Y = X$ in β.*

Proof: Let us first show that for each k, $h_m^* * \psi_k \to g_k$ in $L^1(T)$ as $m \to \infty$ where $\{\psi_k\}$ is a Fejer's kernel and $g_k = X * \psi_k$. For $m, k \in \mathbb{N}$,

$$(h_m^* * \psi_k)(\theta) = \sum_{j=0}^{k} \left(1 - \frac{j}{k+1}\right) e^{ij\theta} (h_m^*)\hat{}(j) \quad (\theta \in [0, 2\pi]).$$

Since $h_m \to f$ in H as $m \to \infty$, $h_m^{(j)} \to f^{(j)}$ in H as $m \to \infty$ ($f^{(j)}$- denote the j-th derivative of f). Also $(h_m^*)\hat{}(j) = \frac{h_m^{(j)}(0)}{j!}$ and hence $(h_m^*)\hat{}(j) \to a_j$ as $m \to \infty$ for each j. This proves that $h_m^* * \psi_k \to g_k$ in $L^1(T)$ as $m \to \infty$ for each $k \in \mathbb{N}$. By hypothesis $h_m^* \xrightarrow{\delta} Y$ in β as $m \to \infty$ and hence there exists $\{\eta_l\} \in \Delta$ such that $h_m^* * \eta_l, Y * \eta_l = \chi_l \in L^1(T)$ and $h_m^* * \eta_l \to \chi_l$ in $L^1(T)$ as $m \to \infty$ for each $l \in \mathbb{N}$. Thus the sequence $\{h_m^* * \eta_l * \psi_k\}$ converges to $g_k * \eta_l$ and $\chi_l * \psi_k$ for each fixed $l, k \in \mathbb{N}$ as $m \to \infty$. It follows that

$$g_k * \eta_l = \chi_l * \psi_k \quad (\forall\ k, l \in \mathbb{N}) \text{ or that } X = Y.$$

This completes the proof of our theorem.

Note 2.4 The above theorems give the existence and the uniqueness of a Boehmian in β which can be naturally associated to the boundary value of $f \in H$. We now remark that in the question raised in [1], the delta sequences constructed in that context form a subclass of Δ defined above. Hence the Boehmian space we have constructed is larger than the one constructed in [1]. Thus it may still be interesting to ask the same question with reference to the original Boehmian space. We can also answer this as follows. If we take a delta sequence $\{\phi_k\}$ (given in [1]) with the property that all ϕ_k's are infinitely differentiable then we have the following inequality (see [2])

$$|\hat{\phi}_k(n)| \leq \frac{b_k}{n^j} \text{ for each } n, j = 1, 2, 3, \cdots .$$

Using this estimate in (2) we can easily show that

$$(S_m^* * \phi_k)(\theta) \to \xi_k(\theta) = \sum_{n=0}^{\infty} a_n e^{in\theta} \hat{\phi}_k(n) \text{ in } L^1(T) \text{ as } m \to \infty \, (k \in \mathbb{N}).$$

However, for this to happen we have to assume that $|a_n| = O(n^p)$ as $n \to \infty$ (where p is any positive integer). Thus we can associate a Boehmian of analytic type as the boundary value of each such analytic function. Further it can be noted that there is a large class of analytic functions in the open unit disc satisfying the above condition. For example, injective analytic functions (in view of the proof of Bieberbach's conjecture) or the so called p-valent functions do satisfy such a condition. We have chosen summability kernels as our delta sequences just to include all analytic functions in the open unit disc.

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